Vertex-Neighbor-Integrity of Composition Graphs of Paths

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Abstract. A vertex subversion strategy of a graph G is a set of vertices $X \subseteq V(G)$ whose closed neighborhood is deleted from G. The survival subgraph is denoted by G/X. The vertex-neighbor-integrity of G is defined to be $VNI(G) = \min\{|X| + \tau(G/X) : X \subseteq V(G)\}$, where $\tau(G/X)$ is the order of a largest component in G/X. This graph parameter was introduced by Cozzens and Wu to measure the vulnerability of spy networks. It was proved by Gambrell that the decision problem of computing the vertex-neighbor-integrity of a graph is $\mathcal{NP}$-complete. In this paper we evaluate the vertex-neighbor-integrity of the composition graph of two paths.

1 Introduction

All graphs considered in this paper are finite and simple. We use Bondy and Murty [2] for terminology and notations not defined here.

In 1987, Barefoot, Entringer and Swart introduced the notion of integrity to measure the vulnerability of graphs [1]. Incorporating the concept of the integrity and the idea of the vertex-neighbor-connectivity [6], in 1996 Cozzens and Wu [3] introduced a new graph parameter called vertex-neighbor-integrity to measure the vulnerability of spy networks. Gambrell [5] proved that the decision problem of computing the vertex-neighbor-integrity of a graph is $\mathcal{NP}$-complete. The vertex-neighbor-integrity of several specific classes of graphs has been investigated. Cozzens and Wu [3, 4] obtained the vertex-neighbor-integrity of paths, cycles and powers of cycles.

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Gambrell [5] studied the vertex-neighbor-integrity of magnifiers, expanders and hypercubes. In this paper we evaluate the vertex-neighbor-integrity of the composition graph of two paths.

Let $G = (V, E)$ be a graph and u a vertex in G. We call $N(u) = \{v : v \in V(G), uv \in E(G)\}$ the *open neighborhood* of u, and $N[u] = N(u) \cup \{u\}$ the *closed neighborhood* of u. We say that u is *subverted* if $N[u]$ is deleted from G. A set of vertices $X \subseteq V(G)$ is called a *vertex subversion strategy* of G if each of the vertices in X has been subverted. By G/X we denote the survival subgraph left after each vertex of X has been subverted. The *vertex-neighbor-integrity* of G is defined as

$$VNI(G) = \min\{|X| + \tau(G/X) : X \subseteq V(G)\},$$

where $\tau(G/X)$ stands for the order of a largest component in G/X.

The *composition graph* of two graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$, and $(u_1, v_1) \in V(G[H])$ adjacent with $(u_2, v_2) \in V(G[H])$ whenever $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

A *vertex dominating set* of the graph G is a set $S \subseteq V(G)$ such that every vertex in G belongs to S or is adjacent to a vertex of S. The cardinality of a smallest vertex dominating set in G is called the *vertex dominating number* of G and is denoted by $\sigma(G)$.

Throughout this paper, we use Z^+ for the positive integer set. For a real number x, $\lceil x \rceil$ stands for the smallest integer greater than or equal to x, and $\lfloor x \rfloor$ stands for the greatest integer less than or equal to x. For two positive integers m and n, we denote

$$p = \lceil \frac{m+3}{\lceil \sqrt{\frac{m+3}{n}} \rceil} \rceil + n \lceil \sqrt{\frac{m+3}{n}} \rceil - (3n+1),$$

and

$$q = \lceil \frac{m+3}{\lfloor \sqrt{\frac{m+3}{n}} \rfloor} \rceil + n \lfloor \sqrt{\frac{m+3}{n}} \rfloor - (3n+1).$$

Our main result is as follows.

Theorem 1. *Let m and n be two integers at least 2. Then*
(i) If $n = 2, 3$, then

$$VNI(P_m[P_n]) = \begin{cases} \lceil \frac{m}{3} \rceil, & \text{if } m \leq 12n; \\ p, & \text{if } 12n < m < 16n - 3, \\ & \text{or } m \geq 16n - 3 \text{ and } \frac{m+3}{n} > \lfloor \sqrt{\frac{m+3}{n}} \rfloor \lceil \sqrt{\frac{m+3}{n}} \rceil; \\ q, & \text{if } m \geq 16n - 3 \text{ and } \frac{m+3}{n} \leq \lfloor \sqrt{\frac{m+3}{n}} \rfloor \lceil \sqrt{\frac{m+3}{n}} \rceil. \end{cases}$$

(*ii*) *If* $n \geq 4$, *then*

$$VNI(P_m[P_n]) = \begin{cases} \lceil \frac{m}{2} \rceil, & \text{if } 2 \leq m \leq 12\lfloor \frac{n}{4} \rfloor - 8, \\ & m \equiv 0,1,3 (\text{mod} 4) \text{ and } n \equiv 0,1 (\text{mod} 4), \\ & \text{or } 2 \leq m \leq 12\lfloor \frac{n}{4} \rfloor - 4, \\ & m \equiv 0,1,3 (\text{mod} 4) \text{ and } n \equiv 2,3 (\text{mod} 4); \\ \frac{m}{2} + 1, & \text{if } 2 \leq m \leq 12\lfloor \frac{n}{4} \rfloor - 8, \\ & m \equiv 2 (\text{mod} 4), \text{ and } n \equiv 0,1 (\text{mod} 4), \\ & \text{or } 2 \leq m \leq 12\lfloor \frac{n}{4} \rfloor - 4, \\ & m \equiv 2 (\text{mod} 4), \text{ and } n \equiv 2,3 (\text{mod} 4); \\ \lceil \frac{m}{3} \rceil + \lceil \frac{n-3}{2} \rceil, & \text{if } 12\lfloor \frac{n}{4} \rfloor - 8 < m \leq 6(n+1) \text{ and} \\ & n \equiv 0 (\text{mod} 4), \\ & \text{or } 12\lfloor \frac{n}{4} \rfloor - 8 < m \leq 6(n+2) \text{ and} \\ & n \equiv 1 (\text{mod} 4), \\ & \text{or } 12\lfloor \frac{n}{4} \rfloor - 4 < m \leq 6(n+1) \text{ and} \\ & n \equiv 2 (\text{mod} 4), \\ & \text{or } 12\lfloor \frac{n}{4} \rfloor - 4 < m \leq 6(n+2) \text{ and} \\ & n \equiv 3 (\text{mod} 4); \\ \lceil \frac{m-1}{4} \rceil + n, & \text{if } 6(n+1) < m \leq 9n - 3 \text{ and} \\ & n \equiv 0 (\text{mod} 2), \\ & \text{or } 6(n+2) < m \leq 9n - 3 \text{ and} \\ & n \equiv 1 (\text{mod} 2); \\ p, & \text{if } 9n - 3 < m < 16n - 3, \\ & \text{or } m \geq 16n - 3 \text{ and} \\ & \frac{m+3}{n} > \lfloor \sqrt{\frac{m+3}{n}} \rfloor \lceil \sqrt{\frac{m+3}{n}} \rceil; \\ q, & \text{if } m \geq 16n - 3 \text{ and} \\ & \frac{m+3}{n} \leq \lfloor \sqrt{\frac{m+3}{n}} \rfloor \lceil \sqrt{\frac{m+3}{n}} \rceil. \end{cases}$$

We postpone the proof of Theorem 1 to Section 3 and before that, in Section 2, we give some lemmata which will be used in the proof of this result. An analysis of how our results relate to previous results on the vertex-neighbor-integrity of graphs will be presented in the final section.

2 Lemmata

To prove Theorem 1, we need the following lemmata.

Lemma 1 ([4]). *For any positive integer* m, *if* $x \geq \sqrt{m}$, *then* $\lceil \frac{m}{x} \rceil - \lceil \frac{m}{x+1} \rceil \leq 1$.

The proof of the following lemma is straightforward and is thus omitted.

Lemma 2. *Let m, n and x be positive integers, and $f(x) = x + n\lceil \frac{m}{x} \rceil$. Then*
(i) $f(x)$ is a decreasing function if $x \le \lfloor \sqrt{m} \rfloor$;
(ii) $f(x)$ is an increasing function if $x \ge m$.

Lemma 3. *Let m and n be positive integers at least 2. Then*

$$\min_{x\in Z^+}\{x+n\lceil\frac{m}{x}\rceil\} = \begin{cases} \lceil\frac{m}{\lceil\sqrt{\frac{m}{n}}\rceil}\rceil + n\lceil\sqrt{\frac{m}{n}}\rceil, & \text{if } \frac{m}{n} > \lfloor\sqrt{\frac{m}{n}}\rfloor\lceil\sqrt{\frac{m}{n}}\rceil; \\ \lceil\frac{m}{\lfloor\sqrt{\frac{m}{n}}\rfloor}\rceil + n\lfloor\sqrt{\frac{m}{n}}\rfloor, & \text{if } \frac{m}{n} \le \lfloor\sqrt{\frac{m}{n}}\rfloor\lceil\sqrt{\frac{m}{n}}\rceil. \end{cases}$$

Proof. By Lemma 2, we have

$$\min_{x\in Z^+}\{x+n\lceil\frac{m}{x}\rceil\} = \min_{\lfloor\sqrt{m}\rfloor\le x\le m}\{x+n\lceil\frac{m}{x}\rceil\}. \tag{1}$$

Let $R = \{\lceil\frac{m}{x}\rceil : \lceil\sqrt{m}\rceil \le x \le m, x \in Z^+\}$. Then we can prove that $R = \{1, 2, \cdots, \lceil\frac{m}{\lceil\sqrt{m}\rceil}\rceil\}$.

In fact, $1, \lceil\frac{m}{\lceil\sqrt{m}\rceil}\rceil \in R$. If $R \ne \{1, 2, \cdots, \lceil\frac{m}{\lceil\sqrt{m}\rceil}\rceil\}$, assume r^* is an integer with $1 < r^* < \lceil\frac{m}{\lceil\sqrt{m}\rceil}\rceil$ and $r^* \notin R$. Let r_1 be the greatest integer smaller than r^* in R, r_2 be the smallest integer greater than r^* in R. Then $r_2 - r_1 \ge 2$. Choose two integers x_1 and x_2 such that $r_1 = \lceil\frac{m}{x_1}\rceil$, $r_2 = \lceil\frac{m}{x_2}\rceil$ and $\lceil\sqrt{m}\rceil \le x_2 < x_1 \le m$. By Lemma 1, $\lceil\frac{m}{x_2}\rceil - \lceil\frac{m}{x_2+1}\rceil \le 1$. Since there is no integer r with $r_1 < r < r_2$ such that $\lceil\frac{m}{x}\rceil = r$ for any integer x with $\lceil\sqrt{m}\rceil \le x \le m$, we get $\lceil\frac{m}{x_2}\rceil = \lceil\frac{m}{x_2+1}\rceil$. Similarly, we can prove that $\lceil\frac{m}{x_1-1}\rceil = \lceil\frac{m}{x_1}\rceil$. Repeating the above arguments, we can finally get $r_1 = \lceil\frac{m}{x_1}\rceil = \lceil\frac{m}{x_1-1}\rceil = \cdots = \lceil\frac{m}{x_2+1}\rceil = \lceil\frac{m}{x_2}\rceil = r_2$, a contradiction.

If $\lceil\frac{m}{x}\rceil = r > 1$, then

$$\lceil\frac{m}{x}\rceil = r \Longleftrightarrow r-1 < \frac{m}{x} \le r \Longleftrightarrow \frac{m}{r} \le x < \frac{m}{r-1}.$$

So it is not difficult to see that

$$\min_{\lfloor\sqrt{m}\rfloor\le x\le m}\{x+n\lceil\frac{m}{x}\rceil\} = \min_{r\in R^*}\{\lceil\frac{m}{r}\rceil + nr\} = \min_{r\in R^*}\lceil\frac{m}{r} + nr\rceil, \tag{2}$$

where $R^* = \{\lceil\frac{m}{\lfloor\sqrt{m}\rfloor}\rceil\} \cup R$.

Set $g(r) = \frac{m}{r} + nr$. It is clear that $g(r)$ is a decreasing function when $0 < r \le \sqrt{\frac{m}{n}}$, and an increasing function when $r \ge \sqrt{\frac{m}{n}}$. So we have

$$\begin{aligned} &\min_{r\in Z^+}\lceil\frac{m}{r} + nr\rceil \\ &= \min_{r\in Z^+}\lceil g(r)\rceil = \begin{cases} \lceil g(\lceil\sqrt{\frac{m}{n}}\rceil)\rceil, & \text{if } m < n; \\ \min\{\lceil g(\lfloor\sqrt{\frac{m}{n}}\rfloor)\rceil, \lceil g(\lceil\sqrt{\frac{m}{n}}\rceil)\rceil\}, & \text{if } m \ge n. \end{cases} \end{aligned} \tag{3}$$

Case 1. $m < n$.

Since $\lceil\sqrt{\frac{m}{n}}\rceil = 1 \in R^*$, we have

$$\min_{r\in R^*}\lceil\frac{m}{r}+nr\rceil = \min_{r\in Z^+}\lceil\frac{m}{r}+nr\rceil.$$

It follows from (1), (2) and (3) that

$$\min_{x\in Z^+}\{x+n\lceil\frac{m}{x}\rceil\} = \lceil g(\lceil\sqrt{\frac{m}{n}}\rceil)\rceil.$$

On the other hand, it is clear that $\frac{m}{n} > \lfloor\sqrt{\frac{m}{n}}\rfloor\lceil\sqrt{\frac{m}{n}}\rceil = 0$ in this case.

Case 2. $m \geq n$.

First, note that

$$\lceil\frac{m}{\lfloor\sqrt{m}\rfloor}\rceil \geq \lceil\frac{m}{\sqrt{m}}\rceil = \lceil\sqrt{m}\rceil > \lceil\sqrt{\frac{m}{n}}\rceil \geq \lfloor\sqrt{\frac{m}{n}}\rfloor \geq 1.$$

So $\lceil\sqrt{\frac{m}{n}}\rceil, \lfloor\sqrt{\frac{m}{n}}\rfloor \in R^*$. It follows from (3) that

$$\begin{aligned}&\min_{r\in R^*}\{\lceil\frac{m}{r}+nr\rceil\}\\ =\ &\min_{r\in Z^+}\{\lceil\frac{m}{r}+nr\rceil\} = \min\{\lceil g(\lfloor\sqrt{\frac{m}{n}}\rfloor)\rceil, \lceil g(\lceil\sqrt{\frac{m}{n}}\rceil)\rceil\}. \qquad (4)\end{aligned}$$

If $\sqrt{\frac{m}{n}}$ is an integer, then $\lfloor\sqrt{\frac{m}{n}}\rfloor = \lceil\sqrt{\frac{m}{n}}\rceil$, the conclusion holds. If $\sqrt{\frac{m}{n}}$ is not an integer, then $\lceil\sqrt{\frac{m}{n}}\rceil = \lfloor\sqrt{\frac{m}{n}}\rfloor + 1$.

If $\frac{m}{n} > \lfloor\sqrt{\frac{m}{n}}\rfloor\lceil\sqrt{\frac{m}{n}}\rceil$, then

$$\begin{aligned}&\frac{m}{\lfloor\sqrt{\frac{m}{n}}\rfloor} - \frac{m}{\lceil\sqrt{\frac{m}{n}}\rceil} > n\\ \Longrightarrow &\frac{m}{\lfloor\sqrt{\frac{m}{n}}\rfloor} + n\lfloor\sqrt{\frac{m}{n}}\rfloor > \frac{m}{\lceil\sqrt{\frac{m}{n}}\rceil} + n\lceil\sqrt{\frac{m}{n}}\rceil\\ \Longrightarrow &\lceil\frac{m}{\lfloor\sqrt{\frac{m}{n}}\rfloor} + n\lfloor\sqrt{\frac{m}{n}}\rfloor\rceil \geq \lceil\frac{m}{\lceil\sqrt{\frac{m}{n}}\rceil} + n\lceil\sqrt{\frac{m}{n}}\rceil\rceil\\ \Longrightarrow &\lceil\frac{m}{\lfloor\sqrt{\frac{m}{n}}\rfloor}\rceil + n\lfloor\sqrt{\frac{m}{n}}\rfloor \geq \lceil\frac{m}{\lceil\sqrt{\frac{m}{n}}\rceil}\rceil + n\lceil\sqrt{\frac{m}{n}}\rceil\\ \Longrightarrow &\lceil g(\lfloor\sqrt{\frac{m}{n}}\rfloor)\rceil \geq \lceil g(\lceil\sqrt{\frac{m}{n}}\rceil)\rceil.\end{aligned}$$

By (1), (2) and (4), we have

$$\min_{x\in Z^+}\{x+n\lceil\frac{m}{x}\rceil\}=\lceil\frac{m}{\lceil\sqrt{\frac{m}{n}}\rceil}\rceil+n\lceil\sqrt{\frac{m}{n}}\rceil.$$

The case $\frac{m}{n}\le\lfloor\sqrt{\frac{m}{n}}\rfloor\lceil\sqrt{\frac{m}{n}}\rceil$ can be proved similarly. □

Lemma 4. *Let m and n be positive integers with $m>9$, $n\ge 2$, and $f(x)=x+n\lceil\frac{m-3x}{x+1}\rceil$, where $x\in Z^+\cup\{0\}$. Then*

$$\min_{0\le x<\lceil\frac{m}{3}\rceil} f(x)=\begin{cases}\lceil\frac{m-1}{4}\rceil+n, & \text{if } 4\le m\le 9n-3;\\ p, & \text{if } 9n-3<m<16n-3,\\ & \text{or } m\ge 16n-3 \text{ and}\\ & \quad \frac{m+3}{n}>\lfloor\sqrt{\frac{m+3}{n}}\rfloor\lceil\sqrt{\frac{m+3}{n}}\rceil\,;\\ q, & \text{if } m\ge 16n-3 \text{ and}\\ & \quad \frac{m+3}{n}\le\lfloor\sqrt{\frac{m+3}{n}}\rfloor\lceil\sqrt{\frac{m+3}{n}}\rceil.\end{cases}$$

Proof. First, note that

$$f(x)=x+n\lceil\frac{m-3x}{x+1}\rceil=(x+1)+n\lceil\frac{m+3}{x+1}\rceil-(3n+1).$$

So, by Lemma 3, we have

$$\min_{x\in Z^+\cup\{0\}} f(x)$$

$$=\begin{cases} f(\lceil\frac{m+3}{\lceil\sqrt{\frac{m+3}{n}}\rceil}\rceil-1)=p, & \text{if } \frac{m+3}{n}>\lfloor\sqrt{\frac{m+3}{n}}\rfloor\lceil\sqrt{\frac{m+3}{n}}\rceil;\\ f(\lceil\frac{m+3}{\lfloor\sqrt{\frac{m+3}{n}}\rfloor}\rceil-1)=q, & \text{if } \frac{m+3}{n}\le\lfloor\sqrt{\frac{m+3}{n}}\rfloor\lceil\sqrt{\frac{m+3}{n}}\rceil.\end{cases}\tag{5}$$

Case 1. $9<m\le 9n-3$.

When $x<\lceil\frac{m}{3}\rceil$, we have $\lceil\frac{m+3}{x+1}\rceil=r\ge 4$. Then

$$\lceil\frac{m+3}{x+1}\rceil=r\Longleftrightarrow r-1<\frac{m+3}{x+1}\le r\Longleftrightarrow\frac{m+3}{r}\le x+1<\frac{m+3}{r-1}.$$

Hence,

$$\min_{0\le x<\lceil\frac{m}{3}\rceil} f(x)=\min_{r\ge 4}\{\lceil\frac{m+3}{r}\rceil+nr-(3n+1)\}.\tag{6}$$

Let $g(r) = \lceil \frac{m+3}{r} \rceil + nr$. It follows from $m \le 9n-3$ and $r \ge 4$ that $\frac{m+3}{r(r+1)} \le \frac{9n}{20}$. Since $n \ge 2$, we get

$$\begin{aligned} g(r+1) - g(r) &= n + \lceil \frac{m+3}{r+1} \rceil - \lceil \frac{m+3}{r} \rceil \\ &\ge n + \frac{m+3}{r+1} - \frac{m+3}{r} - 1 \\ &= n - 1 - \frac{m+3}{r(r+1)} \\ &\ge \frac{11n}{20} - 1 \\ &> 0, \end{aligned}$$

which implies that $g(r)$ is an increasing function when $r \ge 4$.

When $m > 9$, $x^* = \lceil \frac{m}{3} \rceil - 1 < \lceil \frac{m}{3} \rceil$ and $\lceil \frac{m+3}{x^*+1} \rceil = 4$. Then, from (6) we obtain

$$\min_{0 \le x < \lceil \frac{m}{3} \rceil} f(x) = g(4) - (3n+1) = \lceil \frac{m-1}{4} \rceil + n.$$

Case 2. $9n - 3 < m < 16n - 3$.

In this case, we have $\lceil \sqrt{\frac{m+3}{n}} \rceil = 4$ and $\lfloor \sqrt{\frac{m+3}{n}} \rfloor = 3$. So,

$$\lceil \frac{m+3}{\lceil \sqrt{\frac{m+3}{n}} \rceil} \rceil - 1 < \lceil \frac{m}{3} \rceil = \lceil \frac{m+3}{3} \rceil - 1 = \lceil \frac{m+3}{\lfloor \sqrt{\frac{m+3}{n}} \rfloor} \rceil - 1.$$

It follows from (5) that $\min_{0 \le x < \lceil \frac{m}{3} \rceil} f(x) = f(\lceil \frac{m+3}{\lceil \sqrt{\frac{m+3}{n}} \rceil} \rceil - 1) = p$.

Case 3. $m \ge 16n - 3$.

Note that $\lfloor \sqrt{\frac{m+3}{n}} \rfloor \ge 4$. Then we have

$$\lceil \frac{m}{3} \rceil = \lceil \frac{m+3}{3} \rceil - 1 > \lceil \frac{m+3}{\lfloor \sqrt{\frac{m+3}{n}} \rfloor} \rceil - 1 \ge \lceil \frac{m+3}{\lceil \sqrt{\frac{m+3}{n}} \rceil} \rceil - 1.$$

From (5), we obtain

$$\min_{0 \le x < \lceil \frac{m}{3} \rceil} f(x) = \begin{cases} p, & \text{if } \frac{m+3}{n} > \lfloor \sqrt{\frac{m+3}{n}} \rfloor \lceil \sqrt{\frac{m+3}{n}} \rceil; \\ q, & \text{if } \frac{m+3}{n} \le \lfloor \sqrt{\frac{m+3}{n}} \rfloor \lceil \sqrt{\frac{m+3}{n}} \rceil. \end{cases}$$

□

The following lemma is interesting itself.

Lemma 5. *Let m be an integer at least 2 and G a graph. Then*

$$\sigma(P_m[G]) = \begin{cases} \lceil \frac{m}{3} \rceil, & \text{if } \sigma(G) = 1; \\ \lceil \frac{m}{2} \rceil, & \text{if } \sigma(G) \geq 2 \text{ and } m \equiv 0, 1, 3 (\text{mod} 4); \\ \frac{m}{2} + 1, & \text{if } \sigma(G) \geq 2 \text{ and } m \equiv 2 (\text{mod} 4). \end{cases}$$

Proof. The case $\sigma(G) = 1$ follows from the fact $\sigma(P_m) = \lceil \frac{m}{3} \rceil$ immediately.

Suppose $\sigma(G) \geq 2$. Denote $V(P_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Assume $m = 4k + r$, $0 \leq r \leq 3$. Let $V_i = \{(u_i, v_j) : 1 \leq j \leq n\}$ and G_i be the subgraph of $P_m[G]$ induced by V_i for $i = 1, 2, \ldots, m$. The result holds obviously when $k = 0$. So we assume $k \geq 1$.

It is not difficult to see that there always exists a vertex dominating set X of $P_m[G]$ such that $|(V_{4i+2} \cup V_{4i+3}) \cap X| \geq 2$ for each $i = 0, 1, \ldots, k-1$, see Figure 1.

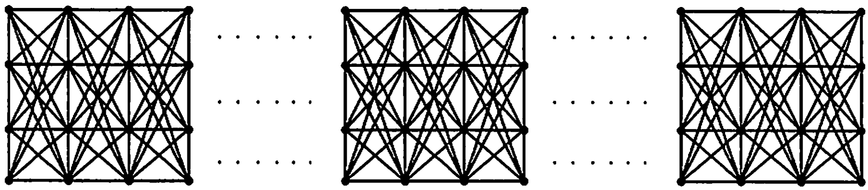

Figure 1: A graph $P_m[G]$ with $m = 4k$

Furthermore, we can assume $|(V_{4k} \cup V_{4k+1}) \cap X| \geq 1$ if $m = 4k + 1$; $|(V_{4k} \cup V_{4k+1} \cup V_{4k+2}) \cap X| \geq 2$ if $m = 4k + 2$; and $|(V_{4k} \cup V_{4k+1} \cup V_{4k+2} \cup V_{4k+3}) \cap X| \geq 2$ if $m = 4k + 3$. So we have $|X| \geq \lceil \frac{m}{2} \rceil$ if $m \equiv 0, 1, 3 (\text{mod} 4)$, and $|X| \geq \frac{m}{2} + 1$ if $m \equiv 2 (\text{mod} 4)$.

On the other hand, it is easy to construct a vertex dominating set X^* of $P_m[G]$ such that $|X^*| = \lceil \frac{m}{2} \rceil$ if $m \equiv 0, 1, 3 (\text{mod} 4)$, and $|X^*| = \frac{m}{2} + 1$ if $m \equiv 2 (\text{mod} 4)$. This completes the proof. □

3 Proof of Theorem 1

We first prove the following claim.

Claim 1. *Let m and n be two integers with $m \geq 4$, $n \geq 2$, and X be any vertex subversion strategy of $P_m[P_n]$ with $|X| = x$. Then*

(a) $\min\limits_{0 \leq x < \lceil \frac{m}{3} \rceil} \{|X| + \tau(P_m[P_n]/X)\}$

$$= \begin{cases} \lceil \frac{m-1}{4} \rceil + n, & \text{if } 4 \leq m \leq 9n - 3; \\ p, & \text{if } 9n - 3 < m < 16n - 3, \\ & \text{or } m \geq 16n - 3 \text{ and } \frac{m+3}{n} > \lfloor \sqrt{\frac{m+3}{n}} \rfloor \lceil \sqrt{\frac{m+3}{n}} \rceil; \\ q, & \text{if } m \geq 16n - 3 \text{ and } \frac{m+3}{n} \leq \lfloor \sqrt{\frac{m+3}{n}} \rfloor \lceil \sqrt{\frac{m+3}{n}} \rceil. \end{cases}$$

(b) $\min_{\lceil \frac{m}{3}\rceil \le x < \lceil \frac{m}{2}\rceil} \{|X| + \tau(P_m[P_n]/X)\} = \lceil \frac{m}{3}\rceil + \lceil \frac{n-3}{2}\rceil$ *if* $m \equiv 0,1,3(\text{mod}4)$ *and* $n \ge 4$; $\min_{\lceil \frac{m}{3}\rceil \le x < \frac{m}{2}+1} \{|X| + \tau(P_m[P_n]/X)\} = \lceil \frac{m}{3}\rceil + \lceil \frac{n-3}{2}\rceil$ *if* $m \equiv 2(\text{mod}4)$ *and* $n \ge 4$.

(c) $\min_{x \ge \lceil \frac{m}{2}\rceil} \{|X| + \tau(P_m[P_n]/X)\} = \lceil \frac{m}{2}\rceil$ *if* $m \equiv 0,1,3(\text{mod}4)$ *and* $n \ge 4$.

(d) $\min_{x \ge \frac{m}{2}+1} \{|X| + \tau(P_m[P_n]/X)\} = \frac{m}{2} + 1$ *if* $m \equiv 2(\text{mod}4)$ *and* $n \ge 4$.

Proof. Let $V(P_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Denote $\{(u_i, v_j) : 1 \le j \le n\}$ by V_i and the subgraph of $P_m[P_n]$ induced on V_i by G_i for $i = 1, 2, \ldots, m$. Then $G_i \cong P_n$. So, $P_m[P_n]$ contains m copies of disjoint P_n's.

(a) When $|X| = x < \lceil \frac{m}{3}\rceil$, it is clear that $P_m[P_n]/X$ contains at least $m - 3x$ copies of disjoint P_ns, and each component of $P_m[P_n]/X$ with at least n vertices is a $P_k[P_n]$. Denote the number of components of $P_m[P_n]/X$ with at least n vertices by $\omega_n(P_m[P_n]/X)$. Then $\omega_n(P_m[P_n]/X) \le x + 1$. So we have

$$\tau(P_m[P_n]/X) \ge n\lceil \frac{m-3x}{x+1}\rceil.$$

At the same time, it is easy to see that for a given integer x with $0 \le x < \lceil \frac{m}{3}\rceil$, there always exists $X \subset V(P_m[P_n])$ such that

$$\tau(P_m[P_n]/X) = n\lceil \frac{m-3x}{x+1}\rceil.$$

Therefore,

$$\min_{0 \le x < \lceil \frac{m}{3}\rceil} \{|X| + \tau(P_m[P_n]/X)\} = \min_{0 \le x < \lceil \frac{m}{3}\rceil} \{x + n\lceil \frac{m-3x}{x+1}\rceil\}.$$

The result follows from Lemma 4 immediately.

(b) We first consider the case $m \equiv 0,1,3(\text{mod}4)$.

If $|X| = \lceil \frac{m}{3}\rceil$, then it is not difficult to see that

$$\tau(P_m[P_n]/X) \ge \lceil \frac{n-3}{2}\rceil.$$

On the other hand, if we set $X^* = \{(u_i, v_{\lceil \frac{n}{2}\rceil}) : i = 3k+2,\ k = 0, 1, \ldots, \lceil \frac{m}{3}\rceil - 1\}$, then

$$\tau(P_m[P_n]/X^*) = \lceil \frac{n-3}{2}\rceil.$$

So,

$$\min_{x = \lceil \frac{m}{3}\rceil} \{|X| + \tau(P_m[P_n]/X)\} = \lceil \frac{m}{3}\rceil + \lceil \frac{n-3}{2}\rceil. \tag{7}$$

If $\lceil \frac{m}{3} \rceil < x < \lceil \frac{m}{2} \rceil$, let H be a maximum component of $P_m[P_n]/X$. Assume that $V(H) \cap V_i \neq \emptyset$ for some i. Then we have $X \cap V_{i-1} = \emptyset$ if $i \geq 2$ and $X \cap V_{i+1} = \emptyset$ if $i \leq m-1$. By the definition of composition graphs, we see that at least one of G_{i-1} (if $i \geq 2$) and G_{i+1} (if $i \leq m-1$) is a subgraph of H. Then

$$\tau(P_m[P_n]/X) = |V(H)| > n > \lceil \frac{n-3}{2} \rceil.$$

Therefore,

$$\min_{\lceil \frac{m}{3} \rceil < x < \lceil \frac{m}{2} \rceil} \{|X| + \tau(P_m[P_n]/X)\} > \lceil \frac{m}{3} \rceil + \lceil \frac{n-3}{2} \rceil. \tag{8}$$

It follows from (7) and (8) that

$$\min_{\lceil \frac{m}{3} \rceil \leq x < \lceil \frac{m}{2} \rceil} \{|X| + \tau(P_m[P_n]/X)\} = \lceil \frac{m}{3} \rceil + \lceil \frac{n-3}{2} \rceil.$$

The other assertion can be proved similarly.

The results in (c) and (d) follow from Lemma 5 immediately. □

Proof of Theorem 1.

(i) If $2 \leq m \leq 9$, then it is obvious that $VNI(P_m[P_n]) = \lceil \frac{m}{3} \rceil$. So we assume $m > 9$.

Let X be an arbitrary subversion strategy of $P_m[P_n]$ with $|X| = x$. By Lemma 5, $\sigma(P_m[P_n]) = \lceil \frac{m}{3} \rceil$. So we have

$$\min_{x \geq \lceil \frac{m}{3} \rceil} \{|X| + \tau(P_m[P_n]/X)\} = \lceil \frac{m}{3} \rceil. \tag{9}$$

If $m \leq 9n-3$, then it follows from Claim 1 (a) that

$$\min_{0 \leq x < \lceil \frac{m}{3} \rceil} \{|X| + \tau(P_m[P_n]/X)\} = \lceil \frac{m-1}{4} \rceil + n.$$

Thus, $VNI(P_m[P_n]) = \min\{\lceil \frac{m}{3} \rceil, \lceil \frac{m-1}{4} \rceil + n\} = \lceil \frac{m}{3} \rceil$.

If $9n-3 < m < 16n-3$, then $\lceil \sqrt{\frac{m+3}{n}} \rceil = 4$ and $p = \lceil \frac{m-1}{4} \rceil + n$. From Claim 1 (a), we have

$$\min_{0 \leq x < \lceil \frac{m}{3} \rceil} \{|X| + \tau(P_m[P_n]/X)\} = p = \lceil \frac{m-1}{4} \rceil + n.$$

Thus, by (9), $VNI(P_m[P_n]) = \min\{\lceil \frac{m}{3} \rceil, p\} = \min\{\lceil \frac{m}{3} \rceil, \lceil \frac{m-1}{4} \rceil + n\} = p$.

If $m \geq 16n-3$, by Claim 1 (a), we have

$$\min_{0\le x<\lceil \frac{m}{3}\rceil}\{|X|+\tau(P_m[P_n]/X)\}$$

$$= \begin{cases} p, & \text{if } m \ge 16n-3 \text{ and } \frac{m+3}{n} > \lfloor\sqrt{\frac{m+3}{n}}\rfloor\lceil\sqrt{\frac{m+3}{n}}\rceil; \\ q, & \text{if } m \ge 16n-3 \text{ and } \frac{m+3}{n} \le \lfloor\sqrt{\frac{m+3}{n}}\rfloor\lceil\sqrt{\frac{m+3}{n}}\rceil. \end{cases} \quad (10)$$

In a manner similar to that of the proof of Lemma 3, we can show that the function $h(r) = \lceil\frac{m+3}{r}\rceil + nr - (3n+1)$ attains its minimum value when $r = \lceil\sqrt{\frac{m+3}{n}}\rceil$ or $\lfloor\sqrt{\frac{m+3}{n}}\rfloor$. Note that $\lceil\sqrt{\frac{m+3}{n}}\rceil \ge \lfloor\sqrt{\frac{m+3}{n}}\rfloor \ge 4$ when $m \ge 16n-3$, $h(\lceil\sqrt{\frac{m+3}{n}}\rceil) = p$, $h(\lfloor\sqrt{\frac{m+3}{n}}\rfloor) = q$, and $h(3) = \lceil\frac{m}{3}\rceil$. So we have $p \le \lceil\frac{m}{3}\rceil$ and $q \le \lceil\frac{m}{3}\rceil$. The result follows from (9) and (10) immediately.

(*ii*) The case $m = 2, 3$ is trivial. So we assume $m \ge 4$.

Case 1. $4 \le m \le 12\lfloor\frac{n}{4}\rfloor - 8$, $m \equiv 0,1,3(\bmod 4)$ and $n \equiv 0,1(\bmod 4)$; or $4 \le m \le 12\lfloor\frac{n}{4}\rfloor - 4$, $m \equiv 0,1,3(\bmod 4)$ and $n \equiv 2,3(\bmod 4)$.

It can be proved that $\lceil\frac{m}{2}\rceil < \lceil\frac{m-1}{4}\rceil + n$ and $\lceil\frac{m}{2}\rceil \le \lceil\frac{m}{3}\rceil + \lceil\frac{n-3}{2}\rceil$. By Claim 1, we have

$$VNI(P_m[P_n]) = min\{\lceil\frac{m-1}{4}\rceil + n,\ \lceil\frac{m}{3}\rceil + \lceil\frac{n-3}{2}\rceil, \lceil\frac{m}{2}\rceil\} = \lceil\frac{m}{2}\rceil.$$

Case 2. $4 \le m \le 12\lfloor\frac{n}{4}\rfloor - 8$, $m \equiv 2(\bmod 4)$ and $n \equiv 0,1(\bmod 4)$; or $4 \le m \le 12\lfloor\frac{n}{4}\rfloor - 4$, $m \equiv 2(\bmod 4)$ and $n \equiv 2,3(\bmod 4)$.

It can be proved that $\frac{m}{2} + 1 < \lceil\frac{m-1}{4}\rceil + n$ and $\frac{m}{2} + 1 \le \lceil\frac{m}{3}\rceil + \lceil\frac{n-3}{2}\rceil$. By Claim 1, we have

$$VNI(P_m[P_n]) = min\{\lceil\frac{m-1}{4}\rceil + n,\ \lceil\frac{m}{3}\rceil + \lceil\frac{n-3}{2}\rceil, \frac{m}{2} + 1\} = \frac{m}{2} + 1.$$

Case 3. $12\lfloor\frac{n}{4}\rfloor - 8 < m \le 6(n+1)$ and $n \equiv 0(\bmod 4)$; $12\lfloor\frac{n}{4}\rfloor - 8 < m \le 6(n+2)$ and $n \equiv 1(\bmod 4)$; $12\lfloor\frac{n}{4}\rfloor - 4 < m \le 6(n+1)$ and $n \equiv 2(\bmod 4)$; or $12\lfloor\frac{n}{4}\rfloor - 4 < m \le 6(n+2)$ and $n \equiv 3(\bmod 4)$.

It can be proved that $\lceil\frac{m}{3}\rceil + \lceil\frac{n-3}{2}\rceil \le \lceil\frac{m-1}{4}\rceil + n$, $\lceil\frac{m}{3}\rceil + \lceil\frac{n-3}{2}\rceil \le \lceil\frac{m}{2}\rceil$ and $\lceil\frac{m}{3}\rceil + \lceil\frac{n-3}{2}\rceil \le \frac{m}{2} + 1$. By Claim 1, we have

$$VNI(P_m[P_n]) = min\{\lceil\frac{m-1}{4}\rceil + n, \lceil\frac{m}{3}\rceil + \lceil\frac{n-3}{2}\rceil, \lceil\frac{m}{2}\rceil, \frac{m}{2} + 1\}$$
$$= \lceil\frac{m}{3}\rceil + \lceil\frac{n-3}{2}\rceil.$$

Case 4. $6(n+1) < m \leq 9n-3$ and $n \equiv 0 (\mathrm{mod} 2)$; or $6(n+2) < m \leq 9n-3$ and $n \equiv 1 (\mathrm{mod} 2)$.

It can be proved that $\lceil \frac{m-1}{4} \rceil + n \leq \lceil \frac{m}{3} \rceil + \lceil \frac{n-3}{2} \rceil$, $\lceil \frac{m-1}{4} \rceil + n \leq \lceil \frac{m}{2} \rceil$ and $\lceil \frac{m-1}{4} \rceil + n \leq \frac{m}{2} + 1$. By Claim 1, we have

$$\begin{aligned} VNI(P_m[P_n]) &= min\{\lceil \frac{m-1}{4} \rceil + n, \lceil \frac{m}{3} \rceil + \lceil \frac{n-3}{2} \rceil, \lceil \frac{m}{2} \rceil, \frac{m}{2} + 1\} \\ &= \lceil \frac{m-1}{4} \rceil + n. \end{aligned}$$

Case 5. $9n-3 < m < 16n-3$; or $m \geq 16n-3$ and $\frac{m+3}{n} > \lfloor \sqrt{\frac{m+3}{n}} \rfloor \lceil \sqrt{\frac{m+3}{n}} \rceil$.

As in the proof of (i), if we set $h(r) = \lceil \frac{m+3}{r} \rceil + nr - (3n+1)$, then $h(\lceil \sqrt{\frac{m+3}{n}} \rceil) = p \leq h(4) = \lceil \frac{m-1}{4} \rceil + n$. On the other hand, it is not difficult to prove that $\lceil \frac{m-1}{4} \rceil + n \leq \lceil \frac{m}{3} \rceil + \lceil \frac{n-3}{2} \rceil$, $\lceil \frac{m-1}{4} \rceil + n \leq \lceil \frac{m}{2} \rceil$ and $\lceil \frac{m-1}{4} \rceil + n \leq \frac{m}{2} + 1$. By Claim 1, we have

$$VNI(P_m[P_n]) = min\{p, \lceil \frac{m}{3} \rceil + \lceil \frac{n-3}{2} \rceil, \lceil \frac{m}{2} \rceil, \frac{m}{2} + 1\} = p.$$

Case 6. $m \geq 16n-3$ and $\frac{m+3}{n} \leq \lfloor \sqrt{\frac{m+3}{n}} \rfloor \lceil \sqrt{\frac{m+3}{n}} \rceil$.

This case can be proved by arguments similar to that for Case 5.
The proof of Theorem 1 is complete.

4 Concluding remarks

Cozzens and Wu [3, 4] showed that the maximum vertex-neighbor-integrity for powers of cycle C_n and for trees on n vertices is on the order of $\sqrt{n}$. In [5], Gambrell proved that the vertex-neighbor-integrity of any member of a family of magnifier graphs is linear in the order of the graph. By an analysis of Theorem 1, we can see that the vertex-neighbor-integrity of $P_m[P_n]$ is also on the order of the square root of the number of the vertices of $P_m[P_n]$.

Gambrell [5] conjectured that the vertex-neighbor-integrity of any graph with n vertices is no greater than $\lceil \frac{n}{3} \rceil$. Our results verify this conjecture for the composition graph of two paths.

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