The Vertex Linear Arboricity of Claw-Free Graphs with Small Degree

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Abstract

The vertex linear arboricity $vla(G)$ of a graph G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces a subgraph whose connected components are paths. It is proved here that $\lceil\frac{\omega(G)}{2}\rceil \le vla(G) \le \lceil\frac{\omega(G)+1}{2}\rceil$ for a claw-free connected graph G having $\Delta(G) \le 6$, where $\omega(G)$ is the clique number of G.

Key words: claw-free graph; vertex linear arboricity; linear forest; linear arboricity

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Throughout this paper, all graphs are finite, simple and undirected. For a real number x, $\lceil x\rceil$ is the least integer not less than x. For a graph G, we use $V(G)$, $E(G)$, $\Delta(G)$ and $\omega(G)$ to denote the vertex set, the edge set, the maximum degree and the clique number of G. A graph is called *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph.

A *linear forest* is a graph in which each component is a path. A coloring φ from V(G) to $\{1,2,\cdots,t\}$ is called a *linear t-coloring* if for any color $i(1 \le i \le t)$, the set $\{v : \varphi(v) = i\}$ induces a linear forest. The *vertex linear arboricity* $vla(G)$ of a graph G is the minimum number t such that G has a linear t-coloring. Matsumoto [6] proved the following theorem.

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Theorem A. For a graph G, $vla(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$, moreover, if $\Delta(G)$ is even, then $vla(G) = \lceil \frac{\Delta(G)+1}{2} \rceil$ if and only if G is the complete graph of order $n+1$ or a cycle.

Goddard [4] and Poh [7] proved that the vertex linear arboricity of a planar graph is at most 3. Akiyama *etc.* [1] proved that the vertex linear arboricity of an outerplanar graph is at most 2. In this note, we prove the following theorem.

Theorem 1. *Let G be a claw-free connected graph having $\Delta(G) \leq 6$. Then*

$$\lceil \frac{\omega(G)}{2} \rceil \leq vla(G) \leq \lceil \frac{\omega(G)+1}{2} \rceil.$$

The *linear arboricity* $la(G)$ of a graph G as defined by Harary [5] is the minimum number of linear forests which partition the edges of G. Akiyama, Exoo and Harary [1] conjectured that $\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ for any graph G and they proved the conjecture is true for graphs with $\Delta = 3, 4$[2][3]. Since the line graph of a graph is claw-free and the linear arboricity of a graph is equal to the vertex linear arboricity of its line graph, the result of Akiyama, Exoo and Harary is generalized.

Proof. The lower bound that $vla(G) \geq \lceil \frac{\omega(G)}{2} \rceil$ is obvious and it suffices to prove $vla(G) \leq \lceil \frac{\omega(G)+1}{2} \rceil$. If $\Delta(G) = 1, 2$ or 3, then $\Delta(G) = \omega(G)$. If $\Delta(G) = 4$, then $3 \leq \omega(G) \leq 5$, moreover, $\omega(G) = 5$ if and only if $G = K_5$. If $\Delta(G) = 6$, then $4 \leq \omega(G) \leq 7$, moreover, $\omega(G) = 7$ if and only if $G = K_7$. So it follows from Theorem A that the theorem is true for $\Delta(G) = 1, 2, 3, 4, 6$. If $\Delta(G) = 5$, then $3 \leq \omega(G) \leq 5$. it follows from Theorem A that the theorem is also true for $\Delta(G) = 5$ and $\omega(G) \geq 4$. So it suffices to prove the case $\Delta(G) = 5$ and $\omega(G) = 3$, that is, to prove that if a claw-free graph G has a linear 2-coloring if $\Delta(G) \leq 5$ and $\omega(G) = 3$.

We use induction on $|G|$. Since G is claw-free and $\omega(G) = 3$, neighbors of any vertex of degree 5 induces a cycle. Let $v \in V(G)$ be a vertex of degree 5 and the cycle induced by $N(v)$ be $v_0v_1v_2v_3v_4v_0$. It is obvious that $H = G - v$ is also claw-free. So by the induction hypothesis, H has a linear 2-coloring φ. Let $V_i = \{u : \varphi(u) = i, u \in V(H)\}, i = 1, 2$. Then $V_1 \cup V_2 \cup \{v\} = V(G)$. A vertex $w \in V(H)$ is said to be a *spanning vertex* of V_i if $w \in V_i$ and $d_{G[V_i]}(w) \leq 1$ for some $i (1 \leq i \leq 2)$. In the following, we shall extend φ to a linear 2-coloring of G.

Claim A. *For some $v_i (0 \leq i \leq 4)$, if $\varphi(v_{i-1}) = \varphi(v_{i+1}) \neq \varphi(v_i)$ where the subscripts are taken modulo 5, then v_i must be a spanning vertex.*

Suppose, to the contrary, that v_i has two neighbors w', w'' such that $w', w'' \notin \{v, v_4, v_0, v_1, v_2, v_3\}$ and $\varphi(w') = \varphi(w'') = \varphi(v_i)$. Then $d(v_i) = 5$.

Since G is claw-free, w', w'', v_i form a cycle of length 3, a contradiction. So the contradiction proves Claim A.

By Claim A, if there are four vertices in $N(v)$ receiving the same color, without loss of generality, assume $\{v_0, v_1, v_2, v_3\} \subseteq V_2$, then v_4 must be a spanning vertex of V_1. We may color v with 1 to extend φ to a linear 2-coloring of G. So without loss of generality, assume that $|N(v) \cap V_1| = 2$ and $|V_2 \cap N(v)| = 3$.

Case 1 Two vertices in $V_1 \cap N(v)$ are not adjacent.

Without loss of generality, assume that $\{v_1, v_4\} \subseteq V_1$ and $\{v_0, v_2, v_3\} \subseteq V_2$. It follows from Claim A that v_1 and v_4 are spanning vertices of V_1, and v_0 is the spanning vertex of V_2. Since G is claw-free, $\max\{|N(v_0) \cap V_2|, |N(v_1) \cap V_1|, |N(v_4) \cap V_1|\} \leq 1$. If $N(v_1) \cap V_1 = \emptyset$ or $N(v_4) \cap V_1 = \emptyset$, then we may color v with 1 to extend φ to a linear 2-coloring of G. So in the following, assume that $|N(v_1) \cap V_1| = |N(v_4) \cap V_1| = 1$.

Subcase 1.1 $N(v_1) \cap V_1 = N(v_4) \cap V_1 = \{v_5\}$. Since G is claw-free, $|\{v_0v_5, v_2v_5, v_3v_5\} \cap E(G)| \geq 1$, moreover, $|\{v_0v_5, v_2v_5, v_3v_5\} \cap E(G)| \geq 2$ if $\{v_3v_5, v_2v_5\} \cap E(G) \neq \emptyset$.

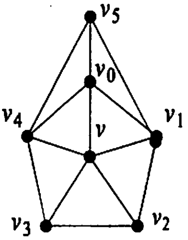

Figure 1

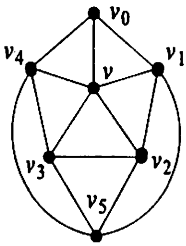

Figure 2

Suppose that $v_0v_5 \in E(G)$ and $v_2v_5, v_3v_5 \notin E(G)$(see Figure 1). Then $N(v_0)$ induces a cycle of length 4. So $d(v_0) = 4$. We may recolor v_2 with 1, color v with 2. Suppose that $v_2v_5, v_3v_5 \in E(G)$ and $v_0v_5 \notin E(G)$(see Figure 2). Then $d(v_2) = d(v_3) = 4$. Recolor v_5 with 2, v_2 and v_3 with 1, and color v with 2. Suppose $v_0v_5, v_2v_5, v_3v_5 \in E(G)$. Then $d_G(v_i) = 4$ for all $i (0 \leq i \leq 4)$, that is, $V(G) = \{v, v_0, \cdots, v_5\}$. We may recolor v_1 with 2, color v with 1. Suppose $v_0v_5 \in E(G)$ and $|\{v_2v_5, v_3v_5\} \cap E(G)| = 1$. Without loss of generality, assume that $v_2v_5 \in E(G)$ and $v_3v_5 \notin E(G)$. Then $d_G(v_0) = 4$. We also have $d_G(v_2) = 4$(for otherwise, if there is another vertex $v_6 \in N(v_2) \backslash \{v, v_1, v_3, v_5\}$, then $v_6v_3, v_5v_6 \in E(G)$. Since $v_6v_2, v_6v_3 \in E(G)$, $v_6 \notin V_2$. Since $v_6v_5 \in E(G)$, $v_6 \notin V_1$. This is impossible). It follows that $d_G(v_5) = d_G(v_4) = 4$ and $d_G(v_3) = 3$. So $V(G) = \{v, v_0, \cdots, v_5\}$. We may recolor v_1 with 2, color v with 1.

Subcase 1.2 $N(v_1) \cap V_1 \neq N(v_4) \cap V_1$, that is to say, there exist two different vertices v_5 and v_6 such that $\varphi(v_5) = \varphi(v_6) = 1$ and $v_5v_1, v_6v_4 \in E(G)$. Then $|\{v_5v_0, v_5v_2\} \cap E(G)| \geq 1$ and $|\{v_0v_6, v_3v_6\} \cap E(G)| \geq 1$.

Suppose that $v_5v_0, v_6v_0 \in E(G)$(see Figure 3). Since $d_G(v_0) = 5$, $v_5v_6 \in E(G)$. If $N(v_1) \cap V_2 = \{v_0, v_2\}$, then $d(v_1) = 4$, v_2 is the spanning vertex of V_2 and $|(N(v_2)\backslash\{v_1\}) \cap V_1| \leq 1$. We can recolor v_1 with 2 and color v with 1. So assume that v_1 and v_2 have a common neighbor $v_7 \in V_2$. By the same argument, assume that v_3 and v_4 have a common neighbor $v_8 \in V_2$. Note that $v_7 \neq v_8$. Thus $d(v_1) = d(v_4) = 5$ and it follows that $v_5v_7, v_6v_8 \in E(G)$. Now we may recolor v_2 with 1, color v with 2.

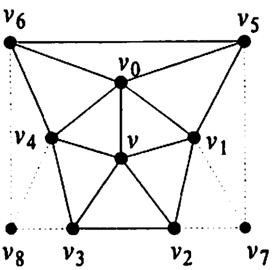

Figure 3

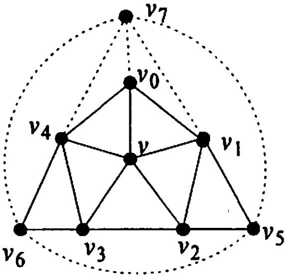

Figure 4

Suppose that $v_5v_2, v_6v_3 \in E(G)$(see Figure 4). Then v_0, v_2 and v_3 are the spanning vertices of V_2. If $N(v_1) \cap V_2 = \{v_0, v_2\}$, then we can recolor v_1 with 2 and color v with 1. Similarly, if $N(v_4) \cap V_2 = \{v_0, v_3\}$, then we can recolor v_4 with 2 and color v with 1. Hence assume $|N(v_1) \cap V_2| = |N(v_4) \cap V_2| = 3$. This implies that v_1 and v_4 has a common neighbor $v_7 \in V_2$. It follows that we have $v_7v_0, v_7v_5, v_7v_6, v_5v_6 \in E(G)$, $d(v_0) = d(v_2) = d(v_3) = d(v_5) = d(v_6) = 4$. So $V(G) = \{v, v_0, \cdots, v_7\}$, and directly color v with 1.

Suppose that $v_5v_0, v_6v_3 \in E(G)$(the case $v_5v_2, v_6v_0 \in E(G)$ can be settled similarly). Then v_0 and v_3 are spanning vertices of V_2. If $N(v_1) \cap V_2 = \{v_0, v_2\}$, then we may recolor v_1 with 2 and color v with 1. If $N(v_4) \cap V_2 = \{v_0, v_3\}$, then we may recolor v_4 with 2 and color v with 1. So assume that v_0 and v_4 have a common neighbor $v_7 \in V_2$, and v_1 and v_2 have a common neighbor $v_8 \in V_2$. Since $d(v_1) = d(v_4) = 5$, $v_5v_7, v_6v_7, v_5v_8 \in E(G)$. If $N(v_2) \cap V_1 = \{v_1\}$, then we may recolor v_2 with 1 and color v with 2; otherwise, recolor v_8 with 1, v_1 with 2 and color v with 1.

Case 2 Two vertices in $V_1 \cap N(v)$ are adjacent.

Without loss of generality, assume that $\{v_0, v_4\} \subseteq V_1$ and $\{v_1, v_2, v_3\} \subseteq V_2$. Since G is claw-free, $N(v_2) \cap V_2 = \{v_1, v_3\}$ and $1 \leq \min\{|N(v_0) \cap V_2|, |N(v_4) \cap V_2|\} \leq 2$. Assume that $1 \leq |N(v_4) \cap V_2| \leq 2$. If $|N(v_2) \cap V_1| \leq 1$, then recolor v_2 with 1, which is changed into Case 1. So assume that v_2 has two neighbors $v_5, v_6 \in V_1$ (refer to Figure 5). Thus we have $v_5v_6, v_1v_5, v_3v_6 \in E(G)$.

If $N(v_4) \cap V_2 = \{v_3\}$, then we may recolor v_4 with 2, color v with 1. So assume that $|N(v_4) \cap V_2| = 2$, that is, v_4 has another neighbor $v_7 \in V_2 \backslash v_3$. If $v_7v_3 \in E(G)$(see Figure 5), then $d(v_4) = 4$, $v_6v_7 \in E(G)$ and v_6 is the spanning vertex of V_1. We may recolor v_3 with 1 and v_4 with 2, which is changed into Case 1. So also assume that $v_7v_0 \in E(G)$(see Figure 6).

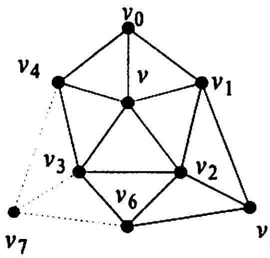

Figure 5

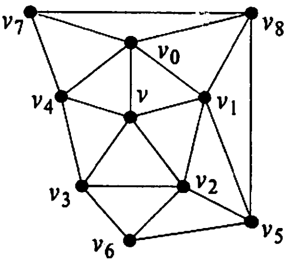

Figure 6

Suppose that $N(v_0) \cap V_2 = \{v_1, v_7\}$. Then v_7 is the spanning vertex of V_2 and we may recolor v_0 with 2, color v with 1. So assume that $|N(v_0) \cap V_2| = 3$, that is, there is a vertex $v_8 \in V_2 \cap (N(v_0) \backslash \{v, v_1, v_4, v_7\})$. Since $v_5, v_6 \in V_1$ and $v_7, v_8 \in V_2$, $v_1v_8, v_7v_8, v_5v_8 \in E(G)$ and $d(v_5) = d(v_8) = 4$. So we may recolor v_8 with 1, v_1 with 2, and color v with 1.

Hence φ is extended to a linear 2-coloring of G. □

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