An Addressing Scheme On Complete Bipartite Graphs

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Abstract. For general graphs G, it is known [6] that the minimal length of an addressing scheme, denoted by $N(G)$, is less than or equal to $|G|-1$. In this paper we prove that for almost all complete bipartite graphs $K_{m,n}$, $N(K_{m,n}) = |K_{m,n}| - 2$.

Key words: eigensharp addressing scheme, address matrix, squashed-cube dimension.

1 Introduction

An address u_i corresponding to vertex v_i is a t-tuple (vector), each coordinate being chosen in $\{0, 1, d\}$, and the vertex set of a graph G is $\{v_1, \dots, v_{|G|}\}$. Let $u_{i,r}$ be the rth coordinate of $\mathbf{u}_i$. Define a function d_1 such that

$$d_1(u_{i,r}, u_{j,r}) = \begin{cases} 1 & \text{if} \quad u_{i,r} = 1 \quad \text{and} \quad u_{j,r} = 0 \\ 1 & \text{if} \quad u_{i,r} = 0 \quad \text{and} \quad u_{j,r} = 1 \\ 0 & \text{otherwise} \end{cases}$$

The distance between u_i and u_j is defined by $d(u_i, u_j) = \sum_{r=1}^{t} d_1(u_{i,r}, u_{j,r})$. Note that the symbol d never contributes to distance.

Theorem 1 [3] For a connected graph G, there exists some integer t so that each vertex has an address of length t such that, for any two vertices v_i and v_j the distance between them in G is equal to $d(u_i, u_j)$.

We call such a correspondence between each vertex and its address, an *addressing scheme*. Hence the next problem is to know how small t can be for a given graph G; the smallest t being denoted by $N(G)$ is called squashed-cube dimension [5] of G. In general the following upper bound on $N(G)$ is known.

Theorem 2 [6] $N(G) \leq |G| - 1$.

The following theorem gives an example saying that the equality can hold.

Theorem 3 [3] $N(T_n) = |T_n| - 1$, where T_n is a tree of order n.

A complete bipartite graph is one whose vertex set can be partitioned into two independent subsets M and N, so that each vertex of M is joined to each vertex of N; if $|M| = m$, $|N| = n$, such a graph is denoted by $K_{m,n}$. We see that for $m, n \geq 2$ and a pair $(m, n) \neq (2, 2)$, unless d is used, we never have valid addressing scheme on complete bipartite graphs. In general, Graham et al. [3] showed that a result of Witsenhausen implies that $N(G) \geq \max\{n_-, n_+\}$, where n_+(resp. n_-) is the number of positive (resp. negative) eigenvalues of the distance matrix of G. In this inequality, when the equality can hold, then an addressing scheme of G is called *eigensharp* [1]. In [2], by use of the above result, Graham et al. gave a lower bound that $N(K_{m,n}) \geq |K_{m,n}| - 2 = m + n - 2$ for any $m, n \geq 2$. In the same literature, they gave a comment as follows.

Theorem 4 [2] $N(K_{m,n}) = |K_{m,n}| - 2$, for any $m \equiv n \equiv 2 \mod 3$.

In this paper we completely determine that for almost all $m, n > 1$, $K_{m,n}$ has an eigensharp addressing scheme, i.e., $N(K_{m,n}) = |K_{m,n}| - 2$.

2 Construction of eigensharp addressing schemes

As is noted in Introduction, Graham et al. [2] proved that for any $m, n \geq 2$, it can hold that $N(K_{m,n}) \geq m + n - 2$. We note that the concept of a tree is included in that of a bipartite graph, and hence $N(K_{1,n}) = |K_{1,n}| - 1$. Here we indeed construct eigensharp addressing schemes of all $K_{m,n}$ except for nine specific cases, where $m, n > 1$. At first we introduce a new term; a *(g,h)-array* is a $g \times h$ matrix whose entries are in $\{0, 1, d\}$.

Lemma 1 For an odd integer $k \geq 3$, there exists a (k+1,k)-array such that any two rows have distance 2.

Proof. Let a row vector a of length k be $(1, \mathbf{0}_{\frac{k-1}{2}}, 1, \boldsymbol{d}_{\frac{k-3}{2}})$, where $\mathbf{0}_i$ (resp. $\boldsymbol{d}_i$) is an all-zero (resp. all-d) row vector of length i. Let $\mathbb{A}$ be a circulant matrix of order k with a in the first row, then the following k+1 $\times$ k matrix yields a (k+1,k)-array, satisfying the above condition:

$$\begin{pmatrix} \mathbf{0}_k \\ \mathbb{A} \end{pmatrix},$$

which completes the proof. $\Box$

Now we shall show the main theorem.

Theorem 5 If $m, n > 1$ and $(m,n) \neq (2,3)$, $(2,4)$, $(2,6)$, $(3,3)$, $(3,4)$, $(3,5)$, $(3,6)$, $(4,4)$, $(4,5)$, then

$$N(K_{m,n}) = |K_{m,n}| - 2.$$

Proof. Suppose that $K_{m,n} = (M \cup N, M \times N)$ has an eigensharp addressing scheme. Let Σ_M (resp. Σ_N) be the set of addresses for the partite set M (resp. N) of the $K_{m,n}$. Let $K_{m+3,n}$ be $(M' \cup N, M' \times N)$, where M' is obtained by adding 3-copies of a given element x in $\sum_M$, say x_1, x_2, x_3, to $\sum_M$; adjoin 000, 011, 101, 110 to x, x_1, x_2, x_3 respectively, and adjoin ddd to each element in $(\Sigma_M \cup \Sigma_N)\backslash x$, see Example 1 with $x = 11$. Then $K_{m+3,n}$ has an eigensharp addressing scheme. We may take the same procedure in extending the original addresses for $K_{m,n}$ to that of $K_{m,n+3}$. Following the proof we demonstrate that $K_{2,2}$, $K_{2,7}$, $K_{2,9}$, $K_{3,7}$, $K_{3,8}$, $K_{3,9}$, $K_{4,6}$, $K_{4,7}$, $K_{4,8}$, $K_{5,6}$, $K_{6,6}$, have eigensharp addressing schemes, we may consider the following six cases.

Case I. By Theorem 4, $K_{m,n}$ has an eigensharp addressing scheme, for $m \equiv n \equiv 2 \pmod 3$.

Case II. Since $K_{2,7}$, $K_{8,4}$ have eigensharp addressing schemes, then similarly $K_{m,n}$ has an eigensharp one for $m \equiv 2 \pmod 3$, $n \equiv 1 \pmod 3$, and $(m,n) \neq (2,4)$, $(5,4)$.

Case III. Since $K_{2,9}$, $K_{5,6}$, and $K_{8,3}$ have eigensharp addressing schemes, then similarly $K_{m,n}$ has an eigensharp one for $m \equiv 2 \pmod 3$, $n \equiv 0 \pmod 3$, and $(m,n) \neq (2,3)$, $(2,6)$, and $(5,3)$.

Case IV. Since $K_{3,7}$, $K_{6,4}$ have eigensharp addressing schemes, then similarly $K_{m,n}$ has an eigensharp one for $m \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$, and $(m,n) \neq (3,4)$.

Case V. Since $K_{4,7}$ has an eigensharp addressing scheme, then similarly $K_{m,n}$ has an eigensharp one for $m \equiv n \equiv 1 \pmod 3$, and $(m,n) \neq (4,4)$.

Case VI. Since $K_{3,9}$, $K_{6,6}$ have eigensharp addressing schemes, then similarly $K_{m,n}$ has an eigensharp one for $m \equiv n \equiv 0 \pmod 3$, and $(m,n) \neq (3,3)$, $(3,6)$, which completes the proof. □

Example 1 {00, 01, 10, 11} is clearly an eigensharp addressing scheme of $K_{2,2}$ [2]. We illustrate the above procedure. For example, an eigensharp addressing scheme of $K_{5,2}$, {00*ddd*, 01*ddd*, 10*ddd*, 11000, 11011, 11101, 11110}, is obtained from the eigensharp addressing scheme of $K_{2,2}$.

In the remaining of this section, eigensharp addressing schemes for $K_{2,7}$, $K_{2,9}$, $K_{3,7}$, $K_{4,6}$, $K_{4,7}$, $K_{3,8}$, $K_{3,9}$, $K_{5,6}$, $K_{6,6}$, $K_{4,8}$ are shown. An *address matrix* of G is a matrix with the ith row consisting of the ith address. We construct an address matrix of $K_{m,n}$ whose rows are arranged by groups Σ_M, Σ_N from the first row.

Lemma 2 For an odd integer $n \neq 1, 3$, an eigensharp addressing scheme of $K_{2,n}$ exists.

Proof. By Lemma 1, let D_{n-1} be a (n-1,n-2)-array such that each any two rows have distance 2, where n is an odd integer. An eigensharp addressing scheme of $K_{2,n}$ is directly constructed:

$$\begin{array}{r|cc|c} m_1 > & 0 & 0 & d_{n-2} \\ m_2 > & 1 & 1 & d_{n-2} \\ \hline n_1 > & 0 & 1 & \\ \vdots & \vdots & \vdots & D_{n-1} \\ n_{n-1} > & 0 & 1 & \\ \hline n_n > & 1 & 0 & d_{n-2}. \end{array}$$

□

From Lemma 2, by taking n to be 7 or 9, we have eigensharp addressing schemes for $K_{2,7}, K_{2,9}$. Finally, let $c_1 = 001dddd$, $c_2 = 11111dddd1$, $c_3 = dd1d001dd$, $c_4 = dd1111111dd$, $c_5 = dd1dd0dd$, $c_6 = 0000dd00dd$, and let $A_{m,n}$ be an address matrix of $K_{m,n}$. Eigensharp addressing schemes for $K_{3,7}$, $K_{4,6}$, $K_{4,7}$, $K_{3,9}$, $K_{5,6}$, $K_{6,6}$, $K_{3,8}$, $K_{4,8}$ are indeed listed in this order.

$$A_{3,7} = \begin{matrix} 0&0&0&0&d&d&d&d \\ 0&1&1&d&1&d&d&d \\ 1&0&1&d&d&d&d&0 \\ 0&0&1&0&d&1&1&0 \\ 0&0&1&d&d&0&0&d \\ 1&d&d&0&1&1&d&1 \\ d&1&d&0&0&0&1&d \\ d&1&d&d&0&1&0&0 \\ d&d&0&1&1&d&1&0 \\ d&d&d&1&0&1&1&1 \end{matrix} \qquad A_{4,6} = \begin{matrix} d&d&0&0&1&0&d&d \\ d&d&0&1&0&0&d&d \\ d&d&1&d&d&1&0&0 \\ d&d&1&d&d&1&1&1 \\ 0&0&0&0&0&d&d&d \\ 0&0&0&0&0&d&d&d \\ 0&0&0&1&1&d&d&d \\ 0&1&1&d&d&0&d&d \\ 1&0&1&d&d&0&d&d \\ 1&1&d&d&d&1&0&1 \\ 1&1&d&d&d&1&1&0 \end{matrix}$$

$$A_{4,7} = \begin{pmatrix} c_1 & 0 \\ A_{3,7} & {c_2}^T \end{pmatrix} \quad A_{3,9} = \begin{pmatrix} A_{3,8} & {c_4}^T \\ c_3 & 0 \end{pmatrix}$$

$$A_{5,6} = \begin{pmatrix} c_5 & 1 \\ A_{4,6} & {c_6}^T \end{pmatrix} \quad A_{6,6} = \begin{pmatrix} c_5 \quad 0 & 1 \\ & 0 \\ A_{5,6} & {c_6}^T \end{pmatrix}$$

$$A_{3,8} = \begin{matrix} 0 & 0 & 0 & 0 & d & d & d & d & d \\ 1 & 1 & 0 & 0 & d & d & d & d & d \\ d & d & 1 & 1 & 0 & 0 & d & d & d \\ 0 & 1 & 0 & d & d & d & 1 & 1 & 1 \\ 1 & 0 & 0 & d & d & d & 1 & 1 & 1 \\ d & d & 0 & 1 & 0 & 0 & d & 0 & 0 \\ d & d & 1 & 0 & 0 & 0 & 0 & d & d \\ d & d & 1 & d & 0 & 1 & 1 & d & 0 \\ d & d & 1 & d & 1 & 0 & 1 & 0 & d \\ d & d & d & 1 & 1 & d & 0 & 1 & 0 \\ d & d & d & 1 & d & 1 & 0 & 0 & 1 \end{matrix} \quad A_{4,8} = \begin{matrix} d & d & 0 & 1 & d & d & d & d & d & 0 \\ 0 & 0 & 0 & 0 & d & d & d & d & d & 1 \\ 1 & 1 & 0 & 0 & d & d & d & d & d & 1 \\ d & d & 1 & 1 & 0 & 0 & d & d & d & 1 \\ 0 & 1 & 0 & d & 0 & 0 & 0 & 0 & d & 1 \\ 0 & 1 & d & 0 & d & d & 1 & 1 & 1 & d \\ 1 & 0 & d & 0 & 0 & 0 & d & d & 1 & d \\ d & d & 0 & 1 & d & d & 1 & 1 & 0 & 1 \\ d & d & 1 & d & 0 & 1 & d & 0 & 0 & d \\ d & d & 1 & d & 1 & 0 & 0 & d & 0 & d \\ d & d & d & 1 & 1 & d & 1 & 0 & 1 & 1 \\ d & d & d & 1 & d & 1 & 0 & 1 & 1 & 1 \end{matrix}$$

3 Nonexistence of eigensharp addressing schemes

In this section we discuss the nonexistence of eigensharp addressing schemes for the nine remaining cases; $(m, n) = (2,3)$, $(2,4)$, $(2,6)$, $(3,3)$, $(3,4)$, $(3,5)$, $(3,6)$, $(4,4)$, $(4,5)$. The next lemma is useful in shortening our procedure. As results on nonexistence of an eigensharp addressing scheme, it is known that the Petersen graph has no eigensharp addressing scheme [1]. Also it is known that $K_{2,3}$ and $K_{3,3}$ has no eigensharp addressing schemes in [2] and in [5] respectively. But, here we show by use of Lemma 3 that $K_{2,4}$ has no eigensharp addressing scheme. We determined by computer search together with Lemma 3, that for the other six cases, there are no eigensharp addressing schemes. Here we denote the number of d's contained in an address $\boldsymbol{m}_i$ by $D(\boldsymbol{m}_i)$.

Lemma 3 Suppose that there exists an eigensharp addressing scheme for $K_{m,n}$ and $n \geq 3$. If $m = 2$ or $m = 3$, then $D(\boldsymbol{m}_i) \geq 2$ for each address $\boldsymbol{m}_i$ of the scheme corresponding to the part of size m, $1 \leq i \leq m$.

Proof of the case $m = 2$.

Case 1. We assume that there exists an eigensharp addressing scheme. Suppose that $D(\boldsymbol{m}_1) = 1$. We may assume that all but nth coordinate of $\mathbf{m}_1$ are zero. Since $d(\boldsymbol{n}_i, \boldsymbol{m}_1) = 1$, each $\boldsymbol{n}_i$ has only one 1 between the first coordinate to the (n-1)th. For $1 \leq l_i, l_j \leq n-1$, $\boldsymbol{n}_i$ and $\boldsymbol{n}_j$ must have single one in distinct coordinates l_i, l_j; otherwise, $d(\boldsymbol{n}_i, \boldsymbol{n}_j) \leq 1$. However, since there are exactly n addresses $\boldsymbol{n}_i$, such a situation never occur.

Case 2. Suppose that $D(\boldsymbol{m}_i) = 0$ holds for both $i = 1, 2$. Without loss of generality, assume that row $\boldsymbol{m}_1$ is all zero. Since $d(\boldsymbol{m}_1, \boldsymbol{n}_i) = 1$, only one coordinate in each $\boldsymbol{n}_i$ is equal to 1, and all the other coordinates are zero or d. Note that $d(\boldsymbol{n}_i, \boldsymbol{n}_j) = 2$ for $i, j = 1, \ldots, n(i \neq j)$. Since $n \geq 3$, we can not decide $\boldsymbol{m}_2$. □

The proof in case of $m = 3$ can be made similarly but is a little more tedious, so we omit it. From Lemma 3 we easily see that $N(K_{2,3}) = 4$ and $N(K_{3,3}) = 5$. We show nonexistence of an eigensharp addressing scheme of $K_{2,4}$.

Proof of $N(K_{2,4}) \neq 4$. Suppose that $N(K_{2,4}) = 4$. Lemma 3 implies that $D(\boldsymbol{m}_1) = D(\boldsymbol{m}_2) = 2$. There exist at least two $\boldsymbol{n}_j$ having one 0 and one 1 in the coordinates where $\boldsymbol{m}_1$ does not have d. We may consider them as $\boldsymbol{n}_1, \boldsymbol{n}_2$. Since $d(\boldsymbol{n}_1, \boldsymbol{n}_2)$ is equal to 2, we see that the third and fourth coordinates of $\boldsymbol{n}_1, \boldsymbol{n}_2$ are 00, 11 respectively. We illustrate this situation:

$\boldsymbol{m}_1 >$	0	0	d	d
$\boldsymbol{m}_2 >$	1	1	d	d
$\boldsymbol{n}_1 >$	0	1	0	0
$\boldsymbol{n}_2 >$	0	1	1	1
$\boldsymbol{n}_3 >$	$*_1$	$*_2$	$*_3$	$*_4$
$\boldsymbol{n}_4 >$				

If $*_1 = 0$ and $*_2 = 1$, then $d(\boldsymbol{n}_j, \boldsymbol{n}_3) \neq 2$ either for $j = 1$ or $j = 2$. Hence $*_1 = 1$ and $*_2 = 0$, $*_3 = *_4 = d$ because $d(\boldsymbol{n}_j, \boldsymbol{n}_3) = 2$. There is no way to decide an address $\boldsymbol{n}_4$. □

Remark 1 In case of $(m, n) = (2, 5)$, the equality in Lemma 3 can hold as follows:

$m_1 >$	0	0	0	d	d
$m_2 >$	d	1	1	0	d
$n_1 >$	1	d	0	0	0
$n_2 >$	1	d	d	1	1
$n_3 >$	0	1	0	0	1
$n_4 >$	0	1	d	1	0
$n_5 >$	0	0	1	0	d

Two addressing schemes of $K_{m,n}$ are called isomorphic if permuting rows in each partite set, permuting columns or replacing symbol 1 by symbol 0 of one, yields another. The above solution is nonisomorphic to the one in Example 1.

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