A CHARACTERIZATION OF CONNECTED MATROIDS

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Abstract

In this paper, we prove that a matroid with at least two elements is connected if and only if it can be obtained from a loop by a nonempty sequence of non-trivial single-element extensions and series extensions.

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The subdivision of an edge, addition of an edge and vertex addition in graphs are well known techniques and have several important applications (see [1, 2, 3, 4]). Hedetniemi [1] characterized 2-connected simple graphs in terms of these operations. In fact, he proved the following theorem.

Theorem 1. A simple graph G is 2-connected if and only if G can be obtained from K_3 by a finite sequence of subdivisions, vertex additions and edge additions.

The graph theoretic operations of subdivision of an edge and addition of an edge have been generalized to matroids and are known as series extension and single-element extension, respectively in matroids. By considering the matrix approach to these operations in binary matroids we could prove that every connected binary matroid with at least two elements can be generated from a loop by a finite sequence of single-element extensions and series extensions. This generalizes Hedetniemi's result to binary matroids. James Oxley in a private communication suggested that the above result can be further generalized to all matroids using well-known results of Murty [3] and Tutte [5]. The proof for a general result is provided.

For undefined notation and terminology in graphs and matroids, we refer the reader to [4].

A matroid M is a *single-element extension* of a matroid N if M has an element e such that the deletion of e from M is N. This extension is non-trivial if e is neither a loop nor a coloop of M. A matroid M is a *series-extension* of a matroid N, and N is a series contraction of M, if M has a 2-element cocircuit $\{x, y\}$ such that the contraction M/x of x from M is N.

The following result of Tutte [5] is well-known.

Lemma 2. Let e be an element of a connected matroid M. Then $M \setminus e$ or M/e is connected.

A consequence of this is that, from every connected matroid M, one can obtain a loop or coloop by removing elements one at a time via deletion or contraction while always maintaining a connected matroid. The purpose here is to prove that this sequence can be chosen so that all contractions are series contractions.

The next result of Murty [3] is also known.

Lemma 3. Let M be a connected matroid with at least two elements such that, for every element e, the deletion $M \setminus e$ is disconnected. Then M has a 2-element cocircuit.

Now, we prove the following theorem.

Theorem 4. Let M be a nonempty matroid that is not a coloop. Then M is connected if and only if M is a loop, or can be obtained from a loop by a sequence of operations each consisting of a non-trivial single-element extension or a series extension.

Proof. It is clear that a matroid that is obtained from a connected matroid by series extensions or single-element extensions is also connected. Since a loop is connected, every matroid obtainable from a loop by a sequence of series extensions and non-trivial single-element extensions is connected.

Suppose that M is connected. If $r(M) \leq 1$, then, as M is not a coloop, either M is a loop, or M is a graph consisting of parallel edges. In either case, the theorem holds. Now assume that M has rank at least two. If M is a circuit, then M can be obtained from a loop by a sequence of series extensions. Suppose that M is not a circuit. Delete elements one at a time from M maintaining a connected matroid until a matroid N is obtained such that every single element deletion from N is disconnected. Then N has the same rank as M. If N is a circuit, then the theorem follows. If not, then, by Lemma 3, N has a 2-element cocircuit $\{x, y\}$ and $N \setminus x$ has y as a coloop and so is disconnected. Thus, by Lemma 2, N/x is connected. Repeat the above procedure using N/x in place of M. Continuing this process, we eventually obtain a circuit and the theorem follows.

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References

[1] S. Hedetniemi, Characterizations and constructions of minimally 2-connected graphs and minimally strong digraphs, *Proceedings of the second Lousiana Conference on Combinatorics, Graph Theory and Computing* (1971), 257-282.

[2] F. Harary, *Graph theory*, Addison-Wesley, Reading (1969).

[3] U.S.R Murty, Extremal critically connected matroids, *Discrete Math.* 8 (1974), 49-58.

[4] J. G. Oxley, *Matroid theory*, Oxford University Press, Oxford (1992).

[5] W. T. Tutte, Connectivity in matroids, *Canad. J. Math.* 18 (1966), 1301-1324.