ON THE EMBEDDING OF COMPLEMENTS OF SOME HYPERBOLIC PLANES III

Pınar ANAPA and İbrahim GÜNALTILI
Eskişehir Osmangazi University
Departmant of Mathematics
26480 Eskişehir-Türkiye
E-mail : panapa@ogu.edu.tr, igunalti@ogu.edu.tr.

Abstract

In this study, we showed that an $(n+1)-$regular linear space, which is the complement of linear space having points are not on $m+1$ lines such that no three are concurrent in a projective subplane of odd order $m, m \geq 9$, could be embedded into a projective plane of order n as the complement of the Ostrom's hyperbolic plane.

Key Words: Linear space, affine and projective planes, hyperbolic planes

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1 Introduction

The complementation problem with repect to a projective plane is to determine the parameters of linear space obtained from projective plane by removing a certain configuration of points and lines. The problem of embedding the " *complements* " of various configuration in a projective plane has been studied by various authors ([1], [2], [3], [4], [5],.....). In 1970, Dickey solved the problem for the case where the configuration removed was a unital [5]. In 1987, L.M. Batten characterized linear spaces which are the complements of affine or projective subplanes of finite projective planes and showed that these spaces can be embeddable in a unique way in a projective plane of order n [1]. A generalization of Batten's Theorem [1] was given by Günaltılı and Olgun [7]. In [8], Günaltılı, Anapa and Olgun showed that a linear space, which is the complement of a linear space whose points are not on a trilateral or quadrilateral in a projective subplane of order m, is embeddable in a unique way in a projective plane of order n. In addition, it was determined that this linear space is the complement of certain regular hyperbolic plane in the sense of Graves [6] with respect to a finite projective plane. In [9], Günaltılı, Anapa and Olgun showed that a linear space, which is the complement of a linear space whose points are not on pentagon, hexagon and heptagon in a projective subplane of order m, is embeddable in a unique way in a projective plane of order n. In addition; it was determined that this linear space is the complement of certain regular hyperbolic plane in the sense of Graves [6] with respect to a finite projective plane.

In this study, it is shown that an $(n+1)-$regular linear space, which is the complement of a linear space whose points are not on $m+1$ lines such that no three are concurrent in a projective subplane of odd order m, $m \geq 9$, is embeddable in an unique way in a projective plane of order n. In addition; it is determined that this linear space is the complement of Ostrom's hyperbolic plane [14].

Now, we give some definitions required.

Definition 1.1 : Let $\mathcal{P}$ be a set of points and $\mathcal{L}$ be a subset of power set of $\mathcal{P}$. Then $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is called a *linear space* if:

L1 Any two points belong to an unique line.

L2 Every line contains at least two points.

If $v = |\mathcal{P}|$ and $b = |\mathcal{L}|$ are finite then $\mathcal{S}$ is called finite. The total number of lines through P is denoted by $b(P)$, and the total number of points on l is denoted by $v(l)$. Thus; if $b(P) = k$ and $v(l) = k$ then P is called a $k-$point and l is called a $k-$line. Furthermore; the total number of $k-$lines is denoted by b_k and the parameters k_m, k_M, r_m and r_M are defined as stated below:

$$\begin{aligned} k_m &= \min\{v(l) \mid l \in \mathcal{L}\} \\ k_M &= \max\{v(l) \mid l \in \mathcal{L}\} \\ r_m &= \min\{b(P) \mid P \in \mathcal{P}\} \\ r_M &= \max\{b(P) \mid P \in \mathcal{P}\} \end{aligned}$$

If every point of $\mathcal{S}$ lies on exactly t lines then $\mathcal{S}$ is called $t-$regular. ($t \geq 1$).

The *order* of a non-trivial finite linear space is defined as one less than the highest degree of points.

A *finite projective plane of order* $n \geq 2$ is a finite linear space with n^2+n+1 points in which $v(l) = b(P) = n+1$, for every line l and every point P.

A set of $\mathcal{O}$ of $m+1$ points in a projective plane π of order m is called an " *oval* " if no three points of $\mathcal{O}$ are collinear. A line of π which contains exactly one point, two points and no points of $\mathcal{O}$ is called " *tangent line,* " " *secant line,* " and " *exterior line,* " respectively. A point of π is called an " *exterior point* " and " *interior point* " if it lies on exactly two tangent lines and no tangent lines, respectively.

Beniamino Segre [17] characterized ovals in a projective plane of order m. If m is even, then all tangent lines pass through a common point. If m is odd, then any point R outside $\mathcal{O}$ is either on 0 or 2 tangents.

Let π be a projective plane of odd order m. A set of $\mathcal{M}$ of $m+1$ lines in π is called a " *(m+1)*-gon " if no three lines of $\mathcal{M}$ are concurrent. Any point of π is called " *corner point* " if it is intersection of any two lines of $\mathcal{M}$. It is clear from the definitions of corner and exterior points that a corner point of π is also an exterior point. Also; from the mentioned above argument, lines of $\mathcal{M}$ is also tangent lines with respect to an oval of π. Thus; an oval can also be described as an $(m+1)-$gon in a finite projective plane of odd order m.

Proposition 1.1 : (Ostrom, [14]) Let $\mathcal{O}$ be an oval in a projective plane of odd order m. The following are valid:

(i) The number of interior points is $\frac{1}{2}m(m-1)$

(ii) The number of exterior points is $\frac{1}{2}m(m+1)$

(iii) A secant line contains $\frac{1}{2}(m-1)$ interior points

(iv) An exterior line contains $\frac{1}{2}(m+1)$ interior points

(v) The number of secant lines is $\frac{1}{2}m(m+1)$

(vi) The number of exterior lines is $\frac{1}{2}m(m-1)$

(vii) The number of tangent lines is $m+1$.

Definition 1.2 : (Graves, [6]) A finite $(m+1)-$regular hyperbolic plane $(\mathcal{P},\mathcal{L})$, in the sense of Graves, is a non-trivial $(m+1)-$*regular linear space* such that :

H1: There are four points, no three of which are collinear.

H2 If P is a point not on a line l, then there exist at least two lines, not meeting l through P.

H3 If a subset $\mathcal{P}'$ of $\mathcal{P}$ contains three non-collinear points and contains all points on the lines through pairs of distinct points of $\mathcal{P}'$, then $\mathcal{P}'$ contains all points of $\mathcal{P}$.

Example 1.1 : Let π be a projective plane of odd order m, $m \geq 9$ and $\mathcal{O}$ an oval in π. Let Ω be the set of interior points of $\mathcal{O}$ and consider the restrictions of the secant and exterior lines of π to the interior points of $\mathcal{O}$. Hence the restrictions of these lines are the set theorical intersections of the secant and exterior lines of π with Ω. The geometrical structure so obtained is a hyperbolic plane known as Ostrom's model [14].

Proposition 1.2 : (Bumcrot, [4]) Any finite linear space satisfying the following conditions :

(i) $r_m \geq k_M + 2$

(ii) $k_m(k_m - 1) \geq r_M$

is a hyperbolic plane in the sense of Graves [6].

A linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is said to be embeddable in a linear space $\mathcal{S}' = (\mathcal{P}', \mathcal{L}')$ if $\mathcal{S}'$ can be obtained from $\mathcal{S}$ by addition of some points called as ideal points and some lines called as ideal lines.

2 MAIN RESULTS

In this section, we show that an $(n+1)-$regular linear space, which is the complement of linear space whose points are not on an $(m+1)-$gon in a projective subplane of odd order $m, m \geq 9$, is embeddable in a unique way in a projective plane of order n. In addition; we determined that this linear space is the complement of the Ostrom's hyperbolic plane.

Proposition 2.1 : Any $(m+1)-$regular linear space with line degree $\frac{m-1}{2}$ or $\frac{m+1}{2}, m \geq 9$, is a hyperbolic plane in the sense of Graves [6].

Proof : Let $\mathcal{S}$ be an $(m+1)-$regular linear space satisfying the condition (i) and (ii). It is clear that $r_m \geq k_M + 2$ and $k_m(k_m - 1) = (\frac{m-1}{2})(\frac{m-3}{2}) \geq m+1, m \geq 9$. By the Proposition 1.2, $\mathcal{S}$ is a hyperbolic plane in the sense of Graves.

We give the following result also given by Ostrom [14] in a some different way.

Proposition 2.2 : A real complement of linear space whose points are on an $(m+1)-$gon in a projective plane of odd order $m, m \geq 9$, is a hyperbolic plane.

Proof : Let π be a projective plane of odd order m and $\mathcal{S}$ be a real complement of an $(m+1)-$gon in π, $m \geq 9$. The total number of points on $(m+1)-$gon is $\dfrac{m^2+3m+2}{2}$. Thus, the total number of points of S is $\dfrac{m(m-1)}{2}$. Since an oval can also be described as $(m+1)-$gon in π, any point P of π not an $(m+1)-$gon is either a 0 or 2 tangents. In addition; from the definitions of corner and exterior points that a corner point of S is also an exterior point which is deleted from π. It follows every line of S has $\dfrac{m-1}{2}$ or $\dfrac{m+1}{2}$ points, from the Proposition 1.1. Therefore, S is a hyperbolic plane by the Proposition 2.1.

Corollary 2.1 : A hyperbolic plane obtained from a projective plane by deleting an $(m+1)-$gon in a projctive plane of odd order m, $m \geq 9$, is the Ostrom's hyperbolic plane.

Theorem 2.1 : Let $S = (\mathcal{P}, \mathcal{L})$ be an $(n+1)-$regular linear space such that :

(i) $v = n^2+n+1-\binom{m}{2}$, $b = n^2+n+1, 9 \leq m < n$, m is odd integer,

(ii) $b_{n-\frac{m-3}{2}} = \binom{m+1}{2}$,

(iii) every line has $n+1, n, n-\frac{m-3}{2}$ or $n-\frac{m-1}{2}$ points.

Then S is embeddable in a unique way in a projective plane of order n and complement of Ostrom's hyperbolic plane.

Proof : Let $\mathcal{P}_{ij}$ be the set of points of S such that there are i lines of degree $n-\frac{m-3}{2}$, j lines of degree $n-\frac{m-1}{2}$, k lines of degree n and t lines of degree $(n+1)$ through every point P of it. Then;

$$i(n-\frac{m-3}{2}-1)+j(n-\frac{m-1}{2}-1)+kn+(m+1-i-j-k)n = v-1 \quad (2.1)$$

$$i+j+k+t = m+1 \quad (2.2)$$

$$\sum_{i,j} |\mathcal{P}_{ij}| = v, \ \sum_t b_t = b, \quad t \in \{n+1, n, n-\frac{m-3}{2}, n-\frac{m-1}{2}\}$$

>From the (2.1) and (2.2), the following results are obtained.

$$\begin{aligned} k &= \frac{m(m-1)}{2} - \frac{m-1}{2}i - \frac{m+1}{2}j \\ t &= n+1-\frac{m(m-1)}{2}+\frac{m-3}{2}i+\frac{m-1}{2}j \end{aligned}$$

Also; by the simple counting methods;

$$\begin{aligned}\sum_{i,j} i\,|\mathcal{P}_{ij}| &= n - \frac{m-3}{2} - \frac{(m+1)m}{2}\\ \sum_{i,j} j\,|\mathcal{P}_{ij}| &= \left(n - \frac{m-1}{2}\right) b_{n-\frac{m-1}{2}}\\ \sum_{i,j} k\,|\mathcal{P}_{ij}| &= nb_n \text{ and } \sum_{i,j} t\,|\mathcal{P}_{ij}| = (n+1)b_{n+1}\end{aligned}$$

and the following results are obtained.

$$\begin{aligned}b_n &= \frac{m(m-1)}{2}(n-m)\\ b_{n+1} &= (n^2+n+1-m^2) - \frac{m(m-1)(n-m)}{2}\\ b_{n-\frac{m-3}{2}} &= \frac{(m+1)m}{2}\\ b_{n-\frac{m-1}{2}} &= \frac{m(m-1)}{2}\end{aligned}$$

Let l be an n–line. The number of lines not meeting l is n, since S is $(n+1)$–regular linear space with n^2+n+1 lines. Therefore; every n–line induces a parallel class having $n+1$ lines none of which is an $(n+1)$–line.

Let c and d be the numbers of $\left(n-\frac{m-3}{2}\right)$ –line and $\left(n-\frac{m-1}{2}\right)$ –line in a fixed class, respectively. Then;

$$c\left(n-\frac{m-3}{2}\right) + d\left(n-\frac{m-1}{2}\right) + (n+1-c-d)n = n^2+n+1-\binom{m}{2}$$

implies that $c = (m+2-d) - \dfrac{2d-4}{(m-3)}$.

Since $c \in \mathbb{Z}$ and m is odd, $\dfrac{2(d-2)}{m-3} \in \mathbb{Z}$. It follows there is a integer number k such that $\dfrac{2(d-2)}{m-3} = k$. Thus the following equalities are obtained.

$$d = \frac{m-3+4k}{2k} \tag{2.3}$$

$$c = (m+2) - \left(\frac{m-3+4k}{2}\right) - \frac{1}{k} \tag{2.4}$$

From (2.3) and (2.4), $k=1$ and $c=d=\dfrac{m+1}{2}$. It requires that the number of n–lines in a parallel class is $n-m$. Thus, the number of distinct parallel classes is $\binom{m}{2}$.

Consider the structure $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*)$ where $\mathcal{P}^*$ is $\mathcal{P}$ along with the parallel classes and $\mathcal{L}^*$ consists the lines of $\mathcal{L}$ extended by those parallel classes to which they belong. We shall prove that $\mathcal{S}^*$ is a linear space. It is clear that two old points (points of $\mathcal{P}$) or an old and a new point are on unique line of $\mathcal{L}^*$, since $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a linear space.

Let X and Y be two new distinct points. We must show that they determine a unique line of $\mathcal{L}^*$. Let l_X and l_Y be $n-$lines which determine the parallel classes corresponding to X and Y, respectively. If l_X and l_Y do not meet, then $X = Y$ which is a contradiction. So l_X and l_Y meet. Each point of l_Y is on a unique line of the parallel class determined by l_X. Thus, l_Y does not meet precisely one line of the parallel class determined by l_X. This leaves precisely one line parallel to both l_X and l_Y. Thus; $\mathcal{S}^*$ is a linear space with n^2+n+1 points and n^2+n+1 lines. So $\mathcal{S}^*$ is a projective plane of order n, by [8].

Consider the complement of $\mathcal{S}$ in $\mathcal{S}^*$. The lines of $\mathcal{S}^* \setminus \mathcal{S}$ are sets of $\frac{m-1}{2}$ or $\frac{m+1}{2}$ points, the extensions of the $(n - \frac{m-3}{2})$ $-$lines or $(n - \frac{m-1}{2})$ $-$lines of $\mathcal{S}$, respectively. It is clear that $\mathcal{S}^* \setminus \mathcal{S}$ is a linear space and there is at least one point not on a given line in $\mathcal{S}^* \setminus \mathcal{S}$ .It is known that there are exactly $\frac{m+1}{2}$ lines of degree $\frac{m-1}{2}$ and $\frac{m+1}{2}$ lines of degree $\frac{m+1}{2}$ through any new point added to $\mathcal{S}$ (any point of $\mathcal{S}^* \setminus \mathcal{S}$). Thus $\mathcal{S}^* \setminus \mathcal{S}$ is a $(m+1)-$regular linear space with $\binom{m}{2}$ points and m^2 lines such that every line has degree $\dfrac{m-1}{2}$ or $\dfrac{m+1}{2}$. Therefore; $\mathcal{S}^* \setminus \mathcal{S}$ is a $(m+1)-$regular hyperbolic plane as known the Ostrom's hyperbolic plane, by the Corollary 2.1.

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