A NOTE ON THE (g,f)-CHROMATIC INDEX OF GRAPHS

Changqing XU[1] Yatao DU[2]
1. Department of Applied Mathematics, Hebei University of Technology
Tianjin, 300130, China
2. Department of Mathematics, Shijiazhuang Mechanical Engineering College
Shijiazhuang 050003, China

ABSTRACT. A (g,f)-coloring is a generalized edge-coloring in which each color appears at each vertex v at least $g(v)$ and at most $f(v)$ times, where $g(v)$ and $f(v)$ are nonnegative and positive integers assigned to v, respectively. The minimum number of colors used by a (g,f)-coloring of G is called the (g,f)-chromatic index of G. The maximum number of colors used by a (g,f)-coloring of G is called the upper (g,f)-chromatic index of G. In this paper, we determine the (g,f)-chromatic index and the upper (g,f)-chromatic index in some cases.

This paper deals with finite undirected graphs without isolated vertices, or loops. Multiple edges are allowed. G is called a simple graph if G has no multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G the degree of v in G is denoted by $d_G(v)$ and by $d(v)$ if there is no confusion. Let g and f be respectively nonnegative and positive integer-valued functions defined on $V(G)$ such that $g(v) \leq f(v)$ for each vertex v of $V(G)$. A generalized edge-coloring, called a (g,f)-coloring is considered in [3, 4] and [11]. A (g,f)-coloring is to color all the edges of G such that each color appears at each vertex v at least $g(v)$ and at most $f(v)$ times. Thus the ordinary edge-coloring is a (g,f)-coloring where $g(v) = 0$ and $f(v) = 1$ for each vertex $v \in V(G)$. If $g(v) = 0$ for each vertex v of G, then a (g,f)-coloring is simply called an f-coloring in [5] and [7]. An edge coloring of G in which each color appears at each vertex v at least $g(v)$

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times is called a g-edge cover-coloring. Clearly a g-edge cover-coloring is a special case of (g, f)-coloring. g-edge cover-coloring is considered in [9].

Note that a graph may have no (g, f)-coloring when $g(v) > 0$ for some $v \in V(G)$. So it is important to determine the existence of (g, f)-coloring of a given graph. The existence of (g, f)-coloring is wildly discussed in many papers (see [1], [3] and etc). If G has a (g, f)-coloring, one of the (g, f)-coloring problem is to find a (g, f)-coloring of G with minimum number of colors, which arises in many applications, such as the network design, the file transfer problem on computer networks, and so on [5], [6]. On the other hand, if G has a (g, f)-coloring, it is also important to find a (g, f)-coloring of G with maximum number of colors. The minimum number of colors used by a (g, f)-coloring of G is denoted by $\chi'_{gf}(G)$ which is called the (g, f)-chromatic index of G. The maximum number of colors used by a (g, f)-coloring of G is denoted by $\overline{\chi'}_{gf}(G)$ which is called the upper (g, f)-chromatic index of G.

Without loss of generality, we assume that $f(v) \leq d(v)$ for each vertex $v \in V(G)$. We call $d_f(v) = \lceil d_G(v)/f(v) \rceil$ the f-degree of vertex v. $\Delta_f(G) = \max\{d_f(v) : v \in V(G)\}$ is called the maximum f-degree of G. Similarly, the g-degree $d_g(v)$ of vertex $v \in V(G)$ is $d_g(v) = \lfloor d_G(v)/g(v) \rfloor$ when $g(v) > 0$ and $d_g(v) = |E(G)|$ when $g(v) = 0$. We call $\delta_g(G) = \min\{d_g(v) : v \in V(G)\}$ the minimum g-degree of G. The minimum number of colors needed to an f-coloring of G is called the f-chromatic index of G and is denoted by $\chi'_f(G)$. Let $\chi'_{gc}(G)$ denote the maximum positive integer k for which a g-edge cover-coloring of G exists. We call $\chi'_{gc}(G)$ the g-cover chromatic index of G. If G does not have a g-edge cover-coloring, we set $\chi'_{gc}(G) = 0$. The existence of $\chi'_{gc}(G)$ is considered in [9]. It is trivially true that $\Delta_f(G) \leq \chi'_f(G)$ and $\chi'_{gc}(G) \leq \delta_g(G)$. If G has a (g, f)-coloring, from the definition of $\chi'_f(G)$, $\chi'_{gc}(G)$, $\chi'_{gf}(G)$ and $\overline{\chi'}_{gf}(G)$, it is easy to see that

$$\chi'_f(G) \leq \chi'_{gf}(G) \leq \overline{\chi'}_{gf}(G) \leq \chi'_{gc}(G) \qquad (*)$$

In [10], it was proved that $\chi'_{gf}(G) = \Delta_f(G)$ and $\overline{\chi'}_{gf}(G) = \delta_g(G)$ when G is bipartite. In this note, we determine $\chi'_{gf}(G)$ and $\overline{\chi'}_{gf}(G)$ for some graphs. To get our main results, we need some Lemmas.

Lemma 1. [2] *Let G be a graph. If $f(v)$ is positive and even for all $v \in V(G)$, then $\chi'_f(G) = \Delta_f(G)$.*

Lemma 2. [9] *Let G be a graph. If $g(v)$ is positive and even for all $v \in V(G)$, then $\chi'_{gc}(G) = \delta_g(G)$.*

A (g,f)-factor of G is a spanning subgraph H of G satisfying $g(v) \le d_H(v) \le f(v)$ for each $v \in V(G)$. If a graph G itself is a (g,f)-factor, then G is called a (g,f)-graph.

Lemma 3. [1] *Let G be a graph and $g(v)$ and $f(v)$ be positive and even for all $v \in V(G)$. If G is an (mg, mf)-graph for some positive integer m, then G can be decomposed into m (g,f)-factors.*

Theorem 4. *Let G be a graph. If $g(v)$ and $f(v)$ are positive and even for all $v \in V(G)$, and $\delta_g(G) \ge \Delta_f(G)$. Then $\chi'_{gf}(G) = \Delta_f(G)$ and $\overline{\chi'}_{gf}(G) = \delta_g(G)$.*

Proof. By Lemma 1 and Lemma 2, $\chi'_f(G) = \Delta_f(G)$ and $\chi'_{gc}(G) = \delta_g(G)$. Set $k = \Delta_f(G)$. Then $d(v) \le kf(v)$ for any $v \in V(G)$. Since $\delta_g(G) \ge \Delta_f(G)$, we have $d(v) \ge kg(v)$ for any $v \in V(G)$. It follows that G is a (kg, kf)-graph. By Lemma 3, G has a (g,f)-coloring with k colors. Combining $(*)$, we have $\chi'_{gf}(G) = \Delta_f(G)$.

Set $k_1 = \delta_g(G)$. Then $d(v) \ge k_1 g(v)$ for any $v \in V(G)$. Since $\delta_g(G) \ge \Delta_f(G)$, we have $d(v) \le k_1 f(v)$ for any $v \in V(G)$. It follows that G is a $(k_1 g, k_1 f)$-graph. By Lemma 3, G has a (g,f)-coloring with k_1 colors. Combining $(*)$, we have $\chi'_{gf}(G) = \delta_g(G)$.

□

Lemma 5. [3] *Let G be a graph and m be a positive integer.*

(1) *If $g(v)$ is positive and even for all $v \in V(G)$ and G is an $(mg, mf - m + 1)$-graph, then G has a (g,f)-coloring with m colors;*

(2) *If $f(v)$ is positive and even for all $v \in V(G)$ and G is an $(mg + m - 1, mf)$-graph, then G has a (g,f)-coloring with m colors.*

Theorem 6. *Let G be a graph and m be a positive integer.*

(1) *If $g(v)$ is positive and even for all $v \in V(G)$ and G is an $(mg, mf - m + 1)$-graph, then $\overline{\chi'}_{gf}(G) = \delta_g(G)$.*

(2) *If $f(v)$ is positive and even for all $v \in V(G)$ and G is an $(mg + m - 1, mf)$-graph, then $\chi'_{gf}(G) = \Delta_f(G)$.*

Proof. Suppose that $g(v)$ is positive and even for all $v \in V(G)$ and G is an $(mg, mf - m + 1)$-graph. By Lemma 5, G has a (g,f)-coloring with m colors. So $m \le \overline{\chi'}_{gf}(G)$. And by $(*)$, $m \le \chi'_{gc}(G)$. Since $g(v)$ is positive and even for all $v \in V(G)$, by Lemma 2, $\chi'_{gc}(G) = \delta_g(G)$. Set $\delta_g(G) = k$. So $d(v) \ge kg(v)$ for all $v \in V(G)$. Thus for all $v \in V(G)$, we get that

$$kg(v) \le d(v) \le mf(v) - m + 1.$$

Note that $m \le k$ and $0 < g(v) \le f(v)$, $mf(v) - m + 1 \le kf(v) - k + 1$. Which deduce that $kg(v) \le d(v) \le kf(v) - k + 1$. By Lemma 5, G has a (g,f)-coloring with k colors. Combining $(*)$, we have $\overline{\chi'}_{gf}(G) = \delta_g(G)$.

Now suppose that $f(v)$ is positive and even for all $v \in V(G)$ and G is an $(mg+m-1, mf)$-graph. By Lemma 5, G has a (g,f)-coloring with m colors. So $m \geq \chi'_{gf}(G)$. And by $(*)$, $m \geq \chi'_f(G)$. Since $f(v)$ is positive and even for all $v \in V(G)$, by Lemma 1, $\chi'_f(G) = \Delta_f(G)$. Set $\Delta_f(G) = k$. So $d(v) \leq kf(v)$ for all $v \in V(G)$. Thus for all $v \in V(G)$, we get that

$$mg(v) + m - 1 \leq d(v) \leq kf(v).$$

Note that $m \geq k$, we have $kg(v) + k - 1 \leq mg(v) + m - 1$. Which deduce that $kg(v) + k - 1 \leq d(v) \leq kf(v)$. By Lemma 5, G has a (g,f)-coloring with k colors. Combining $(*)$, we have $\chi'_{gf}(G) = \Delta_f(G)$.

□

Lemma 7. ([2]) *Let G be a simple graph. Then*

$$\max_{v\in V}\{\lceil \frac{d(v)}{f(v)} \rceil\} \leq \chi'_f(G) \leq \max_{v\in V}\{\lceil \frac{d(v)+1}{f(v)} \rceil\}.$$

Lemma 8. ([9]) *Let G be a simple graph. Then*

$$\min_{v\in V}\{\lfloor \frac{d(v)-1}{g(v)} \rfloor\} \leq \chi'_{gc}(G) \leq \min_{v\in V}\{\lfloor \frac{d(v)}{g(v)} \rfloor\}.$$

Lemma 9. ([8]) *Let G be any $(mg+1, mf-1)$-simple graph, then G has a (g,f)-coloring with m colors.*

Theorem 10. *Let G be a simple graph. If $\chi'_f(G) = \max_{v\in V}\{\lceil (d(v)+1)/f(v) \rceil\}$, $\chi'_{gc}(G) = \min_{v\in V}\{\lfloor (d(v)-1)/g(v) \rfloor\}$ and $\chi'_{gc}(G) \geq \chi'_f(G)$. Then G has a (g,f)-coloring and $\chi'_{gf}(G) = \chi'_f(G)$, $\overline{\chi'}_{gf}(G) = \chi'_{gc}(G)$.*

Proof. Let $k = \max_{v\in V}\{\lceil (d(v)+1)/f(v) \rceil\}$. So for any $v \in V(G)$, $d(v) \leq kf(v) - 1$. By the condition of this theorem, $k \leq \min_{v\in V}\{\lfloor (d(v)-1)/g(v) \rfloor\}$. Thus for any $v \in V(G)$, $d(v) \geq kg(v) + 1$. So $kg(v) + 1 \leq d(v) \leq kf(v) - 1$ for any $v \in V(G)$. By Lemma 9, G has a (g,f)-coloring with k colors. And by $(*)$, we have $\chi'_{gf}(G) = \chi'_f(G)$.

Similarly, we obtain that G has a (g,f)-coloring with $\chi'_{gc}(G)$ colors. And by $(*)$, we have $\overline{\chi'}_{gf}(G) = \chi'_{gc}(G)$.

□

If G has a (g,f)-coloring, we believe that $\chi'_{gf}(G) = \chi'_f(G)$ and $\overline{\chi'}_{gf}(G) = \chi'_{gc}(G)$ for any graph G.

References

[1] M. Cai, On some factor theorems of graphs, *Discrete Math.*, **98** (1991) 223-229.

[2] S. L. Hakimi and O. Kariv, A generalization of edge-coloring in graphs, *Journal of Graph Theory*, **10** (1986) 139-154.

[3] G. Liu, On (g,f)-factors and factorizations. *Acta Math. Sinica*, **37**(1994) 230-237.

[4] G. Liu and X. Deng, A polynomial algorithm for finding (g, f)-colorings orthogonal to stars in bipartite graphs. *Science in China, Ser A* **35(3)**(2005) 334-344.
[5] S. Nakano, T. Nishizeki, and N. Saito, On the f-coloring of multigraphs, *IEEE Trans. Circuit and Syst.*, **35(3)** (1988) 345-353.
[6] S. Nakano, T. Nishizeki, Scheduling file transfers under port and channel constraints. *International Journal on Foundations of Computer Science*, **4(2)** (1993) 425-431.
[7] S. Nakano, T. Nishizeki, Edge-coloring problems for graphs, *Interdisciplinary information Sciences*, **1(1)** (1994) 19-32.
[8] H. Song and G. Liu, Applications of an equitable edge-coloring theorem, *The International Conference on Mathematical Programming*, Dec. 2002 19-22.
[9] H. Song and G. Liu, On f-edge cover-coloring in multigraphs, *Acta Mathematica Sinica*, (2005) 4 *to appear*.
[10] C. Xu and G. Liu, The existence of (g, f)-coloring, *submitted to the Electronic J. Comb.*.
[11] X. Zhou, K. Fuse and T. Nishizeki, A linear algorithm for finding $[g, f]$-colorings of partial k-trees, *Algorithmica*, 27 (2000) 227-243.

E-mail address: chqxu@sdu.edu.cn dyt77@eyou.com