Matrices Associated to Biindexed Linear Recurrence Relations

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Abstract

The factorization of matrix A with entries $a_{i,j}$ determined by $a_{i,j} = \alpha a_{i-1,j-1} + \beta a_{i,j-1}$ is derived as $A = TP^T$. A interesting factorization of matrix B with entries $b_{i,j} = \alpha b_{i-1,j} + \beta b_{i,j-1}$ is given by $B = P[\alpha]TP^T[\beta]$. The beautiful factorization of matrix C whose entries satisfy $c_{i,j} = \alpha c_{i-1,j} + \beta c_{i,j-1} + \gamma c_{i-1,j-1}$ is founded to be $C = P[\alpha]DP^T[\beta]$, where T is Toeplitz matrix. Where P and $P[\alpha]$ are Pascal matrices. The matrix product factorization to the problem is solved perfectly so far.

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1. Introduction

The matrix A with entries $a_{i,j}$ determined by recurrence relation $a_{i,j} = \alpha a_{i-1,j-1} + \beta a_{i-1,j}$ was studied by matrix method, and $A = PT$ is derived in [1]. Where T is Toeplitz matrix and P is Pascal matrix. The matrix $C = (c_{i,j})$ defined by $c_{i,j} = \alpha c_{i-1,j} + \beta c_{i,j-1} + \gamma c_{i-1,j-1}$ was investigated in [2], and an interesting recursive relation on its principal minors was obtained.

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In this paper, we study the linear recurrence relations $a_{i,j} = \alpha a_{i-1,j-1} + \beta a_{i,j-1}$ and $b_{i,j} = \alpha b_{i-1,j} + \beta b_{i,j-1}$, and find similar properties as that in [1]. At the same time, the relation $c_{i,j} = \alpha c_{i-1,j} + \beta c_{i,j-1} + \gamma c_{i-1,j-1}$ is investigated and the beautiful product factorization of its corresponding matrix is obtained. Where α, β are arbitrary complex numbers and $\alpha\beta \neq 0$. So far, we solved this matrix product factorization to the problem completely.

2. On $\mathbf{a_{i,j} = \alpha a_{i-1,j-1} + \beta a_{i,j-1}}$

Suppose recursive relation as

$$a_{i,j} = \alpha a_{i-1,j-1} + \beta a_{i,j-1} \tag{2.1}$$

with initial values $a_{i,0}(i = 0, 1, 2, \cdots)$, $a_{0,j}(j = 0, 1, 2, \cdots)$. The $(n+1) \times (n+1)$ matrix with elements $a_{i,j}$ is denoted by $A_n[\alpha, \beta] = A_n$, and the corresponding infinite matrix is A.

Let the generating functions respectively as follows:

$$f_A(x, y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j, \qquad f_A(x) = \sum_{i \geq 0} a_{i,0} x^i, \qquad g_A(y) = \sum_{j \geq 0} a_{0,j} y^j$$

We have

$$f_A(x, y) = \frac{f_A(x) + g_A(y) - \beta y g_A(y) - a_{0,0}}{1 - \beta y - \alpha x y} \tag{2.2}$$

We call $f_A(x, y)$ as the generating function of infinite matrix A also. Notice that

$$\frac{1}{1 - \beta y - \alpha x y} = \sum_{j \geq i \geq 0} \binom{j}{i} \alpha^i \beta^{j-i} x^i y^j$$

Set

$$Q_{i,j} = \binom{j}{i} \alpha^i \beta^{j-i} \quad , \quad Q_n = Q_n[\alpha, \beta] = (Q_{i,j})_{0 \leq i \leq j \leq n}$$

$$P_n = P_n[\alpha, \beta] = Q_n^T[\alpha, \beta] \quad , \quad P^T = Q = (Q_{i,j})_{0 \leq i,j < \infty}$$

where P^T is the transposed matrix of P. $P_n[\alpha, \beta]$ is generalized Pascal matrix as usual.

The coefficients of $x^i y^j$ in

$$\left(\sum_{i \geq 0} a_{i,0} x^i + \sum_{j \geq 1} a_{0,j} y^j - \sum_{j \geq 0} \beta a_{0,j} y^{j+1} \right) \sum_{0 \leq i \leq j} \binom{j}{i} \alpha^i \beta^{j-i} x^i y^j$$

is

$$a_{i,j} = \begin{cases} \sum_{k=0}^{i-1} a_{i-k,0}Q_{k,j} + \alpha \sum_{k=0}^{j-i} a_{0,k}Q_{i-1,j-k-1} & , j \geq i \geq 1 \\ \sum_{k=0}^{j} a_{i-k,0}Q_{k,j} & , j \leq i \end{cases} \tag{2.3}$$

Build a $(n+1) \times (n+1)$ matrix Toeplitz $T_n = (t_{i,j})_{0 \leq i,y \leq n}$(The corresponding infinite matrix T) as below:

$$t_{i,j} = \begin{cases} (-\alpha)^{i-j} \sum_{l=0}^{j-i} (-1)^l \binom{j-i}{l} \beta^{j-i-l} a_{0,l} & , j \geq i \geq 0 \\ a_{i-j,0} & , j \leq i \end{cases}$$

Observe that

$$P_n^{-1}[\alpha, \beta] = P_n[\frac{1}{\alpha}, -\frac{\beta}{\alpha}], P_n[\alpha_1, \beta_1]P_n[\alpha_2, \beta_2] = P_n[\alpha_1\alpha_2, \beta_1 + \alpha_1\beta_2] \tag{2.4}$$

$$P_n^m[\alpha, \beta] = P_n[\alpha^m, \frac{1-\alpha^m}{1-\alpha}\beta](m \geq 1, \alpha \neq 1) \ , \ P_n^m[1, \beta] = P_n[1, m\beta]$$

Theorem 1. A_n has the following factorization:

$$A_n = T_n P_n^T \tag{2.5}$$

Proof When $j \geq i \geq 1$, at first, we compute the following formula:

$$\sum_{k=0}^{j-i-l} (-1)^k \binom{k+i-1}{k} \binom{j}{k+i+l} = \sum_{k=0}^{j-i-l} (-1)^k \frac{j!}{(i-1)!(j-i-l)!(l+1)!} \binom{j-i-l}{k} \frac{1}{\binom{i+l+k}{l+1}}$$

$$= \frac{j!}{(i-1)!(j-i-l)!(l+1)!} \sum_{k=0}^{j-i-l} (-1)^k \binom{j-i-l}{k} \frac{1}{\binom{i+l+k}{l+1}}$$

$$\stackrel{*}{=} \frac{j!}{(i-1)!(j-i-l)!(l+1)!} \frac{l+1}{(j-i-l)+(l+1)} \frac{1}{\binom{j-i-l+i+l}{i+l-l-1}} = \binom{j-i}{l}$$

where $\stackrel{*}{=}$ is due to formula (4.2) in [3].

Now, we prove the general element at (i,j) in $A_n[\alpha, \beta]Q_n^{-1}[\alpha, \beta]$ is just $t_{i,j}$:

$$\sum_{k=0}^{n} \left(\sum_{l=0}^{i-1} a_{i-l,0}Q_{l,k} + \alpha \sum_{l=0}^{k-i} a_{0,l}Q_{i-1,k-l-1} \right) \binom{j}{k} (-1)^{j+k} \alpha^{-j} \beta^{j-k}$$

$$= \sum_{l=0}^{i-1} a_{i-l,0} \alpha^{l-j} \beta^{j-l} \sum_{k=l}^{j} \binom{k}{l} \binom{j}{k} (-1)^{k+j}$$

$$+ \alpha^{i-j} \sum_{l=0}^{j-i} a_{0,l} \beta^{j-i-l} \sum_{k=i+l}^{j} (-1)^{j+k} \binom{k-l-1}{i-1} \binom{j}{k}$$

$$= \alpha^{i-j} \sum_{l=0}^{j-i} a_{0,l} \beta^{j-i-l} (-1)^{j-i+l} \sum_{t=0}^{j-i-l} (-1)^t \binom{t+i-1}{t} \binom{j}{t+i+l}$$

$$= \alpha^{i-j} \sum_{l=0}^{j-i} a_{0,l} (-1)^{j-i+l} \binom{j-i}{l} \beta^{j-i-l} = (-\alpha)^{i-j} \sum_{l=0}^{j-i} a_{0,l} (-1)^l \binom{j-i}{l} \beta^{j-i-l}$$

We can see that this is just $t_{i,j}$.

Clearly, when $i = 0$, that is $j \geq i = 0$, it is still true.

When $j \leq i$, by (2.3), we know (2.5) holds.

Thus, the proof is completed when considering all above cases. □

For example, the factorization of $A_3[\alpha, \beta]$:

$$\begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ a_{1,0} & \alpha a_{0,0} + \beta a_{1,0} & \alpha a_{0,1} + \alpha\beta a_{0,0} + \beta^2 a_{1,0} & \alpha a_{0,2} + \alpha\beta a_{0,1} + \alpha\beta^2 a_{0,0} + \beta^3 a_{1,0} \\ a_{2,0} & \alpha a_{1,0} + \beta a_{2,0} & \alpha^2 a_{0,0} + 2\alpha\beta a_{1,0} + \beta^2 a_{2,0} & \alpha^2 a_{0,1} + 2\alpha^2\beta a_{0,0} + 3\alpha\beta^2 a_{1,0} + \beta^3 a_{2,0} \\ a_{3,0} & \alpha a_{2,0} + \beta a_{3,0} & \alpha^2 a_{1,0} + 2\alpha\beta a_{2,0} + \beta^2 a_{3,0} & \alpha^3 a_{0,0} + 3\alpha^2\beta a_{1,0} + 3\alpha\beta^2 a_{2,0} + \beta^3 a_{3,0} \end{pmatrix}$$

$$= \begin{pmatrix} a_{0,0} & \frac{a_{0,1} - a_{0,0}\beta}{\alpha} & \frac{a_{0,2} + \beta^2 a_{0,0} - 2\beta a_{0,1}}{\alpha^2} & t_{0,3} \\ a_{1,0} & a_{0,0} & \frac{a_{0,1} - a_{0,0}\beta}{\alpha} & \frac{a_{0,2} + \beta^2 a_{0,0} - 2\beta a_{0,1}}{\alpha^2} \\ a_{2,0} & a_{1,0} & a_{0,0} & \frac{a_{0,1} - a_{0,0}\beta}{\alpha} \\ a_{3,0} & a_{2,0} & a_{1,0} & a_{0,0} \end{pmatrix} \begin{pmatrix} 1 & \beta & \beta^2 & \beta^3 \\ 0 & \alpha & 2\alpha\beta & 3\alpha\beta^2 \\ 0 & 0 & \alpha^2 & 3\alpha^2\beta \\ 0 & 0 & 0 & \alpha^3 \end{pmatrix}$$

where $t_{0,3} = \frac{a_{0,3} + 3\beta a_{0,1} - 3\beta a_{0,2} - a_{0,0}\beta^3}{\alpha^3}$.

The following is clear

$$A = TP^T \tag{2.6}$$

We now investigate the generating function $f_T(x, y)$ of Toeplitz matrix T.

Denote the generating function of sequence in the i-th row of A as $r_i(A, y)$ and generating function of sequence in the j-th column as $c_j(A, x)$ respectively. Then

$$r_i(A, y) = \sum_{j=0}^{\infty} a_{i,j} y^j \quad , \quad c_j(A, x) = \sum_{i=0}^{\infty} a_{i,j} x^i$$

$$c_j(T, x) = \sum_{i=0}^{j} t_{i,j} x^i + \sum_{i=j+1}^{\infty} t_{i,j} x^i = \sum_{i=0}^{j} (-\alpha)^{i-j} \sum_{l=0}^{j-i} (-1)^l \binom{j-i}{l} \beta^{j-i-l} a_{0,l} x^i$$
$$+ \sum_{i=j+1}^{\infty} a_{i-j,0} x^i = \sum_{l=0}^{j} b_l \sum_{i=0}^{j-l} \binom{j-i}{l} (-\tfrac{\beta}{\alpha})^{j-l-i} x^i + (f_A(x) - a_{0,0}) x^j$$

So

$$f_T(x, y) = \sum_{j=0}^{\infty} c_j(T, x) y^j = \sum_{j=0}^{\infty} \sum_{l=0}^{j} b_l \sum_{i=0}^{j-l} (-\tfrac{\beta}{\alpha})^{j-i-l} \binom{j-i}{l} x^i y^j + \frac{f_A(x) - a_{0,0}}{1 - xy}$$
$$= \sum_{l=0}^{\infty} b_l \sum_{i=0}^{\infty} x^i y^{i+l} \sum_{k=0}^{\infty} \binom{l+k}{k} (-\tfrac{\beta y}{\alpha})^k = \sum_{l=0}^{\infty} b_l \sum_{i=0}^{\infty} y^{i+l} \frac{\alpha^{l+1}}{(\alpha + \beta y)^{l+1}} x^i + \frac{f_A(x) - a_{0,0}}{1 - xy}$$

Hence

$$f_T(x, y) = \frac{\alpha}{(1 - xy)(\alpha + \beta y)} g_A\left(\frac{\alpha y}{\alpha + \beta y}\right) + \frac{f_A(x) - a_{0,0}}{1 - xy} \tag{2.7}$$

We can compute the generating functions $r_i(A, y)$ and $c_j(A, x)$ associated to A by the same method as the former.

$c_j(A,x) = a_{0,j} + \sum_{i=1}^{j} a_{i,j}x^i + \sum_{i=j+1}^{\infty} a_{i,j}x^i =$

$a_{0,j}+\sum_{i=1}^{j}\sum_{k=0}^{i-1} a_{i-k,0}Q_{k,j}x^i+\alpha\sum_{i=1}^{j}\sum_{k=0}^{j-i} a_{0,k}Q_{i-1,j-k-1}x^i+\sum_{i=j+1}^{\infty}\sum_{k=0}^{j} a_{i-k,0}Q_{k,j}x^i$

$= a_{0,j}+\sum_{k=0}^{j-1} Q_{k,j}\sum_{i=k+1}^{\infty} a_{i-k,0}x^i+Q_{j,j}\sum_{i=j+1}^{\infty} a_{i-j,0}x^i+\alpha x\sum_{k=0}^{j-1} a_{0,k}(\alpha x+\beta)^{j-k-1}$

$= a_{0,j} + (f_A(x) - a_{0,0})\sum_{k=0}^{j} Q_{k,j}x^k + \alpha x\sum_{k=0}^{j-1} a_{0,k}(\alpha x + \beta)^{j-k-1}$

Therefore

$$c_j(A,x) = a_{0,j} + (f_A(x) - a_{0,0})(\alpha x+\beta)^j + \alpha x\sum_{k=0}^{j-1} a_{0,k}(\alpha x+\beta)^{j-k-1} \quad (2.8)$$

$$r_i(A,y) = \sum_{k=0}^{i-1} a_{i-k,0}\frac{(\alpha y)^k}{(1-\beta y)^{k+1}} + \left(\frac{\alpha y}{1-\beta y}\right)^i g_A(y) \quad (2.9)$$

We can find the generating function of $A = (a_{i,j})$ by using $c_j(A,x)$ or $r_i(A,y)$.

$f_A(x,y) = \sum_{j=0}^{\infty} c_j(A,x)y^j$

$= \sum_{j=0}^{\infty} a_{0,j}y^j + (f_A(x) - a_{0,0})\sum_{j=0}^{\infty}(\alpha x+\beta)^j y^j + \alpha x\sum_{j=1}^{\infty}\sum_{k=0}^{j-1} a_{0,k}(\alpha x+\beta)^{j-k-1}y^j$

$= g_A(y) + \frac{f_A(x)-a_{0,0}}{1-\beta y-\alpha xy} + \alpha x\sum_{k=0}^{\infty} a_{0,k}\sum_{j=k+1}^{\infty}(\alpha x+\beta)^{j-k-1}y^j$

That is

$$f_A(x,y) = \frac{f_A(x) + g_A(y) - \beta y g_A(y) - a_{0.0}}{1-\beta y-\alpha xy}$$

$$f_A(x,y) = \sum_{i=0}^{\infty} r_i(A,y)x^i = \sum_{i=1}^{\infty}\sum_{k=0}^{i-1} a_{i-k,0}\frac{(\alpha y)^k}{(1-\beta y)^{k+1}}x^i + g_A(y)\sum_{i=0}^{\infty}\left(\frac{\alpha y}{1-\beta y}\right)^i x^i$$

$$= \frac{1}{1-\beta y}\sum_{k=0}^{\infty}\left(\frac{\alpha xy}{1-\beta y}\right)^k (f_A(x) - a_{0,0}) + g_A(y)\frac{1-\beta y}{1-\beta y-\alpha xy}$$

That is

$$f_A(x,y) = \frac{f_A(x) + g_A(y) - \beta y g_A(y) - a_{0.0}}{1-\beta y-\alpha xy}$$

Moreover, we find the following relation between $c_n(A,x)$ and $c_n(T,x)$ as follows:

$$c_n(A,x) = \sum_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}c_k(T,x) \quad (2.10)$$

Let infinite matrix A_1, A_2, and corresponding generating functions be $f_{A_1}(x,y)$ and $f_{A_2}(x,y)$, $X=(1,x,x^2,\cdots)$, $Y=(1,y,y^2,\cdots)$, then

$$f_{A_1}(x,y)=XA_1Y^T \tag{2.11}$$

Clearly

$$f_{A_1A_2}(x,y)=\sum_{k=0}^{\infty}c_k(A_1,x)r_k(A_2,y) \tag{2.12}$$

So, for matrix A determined by (2.1), this gives

$$f_A(x,y)=\sum_{k=0}^{\infty}\frac{\alpha^k}{(1-\beta y)^{k+1}}c_k(T,x) \tag{2.13}$$

3. On $\mathbf{b_{i,j}=\alpha b_{i-1,j}+\beta b_{i,j-1}}$

Suppose we have the following relation

$$b_{i,j}=\alpha b_{i-1,j}+\beta b_{i,j-1} \tag{3.1}$$

with initial values $b_{k,0}(k=0,1,2,\cdots)$, $b_{0,k}(k=0,1,2,\cdots)$. Set $B_n=B_n[\alpha,\beta]=(b_{i,j})_{0\le i,j\le n}$ and the related infinite matrix B.

Let $(n+1)\times(n+1)$ low triangular matrix $P_n[\alpha]=P_n[\alpha,\alpha]=\left(\binom{i}{j}\alpha^i\right)$, where $0\le j\le i\le n$. $P[\alpha]$ is the corresponding infinite matrix. Thus

$$P_n^{-1}[\alpha]=((-1)^{i+j}\binom{i}{j}\frac{1}{\alpha^j})_{0\le j\le i\le n}$$

Let generating functions:

$$f_B(x,y)=\sum_{i,j\ge 0}b_{i,j}x^iy^j,\qquad f_B(x)=\sum_{i\ge 0}b_{i,0}x^i,\qquad g_B(y)=\sum_{j\ge 0}b_{0,j}y^j$$

then

$$f_B(x,y)=\frac{(1-\alpha x)f_B(x)+(1-\beta y)g_B(y)-b_{0,0}}{1-\alpha x-\beta y} \tag{3.2}$$

For the coefficient $b_{i,j}(ij\ge 1)$ of x^iy^j in $f_B(x,y)$, we have

$$b_{i,j}=\sum_{k=1}^{i}(b_{k,0}-\alpha b_{k-1,0})\binom{i+j-k}{j}\alpha^{i-k}\beta^j+\sum_{k=1}^{j}(b_{0,k}-\beta b_{0,k-1})\binom{i+j-k}{i}\alpha^i\beta^{j-k}$$
$$+b_{0,0}\binom{i+j}{i}\alpha^i\beta^j$$

We can get another expression in general cases by mathematical induction:

$$b_{i,j} = \begin{cases} \alpha^i \sum\limits_{k=1}^{j} \binom{i+j-k-1}{i-1} \beta^{j-k} b_{0,k} + \beta^j \sum\limits_{k=1}^{i} \binom{i+j-k-1}{j-1} \alpha^{i-k} b_{k,0} & , ij \geq 1 \\ b_{0,j} & , i = 0 \\ b_{i,0} & , j = 0 \end{cases} \tag{3.3}$$

Theorem 2. For $B_n[\alpha, \beta]$ from (3.1), we have

$$B_n[\alpha, \beta] = P_n[\alpha] T_n[\alpha, \beta] P_n^T[\beta] \tag{3.4}$$

where, $T_n[\alpha, \beta]$ is Toeplitz matrix.

Proof For convenience, we take $P_n^{-1}[\alpha] B_n[\alpha, \beta] (P_n^{-1}[\beta])^T = T_n[\alpha, \beta] = (t_{i,j})$. It follows that by matrix production:

$t_{i,j} =$

$$\sum_{k=1}^{n} \sum_{l=1}^{n} (-1)^{i+l} \binom{i}{l} \alpha^{-l} \left(\alpha^l \sum_{v=1}^{k} \binom{l+k-v-1}{l-1} \beta^{k-v} b_{0,v} + \beta^k \sum_{v=1}^{l} \binom{l+k-v-1}{k-1} \alpha^{l-v} b_{v,0} \right) (-1)^{j+k} \binom{j}{k} \beta^{-k}$$

$$+ (-1)^{i+j} \sum_{l=0}^{i} (-1)^l \binom{i}{l} \alpha^{-l} b_{l,0} + (-1)^{i+j} \sum_{k=1}^{j} (-1)^k \binom{j}{k} \beta^{-k} b_{0,k}$$

When $ij \geq 1$, firstly, we compute the first product part in $t_{i,j}$:

$$\sum_{k=1}^{j} \sum_{l=1}^{i} (-1)^{i+j} \sum_{v=1}^{k} \binom{i}{l} \binom{l+k-v-1}{l-1} \binom{j}{k} (-1)^{l+k} \beta^{-v} b_{0,v}$$

$$= \sum_{k=1}^{j} \binom{j}{k} \sum_{v=1}^{k} \beta^{-v} b_{0,v} \sum_{l=1}^{i} (-1)^{i+j+l+k} \binom{i}{l} \binom{l+k-v-1}{l-1}$$

$$\overset{*}{=} \sum_{k=1}^{j} \sum_{v=1}^{k} \beta^{-v} b_{0,v} (-1)^{j+k} \binom{j}{k} \binom{k-v-1}{i-1}$$

$$= \sum_{v=1}^{j} \beta^{-v} b_{0,v} \sum_{k=v}^{j} (-1)^{j+k} \binom{j}{k} \binom{k-v-1}{i-1}$$

$$= \begin{cases} 0 & , i \geq j \\ \sum\limits_{v=1}^{j-i} b_{0,v} \beta^{-v} \sum\limits_{t=0}^{j-i-v} (-1)^{i+j+t+k} \binom{(i-1)+t}{t} \binom{(i+v)+(j-i-v)}{(i+v)+t} & , i < j \end{cases}$$

$$\overset{*}{=} \begin{cases} 0 & , i \geq j \\ \sum\limits_{v=1}^{j-i} \beta^{-v} b_{0,v} \binom{-v-1}{j-i-v} & , i < j \end{cases} = \begin{cases} 0 & , i \geq j \\ (-1)^{i+j} \sum\limits_{v=1}^{j-i} (-1)^v \binom{j-i}{v} \beta^{-v} b_{0,v} & , i < j \end{cases}$$

Where $\overset{*}{=}$ is true because of a combinatorial identity $((5d), P_8)$ in [4].

By using the same method as before, we can compute the second product in $t_{i,j}$easily, omitted here. At last, we have

$$t_{i,j} = \begin{cases} 0 & , i \geq j \\ (-1)^{i+j} \sum\limits_{k=1}^{j-i} (-1)^k \binom{j-i}{k} \beta^{-k} b_{0,k} & , i < j \end{cases}$$

$$+\begin{cases} (-1)^{i+j}\sum\limits_{k=1}^{i-j}(-1)^k\binom{i-j}{k}\alpha^{-k}b_{k,0} & ,\ i>j \\ 0 & ,\ i\le j \end{cases} +(-1)^{i+j}\sum_{k=0}^{i}(-1)^k\binom{i}{k}\alpha^{-k}b_{k,0}$$

$$+(-1)^{i+j}\sum_{k=1}^{j}(-1)^k\binom{j}{k}\beta^{-k}b_{0,k}$$

Therefore

$$t_{i,j}=(-1)^{i+j}\sum_{k=0}^{i-j}(-1)^k\binom{i-j}{k}\alpha^{-k}b_{k,0}+(-1)^{i+j}\sum_{k=1}^{j-i}(-1)^k\binom{j-i}{k}\beta^{-k}b_{0,k} \quad (3.5)$$

Clearly, $T_n[\alpha,\beta]$ is Toepolitz matrix also. The proof is completed. □

In addition, we can give the explicit factorization on $B_n[\alpha,\beta]$ easily from our proof.

For example $B_2[\alpha,\beta]$:

$$\begin{pmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & \alpha b_{0,1}+\beta b_{1,0} & \alpha b_{0,2}+\alpha\beta b_{0,1}+\beta^2 b_{1,0} \\ b_{2,0} & \alpha^2 b_{0,1}+\alpha\beta b_{1,0}+\beta b_{2,0} & \alpha^2 b_{0,2}+2\alpha^2\beta b_{0,1}+2\alpha\beta^2 b_{1,0}+\beta^2 b_{2,0} \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & \alpha & 0 & 0 \\ \alpha^2 & 2\alpha^2 & \alpha^2 & 0 \\ \alpha^3 & 3\alpha^3 & 3\alpha^3 & \alpha^3 \end{pmatrix}\begin{pmatrix} b_{0,0} & -b_{0,0}+\frac{b_{0,1}}{\beta} & b_{0,0}-2\frac{b_{0,1}}{\beta}+\frac{b_{0,2}}{\beta^2} \\ -b_{0,0}+\frac{b_{1,0}}{\alpha} & b_{0,0} & -b_{0,0}+\frac{b_{0,1}}{\beta} \\ b_{0,0}-2\frac{b_{1,0}}{\alpha}+\frac{b_{2,0}}{\alpha^2} & -b_{0,0}+\frac{b_{1,0}}{\alpha} & b_{0,0} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \beta & \beta^2 & \beta^3 \\ 0 & \beta & 2\beta^2 & 3\beta^3 \\ 0 & 0 & \beta^2 & 3\beta^3 \\ 0 & 0 & 0 & \beta^3 \end{pmatrix}$$

Furthermore, if we set T is the infinite matrix of $T_n[\alpha,\beta]$, then

$$B=P[\alpha]TP^T[\beta] \quad (3.6)$$

Now we begin to investigate a special case with initial values:

$$b_{k,0}=(b_{0,0}+k\omega)\alpha^k(k=0,1,2,\cdots),b_{0,k}=(b_{0,0}+k\omega)\beta^k(k=0,1,2,\cdots) \quad (3.7)$$

where α,β,ω are arbitrary complex numbers with $\alpha\beta\omega\neq 0$.

Thus, if initial values satisfy the condition (3.7), then the coefficients $b_{i,j}$ is

$$b_{i,j}=\binom{i+j}{i}\left(\frac{i^2+j^2+i+j}{(i+1)(j+1)}\omega+b_{0,0}\right)\alpha^i\beta^j \quad (i,j\ge 0) \quad (3.8)$$

The $(n+1)\times(n+1)$ matrix B_n defined by (3.1) with initial values (3.7) has the following factorization

$$B_n=P_n[\alpha](b_{0,0}I_n+\omega E_n)P_n^T[\beta] \quad (3.9)$$

where matrix I_n is $(n+1)\times(n+1)$ identity matrix, and $E_n = (e_{i,j})$ is $(n+1)\times(n+1)$ matrix with entry $e_{i,j} = 1$ only when $|i-j|=1$ otherwise 0. In fact, E_n is a special *Toeplitz* matrix.

For example $B_3(b_{0,0} = 0, \alpha = \beta = \omega = 1)$:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 4 & 7 \\ 2 & 4 & 8 & 15 \\ 3 & 7 & 15 & 30 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For the corresponding infinite matrix, we have

$$B = P[\alpha](b_{0,0}I + \omega E)P^T[\beta] \tag{3.10}$$

$$f_B(x,y) = \frac{b_{0,0}}{1-\alpha x-\beta y} + \frac{\alpha x+\beta y-2\alpha\beta xy}{(1-\alpha x-\beta y)(1-\alpha x)(1-\beta y)}\omega$$

Based on our reasoning, the generating function of sequence

$$\binom{i+j}{i}\frac{i^2+j^2+i+j}{(i+1)(j+1)}\alpha^i\beta^j$$

(*i.e.* that of matrix $P[\alpha]EP[\beta]$) is derived as:

$$\frac{\alpha x+\beta y-2\alpha\beta xy}{(1-\alpha x-\beta y)(1-\alpha x)(1-\beta y)}$$

Theorem 3. The determinant of B_n determined by (3.7) is

$$|B_n| = \left(\frac{\omega_2^n-\omega_1^n}{\omega_2-\omega_1}b_{0,0} - \frac{\omega_2^{n-1}-\omega_1^{n-1}}{\omega_2-\omega_1}\right)\omega^{2n-2} \tag{3.11}$$

where

$$\omega_1 = \frac{b_{0,0}+\sqrt{b_{0,0}^2-4\omega^2}}{2\omega^2}, \quad \omega_2 = \frac{b_{0,0}-\sqrt{b_{0,0}^2-4\omega^2}}{2\omega^2}$$

Proof When $|b_{0,0}| \neq 2|\omega|$, because of $|B_n| = |b_{0,0}I_n + \omega E_n|$, so we need to study the determinant d_n of matrix $b_{0,0}I_n + \omega E_n$ only. We have the following recursive relation by property of determinant.

$$d_n = b_{0,0}d_{n-1} - \omega^2 d_{n-2} (n \geq 3)\ ,\ d_1 = b_{0,0}\ ,\ d_2 = b_{0,0}^2 - \omega^2 \tag{3.12}$$

Let the generating function of d_n be $d(x) = \sum_{n=1}^{\infty} d_n x^n$. This gives

$$d(x) = \omega_1\omega_2\frac{b_{0,0}x-\omega^2x^2}{(\omega_1-x)(\omega_2-x)}$$

$$= b_{0,0}x + \sum_{n=2}^{\infty} \left(\frac{\omega_2^n - \omega_1^n}{\omega_2 - \omega_1} b_{0,0} - \frac{\omega_2^{n-1} - \omega_1^{n-1}}{\omega_2 - \omega_1} \right) \omega^{2n-2} x^n$$

When $|b_{0,0}| = 2|\omega|$, we have

$$|B_n| = (n+1)(\frac{b_{0,0}}{2})^n (n \geq 1) \tag{3.13}$$

In fact, we can derive the same result from limitation method $\omega_1 \longrightarrow \omega_2$ in (3.11).

So, (3.11) is valid in any cases. The proof is completed. □

4. On $c_{i,j} = \alpha c_{i-1,j} + \beta c_{i,j-1} + \gamma c_{i-1,j-1}$

Suppose recurrence relation

$$c_{i,j} = \alpha c_{i-1,j} + \beta c_{i,j-1} + \gamma c_{i-1,j-1} \tag{4.1}$$

with initial values $c_{i,0}(i = 0, 1, \cdots), c_{0,j}(j = 0, 1, \cdots)$, where $\alpha\beta \neq 0$.

Set $(n+1) \times (n+1)$ matrix $C_n = C_n[\alpha, \beta] = (c_{i,j})_{0 \leq i,j \leq n}$, and the corresponding infinity matrix is noted as C.

Take

$$f_C(x, y) = \sum_{i,j \geq 0} c_{i,j} x^i y^j, \qquad f_C(x) = \sum_{i \geq 0} c_{i,0} x^i, \qquad g_C(y) = \sum_{j \geq 0} c_{0,j} y^j$$

then

$$f_C(x, y) = \frac{(1 - \alpha x) f_C(x) + (1 - \beta y) g_C(y) - c_{0,0}}{1 - \alpha x - \beta y - \gamma xy}$$

Compute the coefficient of $x^i y^j$ in $\frac{1}{1 - \alpha x - \beta y - \gamma xy}$ as follows:

$$\frac{1}{1 - \alpha x - \beta y - \gamma xy} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i+j}{i} x^i y^j \sum_{l=0}^{i} \binom{i}{l} y^l \gamma^l \alpha^{i-l} \beta^j$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{min(i,j)} \binom{i+j-l}{i} \binom{i}{l} \alpha^{i-l} \beta^{j-l} \gamma^l$$

Set $\delta = 1 + \frac{\gamma}{\alpha\beta}, (i, j) = min(i, j)$. So the coefficient of $x^i y^j$ is

$$\sum_{k=0}^{(i,j)} \binom{i+j-k}{i} \binom{i}{k} \alpha^i \beta^j (\frac{\gamma}{\alpha\beta})^k = \alpha^i \beta^j \sum_{k=0}^{(i,j)} \binom{i}{k} \binom{j}{k} \delta^k$$

By comparing, the coefficient $c_{i,j}$ of x^iy^j in $f_C(x,y)$ is

$$\alpha^i\beta^j c_{0,0}\sum_{k=0}^{(i,j)}\binom{i}{k}\binom{j}{k}\left(\frac{k}{i}+\frac{k}{j}-1\right)\delta^k+\sum_{k=1}^{i-1}\sum_{t=1}^{(i-1-k,j)}\binom{i-1-k}{t-1}\binom{j}{t}\delta^t\alpha^{i-k}\beta^j c_{k,0}+$$
$$\beta^j c_{i,0}+\alpha^i c_{0,j}+\sum_{k=1}^{i-1}\binom{j}{i-k}(\delta\alpha)^{i-k}\beta^j c_{k,0}+\sum_{k=1}^{j-1}\sum_{t=1}^{(i,j-k-1)}\binom{j-1-k}{t-1}\binom{i}{t}\delta^t\alpha^i\beta^{j-k}c_{0,k}$$
$$+\sum_{k=1}^{j-1}\binom{i}{j-k}(\delta\beta)^{j-k}\alpha^i c_{0,k}$$

Take

$$d_i=\sum_{k=0}^{i}(-1)^{i+k}\alpha^{-k}\binom{i}{k}c_{k,0}(i=0,1,2,\cdots)\ ,\ d_{-j}=\sum_{k=0}^{j}(-1)^{j+k}\beta^{-k}\binom{j}{k}c_{0,k}$$
$(j=0,1,2,\cdots)$.

$$D_n=(\delta^{min(i,j)}d_{i-j})=\begin{pmatrix} d_0 & d_{-1} & d_{-2} & d_{-3} & \cdots & d_{-n}\\ d_1 & \delta d_0 & \delta d_{-1} & \delta d_{-2} & \cdots & \delta d_{-(n-1)}\\ d_2 & \delta d_1 & \delta^2 d_0 & \delta^2 d_{-1} & \cdots & \delta^2 d_{-(n-2)}\\ d_3 & \delta d_2 & \delta^2 d_1 & \delta^3 d_0 & \cdots & \delta^3 d_{-(n-3)}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ d_n & \delta d_{n-1} & \delta^2 d_{n-2} & \delta^3 d_{(n-3)} & \cdots & \delta^n d_0 \end{pmatrix}$$

where $0\le i,j\le n$.

Let the corresponding infinity matrix of D_n be D, then we have

Theorem 4. Matrix C_n has factorization:

$$C_n=P_n[\alpha]D_nP_n^T[\beta]\ \ ,\ \ C=P[\alpha]DP^T[\beta] \qquad (4.3)$$

Proof We use generating function method to prove the Theorem 4. Because generating function is bijective to sequence.

This is to prove

$$c_{i,j}=\alpha^i\beta^j\sum_{l=0}^{i}\sum_{k=0}^{j}\binom{i}{l}\binom{j}{k}\delta^{(l,k)}d_{l-k}$$

That is $c_{i,j}=$

$$\alpha^i\beta^j\sum_{l=0}^{i}\sum_{k=0}^{l}\binom{i}{l}\binom{j}{k}\delta^k d_{l-k}+\alpha^i\beta^j\sum_{l=0}^{i}\sum_{k=l}^{j}\binom{i}{l}\binom{j}{k}\delta^l d_{l-k}-\alpha^i\beta^j\sum_{l=0}^{i}\binom{i}{l}\binom{j}{l}\delta^l c_{0,0} \quad (4.4)$$

Firstly, compute

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha^i\beta^j x^iy^j\sum_{l=0}^{i}\sum_{k=0}^{l}\binom{i}{l}\binom{j}{k}\delta^k d_{l-k}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{i}\sum_{s=0}^{l}\sum_{k=0}^{l-s}(-1)^{l+k+s}\binom{l-k}{s}\binom{j}{k}\delta^k\binom{i}{l}c_{s,0}\alpha^{-s}(\alpha x)^i(\beta y)^j$$

$$= \sum_{i=0}^{\infty}\sum_{l=0}^{i}\sum_{s=0}^{l}\sum_{k=0}^{l-s}(-1)^{l+k+s}\binom{l-k}{s}\binom{i}{l}\delta^k c_{s,0}\alpha^{-s}(\alpha x)^i\sum_{j=0}^{\infty}\binom{j}{k}(\beta y)^j$$

$$= \sum_{l=0}^{\infty}\sum_{i=l}^{\infty}\sum_{s=0}^{l}\sum_{k=0}^{l-s}(-1)^{l+k+s}\binom{l-k}{s}\binom{i}{l}\delta^k c_{s,0}\alpha^{-s}(\alpha x)^i\frac{(\beta y)^k}{(1-\beta y)^{k+1}}$$

$$= \sum_{l=0}^{\infty}\sum_{s=0}^{l}\sum_{k=0}^{l-s}\sum_{i=l}^{\infty}(-1)^{l+k+s}\binom{l-k}{s}\binom{i}{l}\delta^k c_{s,0}\alpha^{-s}(\alpha x)^i\frac{(\beta y)^k}{(1-\beta y)^{k+1}}$$

$$= \sum_{l=0}^{\infty}\sum_{s=0}^{l}\sum_{k=0}^{l-s}(-1)^{l+k+s}\binom{l-k}{s}\delta^k c_{s,0}\alpha^{-s}\frac{(\alpha x)^l}{(1-\alpha x)^{l+1}}\frac{(\beta y)^k}{(1-\beta y)^{k+1}}$$

$$= \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\delta^k c_{s,0}\alpha^{-s}\frac{(\beta y)^k}{(1-\beta y)^{k+1}}\sum_{l=k+s}^{\infty}(-1)^{l+k+s}\binom{l-k}{s}\frac{(\alpha x)^l}{(1-\alpha x)^{l+1}}$$

$$= \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\delta^k c_{s,0}\alpha^{-s}\frac{(\beta y)^k}{(1-\beta y)^{k+1}}(\alpha x)^s\left(\frac{\alpha x}{1-\alpha x}\right)^k = \frac{(1-\alpha x)f_C(x)}{1-\alpha x-\beta y-\gamma xy}$$

Secondly, compute

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha^i x^i\beta^j y^j\sum_{l=0}^{i}\sum_{k=l}^{j}\binom{i}{l}\binom{j}{k}\delta^l d_{l-k}$$

$$= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=l}^{\infty}\sum_{k=l}^{j}\sum_{s=0}^{k-l}(-1)^{k+l+s}\binom{i}{l}\binom{j}{k}\binom{k-l}{s}(x\alpha)^i(y\beta)^j\delta^l\beta^{-s}c_{0,s}$$

$$= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=l}^{j}\sum_{s=0}^{k-l}(-1)^{k+l+s}\binom{j}{k}\binom{k-l}{s}\delta^l(\beta y)^j\beta^{-s}c_{0,s}\frac{(\alpha x)^l}{(1-\alpha x)^{l+1}}$$

$$= \sum_{l=0}^{\infty}\sum_{k=l}^{\infty}\sum_{s=0}^{k-l}\sum_{j=k}^{\infty}(-1)^{k+l+s}\binom{j}{k}\binom{k-l}{s}\delta^l(\beta y)^j\beta^{-s}c_{0,s}\frac{(\alpha x)^l}{(1-\alpha x)^{l+1}}$$

$$= \sum_{l=0}^{\infty}\sum_{k=l}^{\infty}\sum_{s=0}^{k-l}(-1)^{k+l+s}\binom{k-l}{s}\delta^l\beta^{-s}c_{0,s}\frac{(\alpha x)^l}{(1-\alpha x)^{l+1}}\frac{(\beta y)^k}{(1-\beta y)^{k+1}}$$

$$= \sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\delta^l\beta^{-s}c_{0,s}\frac{(\alpha x)^l}{(1-\alpha x)^{l+1}}\sum_{k=l+s}^{\infty}(-1)^{k+l+s}\binom{k-l}{s}\frac{(\beta y)^k}{(1-\beta y)^{k+1}}$$

$$=\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\delta^l\beta^{-s}c_{0,s}\frac{(\alpha x)^l}{(1-\alpha x)^{l+1}}\frac{(\beta y)^{l+s}}{(1-\beta y)^l}=\frac{(1-\beta y)g_C(y)}{1-\alpha x-\beta y-\gamma xy}$$

At last, compute

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(\alpha x)^i(\beta y)^j\sum_{l=0}^{i}\binom{i}{l}\binom{j}{l}\delta^l c_{0,0}=\frac{c_{0,0}}{1-\alpha x-\beta y-\gamma xy}$$

Through preceding three computations, we can see that two generating functions of two sequences of right side and left side on (4.4) are equal.

It is true obviously for infinity case. Theorem 4 is proved. □

For example C_2:

$$\begin{pmatrix} c_{0,0} & c_{0,1} & c_{0,2} \\ c_{0,1} & \alpha c_{0,1}+\beta c_{1,0}+\gamma c_{0,0} & \alpha c_{0,2}+\alpha\beta c_{0,1}+\beta^2 c_{1,0}+\beta\gamma c_{0,0}+\gamma c_{0,1} \\ c_{2,0} & \alpha^2 c_{0,1}+\alpha\beta c_{1,0}+\alpha\gamma c_{0,0}+\beta c_{2,0}+\gamma c_{1,0} & c_{2,2} \end{pmatrix}=$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & \alpha & 0 \\ \alpha^2 & 2\alpha^2 & \alpha^2 \end{pmatrix}\begin{pmatrix} c_{0,0} & -c_{0,0}+\frac{c_{0,1}}{\beta} & c_{0,0}-\frac{2}{\beta}c_{0,1}+\frac{1}{\beta^2}c_{0,2} \\ -c_{0,0}+\frac{c_{1,0}}{\alpha} & (1+\frac{\gamma}{\alpha\beta})c_{0,0} & (1+\frac{\gamma}{\alpha\beta})(-c_{0,0}+\frac{c_{0,1}}{\beta}) \\ c_{0,0}-\frac{2}{\alpha}c_{1,0}+\frac{1}{\alpha^2}c_{2,0} & (1+\frac{\gamma}{\alpha\beta})(-c_{0,0}+\frac{c_{1,0}}{\alpha}) & (1+\frac{\gamma}{\alpha\beta})^2 c_{0,0} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \beta & \beta^2 \\ 0 & \beta & 2\beta^2 \\ 0 & 0 & \beta^2 \end{pmatrix}$$

where

$c_{2,2}=\alpha^2 c_{0,2}+2\alpha^2\beta c_{0,1}+2\alpha\beta^2 c_{1,0}+2\alpha\beta\gamma c_{0,0}+2\alpha\gamma c_{0,1}+2\beta\gamma c_{1,0}+\beta^2 c_{2,0}+\gamma^2 c_{0,0}$

Now, we investigate the special case by matrix method. Let $c_{i,0}=\alpha^i(i=0,1,\cdots), c_{0,j}=\beta^j(j=0,1,\cdots), \gamma=\alpha\beta$. The $n+1$ order diagonal matrix $D_n=diag\{2^0,2^1,\cdots,2^n\}$.

Corollary. Matrix C_n has following factorization

$$C_n[\alpha,\beta]=P_n[\alpha]D_nP_n^T[\beta]\ ,\ |C_n[\alpha,\beta]|=(2\alpha\beta)^{\binom{n+1}{2}} \qquad (4.5)$$

Proof Use mathematical induction to prove (4.5).
When $n=0,1,2,3$, (4.5) is true clearly
Suppose (4.5) holds for n. Now we investigate case $n+1$.
Let

$$C_{n+1}[\alpha,\beta]=\begin{pmatrix} C_n & V_{n+1} \\ U_{n+1} & c_{n+1,n+1} \end{pmatrix}$$

where, vector matrices:

$$U_{n+1}=(c_{n+1,0}\ \ c_{n+1,1}\ \ \cdots\ \ c_{n+1,n})_{1\times(n+1)}$$

$$V_{n+1} = (c_{0,n+1} \ \ c_{1,n+1} \ \ \cdots \ \ c_{n,n+1})^T_{1\times(n+1)}$$

Set

$$W_{n+1} = \alpha^{n+1}\begin{pmatrix}1 & \binom{n+1}{1} & \cdots & \binom{n+1}{n}\end{pmatrix}_{1\times(n+1)}$$

$$Z_{n+1} = \beta^{n+1}\begin{pmatrix}1 & \binom{n+1}{1} & \cdots & \binom{n+1}{n}\end{pmatrix}^T_{1\times(n+1)}$$

Because

$$\begin{pmatrix} P_n[\alpha] & 0 \\ W_{n+1} & 1 \end{pmatrix}\begin{pmatrix} D_n & 0 \\ 0 & 2^{n+1} \end{pmatrix}\begin{pmatrix} R_n[\beta] & Z_{n+1} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} P_n[\alpha]D_nR_n[\beta] & P_n[\alpha]D_nZ_{n+1} \\ W_{n+1}D_nR_n[\beta] & W_{n+1}D_nZ_{n+1} + 2^{n+1} \end{pmatrix}$$

Therefore, we deed to prove

$$U_{n+1} = W_{n+1}D_nR_n[\beta], \quad V_{n+1} = P_n[\alpha]D_nZ_{n+1}$$

$$W_{n+1}D_nZ_{n+1} + 2^{n+1} = c_{n+1,n+1}$$

The $m-$th$(0 \le m \le n)$ element in $W_{n+1}D_nR_n[\beta]$ is:

$$\alpha^{n+1}\beta^m \sum_{k=0}^{m} \binom{n+1}{k}\binom{m}{k}2^k$$

By (3.17) in [3], when $z = 2$, we have

$$\sum_{k=max(0,i-j)}^{i} \binom{k+j}{k}\binom{j}{i-k} = \sum_{k=0}^{(i,j)} \binom{i}{k}\binom{j}{k}2^k$$

thus

$$c_{n+1,m} = \alpha^{n+1}\beta^m \sum_{k=0}^{m}\binom{k+n+1}{k}\binom{n+1}{m-k} = \alpha^{n+1}\beta^m \sum_{k=0}^{m}\binom{n+1}{k}\binom{m}{k}2^k$$

Clearly, when $m = n+1$, the former is true also.

By using the same method as above, we can get $V_{n+1} = P_n[\alpha]D_nZ_{n+1}$.

So,(4.5) holds by mathematical induction. □

For example, $C_3[\alpha,\beta] = P_3[\alpha]D_3P_3^T[\beta]$:

$$\begin{pmatrix} 1 & \beta & \beta^2 & \beta^3 \\ \alpha & 3\alpha\beta & 5\alpha\beta^2 & 7\alpha\beta^3 \\ \alpha^2 & 5\alpha^2\beta & 13\alpha^2\beta^2 & 25\alpha^2\beta^3 \\ \alpha^3 & 7\alpha^3\beta & 25\alpha^3\beta^2 & 63\alpha^3\beta^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & \alpha & 0 & 0 \\ \alpha^2 & 2\alpha^2 & \alpha^2 & 0 \\ \alpha^3 & 3\alpha^3 & 3\alpha^3 & \alpha^3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}\begin{pmatrix} 1 & \beta & \beta^2 & \beta^3 \\ 0 & \beta & 2\beta^2 & 3\beta^3 \\ 0 & 0 & \beta^2 & 3\beta^3 \\ 0 & 0 & 0 & \beta^3 \end{pmatrix}$$

Remark. We notice that the second recurrence is a special case $\gamma = 0$ in the third one clearly. But we deal them with different methods. In addition, how to apply the results in this paper and how to derive the results in [2] by using Theorem 4 are still need to be investigated in the future.

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