A Note on Combinatorial Identities arising from the Lagrange-Waring Interpolation Formula

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The Lagrange Interpolation Formula may be stated in two ways as follows:

$$f(x+y) = \sum_{k=0}^{n} f(x_k + y) \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}, \tag{1}$$

$$f(x+y) = g(x) \sum_{k=0}^{n} \frac{f(x_k + y)}{(x - x_k) g'(x_k)}, \tag{2}$$

where

$$g(x) = \prod_{i=0}^{n} (x - x_i), \tag{3}$$

and where $x_0, x_1, x_2, \ldots, x_n$ is a sequence of $n+1$ distinct values of x, i.e. $x_i = x_j$ if and only if $i = j$.

The expansion is given as formula (Z.1) in my book [1].

The expansion is exact when f(x) is a polynomial of degree $\leq n$. For other functions there will be a remainder term, which is of concern only when one studies numerical approximations.

Remarks. The Lagrange formula is a specialization of Newton's Divided

Difference Formula. Also, Waring [4] published the formula (1) sixteen years before Lagrange [3, Œuvres, p. 286].

Melzak [4] posed the formula

$$f(x+y) = y\binom{y+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(x-k)}{y+k}, \tag{4}$$

and this follows directly from the Lagrange formula when we choose $x_k = -k$ and apply the Lagrange formula to get

$$f(x+y) = \sum_{k=0}^{n} f(y-k)\prod_{\substack{i=0\\ i\neq k}}^{n}\frac{x+i}{i-k}$$

and then note that

$$\prod_{i=0}^{n}(x+i) = n!\,x\binom{x+n}{n}, \tag{5}$$

and

$$\frac{n!}{\prod_{\substack{i=0\\ i\neq k}}^{n}(i-k)} = (-1)^k\binom{n}{k}. \tag{6}$$

My unpublished (but mentioned) proof [4] of the Melzak formula is exactly what we have just exhibited. Melzak's formula is listed as (Z.5) in [1].

The Melzak formula figured widely in obtaining a number of the combinatorial identities in my book [1].

Our purpose here is to show the proofs of a few of those identities and to exhibit some other novel identities that follow easily from the Lagrange-Waring interpolation formula.

The simplest case of Melzak's formula is when we choose f(x) = 1 identically:

$$y\binom{y+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{y+k} = 1 \tag{7}$$

and this is tabulated in my book as formula (1.41). Also listed in the book as formula (1.42) is the natural inverse of this:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{y+k}{k}^{-1} = \frac{y}{y+n}. \tag{8}$$

Both of these, as well as the original Lagrange-Waring expansion, may be considered as partial-fraction expansions. One solution [4] obtained Melzak's formula in this manner.

Natural inverses follow because of the simple binomial inverse pair:

$$A_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}B_k \tag{9}$$

if and only if

$$B_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}A_k. \tag{10}$$

Relations (7) and (8) have appeared numerous times in the vast mathematical literature.

Just a few of the initial identities in [1] that follow from the

Lagrange or Melzak formulas are: (1.43), (1.44), (1.45), (1.46), (1.47), and the reader is challenged to determine just how many of the 550 identities in [1] can be seen as consequences of the Lagrange formula. We will discuss some of the more curious ones here.

Differentiation of the Melzak formula with respect to y gives

$$f(x+y)\sum_{k=0}^{n}\frac{1}{y+k} - D_y f(x+y)$$

$$= y\binom{y+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(x-k)}{(y+k)^2}, \qquad (11)$$

and this is expansion (Z.6) in [1]. By repeated differentiation with respect to y and using $f(x) = 1$ one can sum the series

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(y+k)^{p+1}}, \qquad (12)$$

in terms of the reciprocal power sums

$$S_p = \sum_{k=0}^{n}\frac{1}{(y+k)^p}.$$

which uses Riordan's cycle indicator, and this will be discussed in [2].

An interesting special case of the Lagrange expansion is

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}\frac{f(y+k^2)}{x^2-k^2}$$

$$= (1)^{n} \frac{f(x^{2}+y)}{2x(x-n)\binom{x+n}{2n}} + \frac{1}{2}\binom{2n}{n}\frac{f(y)}{x^{2}}, \tag{13}$$

and this is formula (Z.10) in [1].

An interesting alternative statement of this is as follows:

$$\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \frac{f(y+k^{2})}{\binom{n+k}{k}(x^{2}-k^{2})}$$

$$= (-1)^{n} \frac{f(x^{2}+y)}{2x(x-n)\binom{2n}{n}\binom{x-n}{2n}} + \frac{f(y)}{2x^{2}}, \tag{14}$$

which is formula (Z.11) in [1].

The derivation of (13) from the Lagrange expansion (2) follows by choosing $x_k = k^2$, and noting the following easy calculations:

$$g(x^{2}) = x(x-n)(2n)!\binom{x+n}{2n},$$

and

$$g'(k^{2}) = \frac{1}{2}(-1)^{k}(n-k)!(n+k)!.$$

Formula (14) then follows by simple factorial rearrangements.

Another variation is formula (Z.14) in [1] which says that

$$\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \frac{f(y-k)}{(k+x)(k+z)}$$

$$= \frac{1}{(n+1)(x-z)} \left\{ \frac{f(y+z)}{\binom{z+n}{n+1}} - \frac{f(y+x)}{\binom{x+n}{n+1}} \right\}, \tag{15}$$

where now $f(x)$ is a polynomial of degree $\leq n+1$ in x.

The following is an easy deduction from the Lagrange formula:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(y-k)}{\binom{k+r}{k}} = -\sum_{k=1}^{r} (-1)^k \binom{n}{k} \frac{f(y+k)}{\binom{k+n}{k}}, \tag{16}$$

where $f(x)$ is a polynomial of degree $\leq n+r-1$ in x. This is formula (Z.16) in [1].

In this, set $r = n$. Then we have the special case [1,(Z.17)]

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(y-k) + f(y+k)}{\binom{k+n}{k}} = f(y), \tag{17}$$

where $f(x)$ is a polynomial of degree $\leq 2n-1$ in x

Now we move to some formulas not tabulated in [1]. Recall that by choosing $x_k = -k$, we found (4) which we have called Melzak's formula.

It is then natural to choose $x_k = k$, and get a companion to Melzak. Now we find easily that

$$\prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x-i}{k-i} = \binom{x}{k}\binom{n-x}{n-k}, \tag{18}$$

so that from the Lagrange expansion (1) we get the novel formula

$$f(x+y) = \sum_{k=0}^{n} \binom{x}{k}\binom{n-x}{n-k} f(k+y), \qquad (19)$$

We may then use this to see that

$$f(k+y) = \sum_{j=0}^{n} \binom{y}{j}\binom{n-y}{n-j} f(j+k),$$

so that we get the second novel formula

$$f(x+y) = \sum_{k=0}^{n}\sum_{j=0}^{n} \binom{x}{k}\binom{n-x}{n-k}\binom{y}{j}\binom{n-y}{n-j} f(j+k), \qquad (20)$$

where in both (19) and (20) $f(x)$ is a polynomial of degree $\leq n$ in x.

We note two interesting special cases of (19):

$$\sum_{k=0}^{n} \binom{x}{k}\binom{n-x}{n-k}\binom{k+y}{r} = \binom{x+y}{r}, \quad \text{for any } 0 \leq r \leq n, \qquad (21)$$

and

$$\sum_{k=0}^{n} \binom{x}{k}\binom{n-x}{n-k}(k+y)^{r} = (x+y)^{r}, \quad \text{or any } 0 \leq r \leq n. \qquad (22)$$

From (20) we have

$$\sum_{k=0}^{n}\sum_{j=0}^{n} \binom{x}{k}\binom{n-x}{n-k}\binom{y}{j}\binom{n-y}{n-j}\binom{j+k}{r} = \binom{x+y}{r}, \qquad (23)$$

where $0 \leq r \leq n$.

Relations (19), (20), (21), and (23) were not listed in [1], and do not seem to be commonly known.

REFERENCES

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