On (a, d)-Antimagic Labelings of Generalized Petersen Graphs $P(n, 3)$ *

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Abstract

A connected graph $G = (V, E)$ is said to be (a, d)- antimagic if there exist positive integers a, d and a bijection $f : E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $g_f : V \to N$, defined by $g_f(v) = \sum f(uv)$, $uv \in E(G)$, is injective and $g_f(V) = \{a, a + d, \ldots, a + (|V| - 1)d\}$. Mirka Miller and Martin Bača proved that the generalized Petersen graph $P(n, 2)$ is $(\frac{3n+6}{2}, 3)$-antimagic for $n \equiv 0 (\mathrm{mod}\ 4), n \geq 8$ and conjectured that the generalized Petersen graph $P(n, k)$ is $(\frac{3n+6}{2}, 3)$-antimagic for even n and $2 \leq k \leq \frac{n}{2} - 1$. In this paper, we show that the generalized Petersen graph $P(n, 3)$ is $(\frac{3n+6}{2}, 3)$-antimagic for even $n \geq 8$.

Keywords: *(a, d)-antimagic labeling, Petersen graph, vertex labeling, edge labeling*

1 Introduction

Hartsfield and Ringel[6] introduced the concept of an antimagic graph. An antimagic graph G is a graph whose edges can be labeled

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with the integers $1, 2, \ldots, |E(G)|$ so that the sum of the labels at any given vertex is different from the sum of the labels at any other vertex, that is, no two vertices receive the same weight. Hartsfield and Ringel conjectured that every tree other than K_2 is antimagic and, more strongly, that every connected graph other than K_2 is antimagic.

Bodendiek and Walther[7] defined the concept of an (a, d)-antimagic graph as a special case of an antimagic graph. Let $G = (V, E)$ be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $p = |V(G)|$, $q = |E(G)|$ be the number of vertices and edges of G, respectively. A connected graph $G = (V, E)$ is said to be (a, d)- antimagic if there exist positive integers a, d and a bijection $f : E \to \{1, 2, \ldots, q\}$ such that the induced mapping $g_f : V \to N$, defined by $g_f(v) = \sum f(uv)$, $uv \in E(G)$, is injective and $g_f(V) = \{a, a + d, \ldots, a + (p - 1)d\}$. In this case f is called an (a, d)-antimagic labeling of G.

Bodendiek and Walther[11] showed that the theory of linear Diophantine equations and other concepts of number theory can be applied to determine the set of all connected (a, d)-antimagic graphs.

Bodendiek and Walther[8] proved some graphs(including even cycles; paths of even order; stars; $C_3^{(k)}$; $C_4^{(k)}$; $K_{3,3}$; tree with odd order $n \geq 5$ and having a vertex that is adjacent to at least three end vertices) are not (a, d)-antimagic. They also proved that P_{2k+1} is $(k, 1)$-antimagic; C_{2k+1} is $(k + 2, 1)$-antimagic; if a tree of odd order $2k+1 (k > 1)$ is (a, d)-antimagic, then $d = 1$ and $a = k$; if $K_{4k} (k \geq 2)$ is (a, d)-antimagic, then d is odd and $d \leq (2k + 1)(4k - 1) + 1$; if $K_{2k+1} (k \geq 2)$ is (a, d)-antimagic, then $d \leq (2k + 1)(k - 1) + 1$. For special graphs called parachutes, (a, d)-antimagic labelings are described in [9, 10].

For the literature on (a, d)-antimagic graphs we refer to [1] and the relevant references given in it.

Let n, k be integers such that $n \geq 3$, $1 \leq k < n$ and $n \neq 2k$. For such n, k, the generalized Petersen graph $P(n,k)$ is defined by

$V(P(n,k)) = \{u_i, v_i | 1 \le i \le n\}$,
$E(P(n,k)) = \{u_i u_{1+(i \bmod n)},\ u_i v_i,\ v_i v_{1+((i+k-1) \bmod n)} | 1 \le i \le n\}$.

Since generalized Petersen graphs $P(n,k)$ form an important class of 3-regular graphs with $2n$ vertices and $3n$ edges, it is desirable to determine which of the $P(n,k)$ are (a,d)-antimagic.

Bodendiek and Walther[11] conjectured that $P(n,1)$ is $(\frac{7n+4}{2}, 1)$-antimagic for even n and $P(n,1)$ is $(\frac{5n+5}{2}, 2)$-antimagic for odd n. In [2] the proofs of the conjectures are given, and it is shown that $P(n,1)$ is $(\frac{3n+6}{2}, 3)$-antimagic for even n. In [3] it is proved that $P(n,k)$ is $(a,1)$-antimagic if and only if n is even, $k \le \frac{n}{2} - 1$ and $a = \frac{7n+4}{2}$.

Mirka Miller and Martin Bača [5] proved that $P(n,2)$ is $(\frac{3n+6}{2}, 3)$-antimagic for $n \equiv 0 \pmod 4), n \ge 8$ and conjectured that $P(n,k)$ is $(\frac{3n+6}{2}, 3)$-antimagic for even n and $2 \le k \le \frac{n}{2} - 1$. In this paper, we show that $P(n,3)$ is $(\frac{3n+6}{2}, 3)$-antimagic for even $n \ge 8$.

2 Main Result

Theorem 2.1 $P(n,3)$ is $(\frac{3n+6}{2}, 3)$-antimagic for even $n \ge 8$.

Proof. We define the edge labeling f of $P(n,3)$ for even $n \ge 8$ as follows:

$$f(u_i u_{1+(i \bmod n)}) = \begin{cases} (n+i+1)/2, & 1 \le i \le n-1 \wedge i \bmod 2 = 1, \\ i/2+1, & 2 \le i \le n-4 \wedge i \bmod 2 = 0, \\ 3n/2, & i = n-2, \\ 1, & i = n. \end{cases}$$

$$f(u_i v_i) = \begin{cases} n + (i+1)/2, & 1 \le i \le n-3 \wedge i \bmod 2 = 1, \\ 5n/2 + 2 + i/2, & 2 \le i \le n-4 \wedge i \bmod 2 = 0, \\ 1 + 2n, & i = n-2, \\ n/2, & i = n-1, \\ 2 + 2n, & i = n. \end{cases}$$

$$f(v_i v_{1+((i+2) \bmod n)}) = \begin{cases} 2n+3+((i+1) \bmod n)/2, \\ \qquad\qquad 1 \le i \le n-1 \wedge i \bmod 2 = 1, \\ 3n/2+1+((i+2) \bmod n)/2, \\ \qquad\qquad 2 \le i \le n \wedge i \bmod 2 = 0. \end{cases}$$

Now we verify that f is a bijection from the edge set $E(P(n,3))$ onto $\{1,2,\ldots,q\}$.

Denote by

$$\begin{aligned} S_1 &= \{f(u_i u_{1+(i \bmod n)}) | 1 \le i \le n\}, \\ S_2 &= \{f(u_i v_i) | 1 \le i \le n\}, \\ S_3 &= \{f(v_i v_{1+((i+2) \bmod n)}) | 1 \le i \le n\}. \end{aligned}$$

Then

$$\begin{aligned} S_1 = &\quad S_{11} \cup S_{12} \cup S_{13} \cup S_{14}, \\ S_{11} = &\quad \{f(u_i u_{1+(i \bmod n)}) | 1 \le i \le n-1 \wedge i \bmod 2 = 1\} \\ = &\quad \{n/2 + (i+1)/2 | 1 \le i \le n-1 \wedge i \bmod 2 = 1\} \\ = &\quad \{n/2+1, n/2+2, \ldots, n\}, \\ S_{12} = &\quad \{f(u_i u_{1+(i \bmod n)}) | 2 \le i \le n-4 \wedge i \bmod 2 = 0\} \\ = &\quad \{i/2+1 | 2 \le i \le n-4 \wedge i \bmod 2 = 0\} = \{2,3,\ldots,n/2-1\}, \\ S_{13} = &\quad \{f(u_i u_{1+(i \bmod n)}) | i = n-2\} = \{3n/2\}, \\ S_{14} = &\quad \{f(u_i u_{1+(i \bmod n)}) | i = n\} = \{1\}, \end{aligned}$$

$$\begin{aligned} S_2 = &\quad S_{21} \cup S_{22} \cup S_{23} \cup S_{24} \cup S_{25}, \\ S_{21} = &\quad \{f(u_i v_i) | 1 \le i \le n-3 \wedge i \bmod 2 = 1\} \\ = &\quad \{n + (i+1)/2 | 1 \le i \le n-3 \wedge i \bmod 2 = 1\} \\ = &\quad \{n+1, n+2, \ldots, 3n/2-1\}, \\ S_{22} = &\quad \{f(u_i v_i) | 2 \le i \le n-4 \wedge i \bmod 2 = 0\} \\ = &\quad \{5n/2+2+i/2 | 2 \le i \le n-4 \wedge i \bmod 2 = 0\} \\ = &\quad \{5n/2+3, 5n/2+4, \ldots, 3n\}, \\ S_{23} = &\quad \{f(u_i v_i) | i = n-2\} = \{1+2n\}, \\ S_{24} = &\quad \{f(u_i v_i) | i = n-1\} = \{n/2\}, \\ S_{25} = &\quad \{f(u_i v_i) | i = n\} = \{2+2n\}, \end{aligned}$$

$$\begin{aligned}
S_3 = \;& S_{31} \cup S_{32}, \\
S_{31} = \;& \{f(v_i v_{1+((i+2) \bmod n)}) | 1 \le i \le n-1 \wedge i \bmod 2 = 1\} \\
= \;& \{2n+3+((i+1) \bmod n)/2 | 1 \le i \le n-1 \wedge i \bmod 2 = 1\} \\
= \;& \{2n+4, 2n+5, \ldots, 5n/2+2, 2n+3\}, \\
S_{32} = \;& \{f(v_i v_{1+((i+2) \bmod n)}) | 2 \le i \le n \wedge i \bmod 2 = 0\} \\
= \;& \{3n/2+1+((i+2) \bmod n)/2 | 2 \le i \le n \wedge i \bmod 2 = 0\} \\
= \;& \{3n/2+3, 3n/2+4, \ldots, 2n, 3n/2+1, 3n/2+2\}.
\end{aligned}$$

Hence, $S_1 \cup S_2 \cup S_3$ is the set of labels of all edges, and

$$\begin{aligned}
& S_1 \cup S_2 \cup S_3 \\
= \;& S_{11} \cup S_{12} \cup S_{13} \cup S_{14} \cup S_{21} \cup S_{22} \cup S_{23} \cup S_{24} \cup S_{25} \cup S_{31} \cup S_{32} \\
= \;& S_{14} \cup S_{12} \cup S_{24} \cup S_{11} \cup S_{21} \cup S_{13} \cup S_{32} \cup S_{23} \cup S_{25} \cup S_{31} \cup S_{22} \\
= \;& \{1,\ 2, 3, \ldots, n/2-1,\ n/2,\ n/2+1, n/2+2, \ldots, n,\ n+1, \\
& n+2, \ldots, 3n/2-1, 3n/2,\ 3n/2+1,\ 3n/2+2,\ 3n/2+3, \\
& 3n/2+4, \ldots, 2n,\ 2n+1,\ 2n+2,\ 2n+3,\ 2n+4, 2n+5, \\
& \ldots, 5n/2+2,\ 5n/2+3, 5n/2+4, \ldots, 3n\} \\
= \;& \{1, 2 \ldots, 3n\}.
\end{aligned}$$

Therefore we conclude that f is a bijection from $E(G)$ onto $\{1, 2, \ldots, 3n\}$.

Denote by

$$\begin{aligned}
& g_f(v) = \textstyle\sum f(uv),\ uv \in E(G), \\
& W = \{g_f(v) | v \in V(G)\}.
\end{aligned}$$

Now, we show that g_f is a bijective mapping from $V(G)$ onto W.

Let us denote the sets of the weights(under an edge labeling f) of vertices u_i and v_i of $P(n,3)$ by

$$\begin{aligned}
W_1 \;& = \{g_f(u_i) | 1 \le i \le n\} \\
& = \{f(u_{1+((i+n-2) \bmod n)} u_i) + f(u_i u_{1+(i \bmod n)}) + f(u_i v_i) \\
& \quad | 1 \le i \le n\}, \\
W_2 \;& = \{g_f(v_i) | 1 \le i \le n\} \\
& = \{f(v_{1+((i+n-4) \bmod n)} v_i) + f(v_i v_{1+((i+2) \bmod n)}) + f(u_i v_i) \\
& \quad | 1 \le i \le n\}.
\end{aligned}$$

Where

$$
\begin{aligned}
W_1 &= W_{11} \cup W_{12} \cup W_{13} \cup W_{14} \cup W_{15}, \\
W_{11} &= \{f(u_{1+((i+n-2) \bmod n)}u_i) + f(u_i u_{1+(i \bmod n)}) + f(u_i v_i) \\
&\qquad |1 \le i \le n-3 \wedge i \bmod 2 = 1\} \\
&= \{3n/2 + 2 + (3i-1)/2 | 1 \le i \le n-3 \wedge i \bmod 2 = 1\} \\
&= \{3n/2+3, 3n/2+6, \ldots, 3n-3\}, \\
W_{12} &= \{f(u_{1+((i+n-2) \bmod n)}u_i) + f(u_i u_{1+(i \bmod n)}) + f(u_i v_i) \\
&\qquad |2 \le i \le n-4 \wedge i \bmod 2 = 0\} \\
&= \{3n + 3 + 3i/2 | 2 \le i \le n-4 \wedge i \bmod 2 = 0\} \\
&= \{3n+6, 3n+9, \ldots, 9n/2-3\}, \\
W_{13} &= \{f(u_{1+((i+n-2) \bmod n)}u_i) + f(u_i u_{1+(i \bmod n)}) + f(u_i v_i) \\
&\qquad |i = n-2\} = \{9n/2\}, \\
W_{14} &= \{f(u_{1+((i+n-2) \bmod n)}u_i) + f(u_i u_{1+(i \bmod n)}) + f(u_i v_i) \\
&\qquad |i = n-1\} = \{3n\}, \\
W_{15} &= \{f(u_{1+((i+n-2) \bmod n)}u_i) + f(u_i u_{1+(i \bmod n)}) + f(u_i v_i) \\
&\qquad |i = n\} = \{3n+3\},
\end{aligned}
$$

$$
\begin{aligned}
W_2 &= W_{21} \cup W_{22} \cup W_{23} \cup W_{24} \cup W_{25}, \\
W_{21} &= \{f(v_{1+((i+n-4) \bmod n)}v_i) + f(v_i v_{1+((i+2) \bmod n)}) + f(u_i v_i) \\
&\qquad |1 \le i \le n-3 \wedge i \bmod 2 = 1\} \\
&= \{9n/2 + 4 + (3i+1)/2 | 1 \le i \le n-3 \wedge i \bmod 2 = 1\} \\
&= \{9n/2+6, 9n/2+9, \ldots, 6n\},
\end{aligned}
$$

$$
\begin{aligned}
W_{22} &= \{f(v_{1+((i+n-4) \bmod n)}v_i) + f(v_i v_{1+((i+2) \bmod n)}) + f(u_i v_i) \\
&\qquad |2 \le i \le n-4 \wedge i \bmod 2 = 0\} \\
&= \{6n + 6 + 3i/2 | 2 \le i \le n-4 \wedge i \bmod 2 = 0\} \\
&= \{6n+9, 6n+12, \ldots, 15n/2\}, \\
W_{23} &= \{f(v_{1+((i+n-4) \bmod n)}v_i) + f(v_i v_{1+((i+2) \bmod n)}) + f(u_i v_i) \\
&\qquad |i = n-2\} = \{6n+3\}, \\
W_{24} &= \{f(v_{1+((i+n-4) \bmod n)}v_i) + f(v_i v_{1+((i+2) \bmod n)}) + f(u_i v_i) \\
&\qquad |i = n-1\} = \{9n/2+3\}, \\
W_{25} &= \{f(v_{1+((i+n-4) \bmod n)}v_i) + f(v_i v_{1+((i+2) \bmod n)}) + f(u_i v_i) \\
&\qquad |i = n\} = \{6n+6\}.
\end{aligned}
$$

Hence, $W = W_1 \cup W_2$ is the set of the weights of all vertices, and

$$\begin{aligned} W &= W_1 \cup W_2 \\ &= W_{11} \cup W_{12} \cup W_{13} \cup W_{14} \cup W_{15} \cup W_{21} \cup W_{22} \cup W_{23} \cup W_{24} \cup W_{25} \\ &= W_{11} \cup W_{14} \cup W_{15} \cup W_{12} \cup W_{13} \cup W_{24} \cup W_{21} \cup W_{23} \cup W_{25} \cup W_{22} \\ &= \{3n/2+3, 3n/2+6, \ldots, 3n-3,\ 3n, 3n+3, 3n+6, 3n+9, \ldots, \\ &\quad 9n/2-3,\ 9n/2,\ 9n/2+3,\ 9n/2+6, 9n/2+9, \ldots, 6n,\ 6n+3, \\ &\quad 6n+6,\ 6n+9, 6n+12, \ldots, 15n/2\} \\ &= \{3n/2+3, 3n/2+6, \ldots, 15n/2\}. \end{aligned}$$

We can see that each vertex of $P(n,3)$ receives exactly one label of weight from W and each number from W is used exactly once as a label of weight and further that the set $W = \{a, a+d, \ldots, a+(|V|-1)d\}$, where $a = \frac{3n+6}{2}$ and $d = 3$. According to the definition of (a,d)-antimagic labeling, we thus conclude that $P(n,3)$ is $(\frac{3n+6}{2}, 3)$-antimagic for even $n \geq 8$. □

In Figure 1, we give a $(\frac{3n+6}{2}, 3)$-antimagic labeling for $P(16,3)$.

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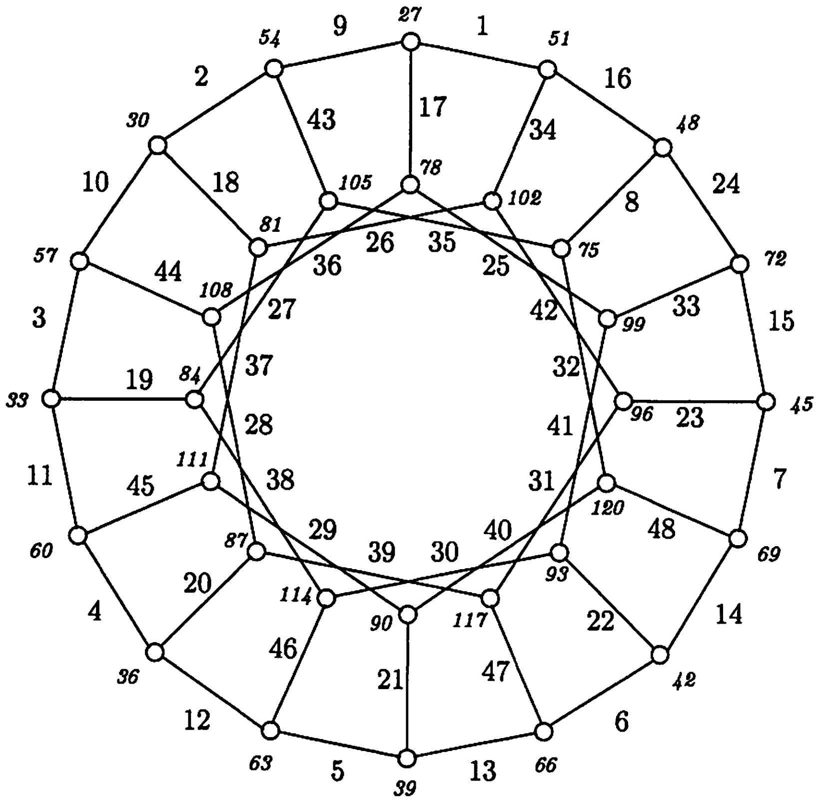

Figure 1 : The $(\frac{3n+6}{2}, 3)$-antimagic labeling of the graph $P(16, 3)$.