One edge union of k shell graphs is cordial

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Abstract. A shell graph of order n denoted by $H(n, n-3)$ is the graph obtained from the cycle C_n of order n by adding $n-3$ chords incident with a common vertex say u. Let v be a vertex adjacent to u in C_n. Sethuraman and Selvaraju [3] conjectured that for all $k \geq 1$, and for all $n_i \geq 4$, $1 \leq i \leq k$, one edge (uv) union of k-shell graphs $H(n_i, n_i - 3)$ is cordial. In this paper we settle this conjecture affirmatively.

Keywords. cordial labeling, shell graph

1 Introduction

In 1987 Cahit [1] introduced a new types of labeling which is called cordial labeling. Cordial labeling may be considered a weaker version of both graceful labeling and harmonious labeling. Suppose G is a graph with vertex set $V(G)$ and edge set $E(G)$. A *binary labeling* of G is a mapping $f : V(G) \to \{0,1\}$. The mapping f induces an edge-labeling $f^* : E(G) \to \{0,1\}$ defined by $f^*(uv) = |f(u) - f(v)|$, for all edges $uv \in E(G)$. Let $V_f(0) = \{v \in V(G) : f(v) = 0\}, V_f(1) = \{v \in V(G) : f(v) = 1\}, E_f(0) = \{e \in E(G) : f^*(e) = 0\}$ and $E_f(1) = \{e \in E(G) : f^*(e) = 1\}$, and let their cardinalities be $v_f(0), v_f(1), e_f(0)$ and $e_f(1)$, respectively. A labeling f of a graph G is *cordial* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is *cordial* if it has a cordial labeling.

A shell $H(n, n-3)$ of width n is a graph obtained by taking $n-3$ concurrent chords in a cycle C_n on n vertices. The vertex at which all the

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chords are concurrent is called apex u. The two vertices adjacent to the apex have degree 2, apex has degree $n-1$ and all the other vertices have degree 3. Let v be a vertex adjacent to u in C_n. A graph G is called one edge union of k shell graphs $H(n_i, n_i-3)$'s, $1 \leq i \leq k$, if G is obtained by taking the union of k shell graphs $H(n_i, n_i-3)$'s, $1 \leq i \leq k$, having the k edges uv's identified. For the literature on cordial graphs we refer to [2] and the relevant reference given in it.

Shee and Ho [4, 5] gave the cordiality of one-point union of n copies of a graph and the path-union of n copies of a graph. Sethuraman and Selvaraju [3] showed that one edge union of k copies of a shell graph $H(n, n-3)$ is cordial, and conjectured for all $k \geq 1$ and $n_i \geq 4$, $1 \leq i \leq k$, one edge union of k shell graphs $H(n_i, n_i-3)$ is cordial. In this paper, we prove that the above conjecture of Sethuraman and Selvaraju is true.

2 One edge union of k shell graphs

Here we prove our main result. For the convenience of labeling we describe the vertex set and edge set of i-th shell graph $H(n_i, n_i-3)$, $1 \leq i \leq k$, in the following way.

$$\begin{aligned}
&V(H_i) = \{u, u_{i,j}, v \ : \ 1 \leq j \leq n_i - 2\},\\
&E_{ci}(H_i) = \{uu_{i,1}, u_{i,j-1}u_{i,j}, u_{i,n_i-2}v, uv \ : \ 2 \leq j \leq n_i - 2\},\\
&E_{ch}(H_i) = \{uu_{i,j} \ : \ 2 \leq j \leq n_i - 2\},\\
&E(H_i) = E_{ci}(H_i) \cup E_{ch}(H_i).
\end{aligned}$$

Let $G_k = G(n_1, n_2, \ldots, n_k) = \cup_{i=1}^{k} H_i$ be an one edge union of k shell graphs $H(n_i, n_{i-3})$, where $k \geq 1, n_i \geq 4$ $(1 \leq i \leq k)$, and

$$\begin{aligned}
V(G_k) &= \cup_{i=1}^{k} V(H_i) = \{u, u_{i,j}, v \ : \ 1 \leq i \leq k, 1 \leq j \leq n_i - 2\},\\
E(G_k) &= \cup_{i=1}^{k} E(H_i) = \{uu_{i,1}, u_{i,j-1}u_{i,j}, u_{i,n_i-2}v, uv, uu_{i,j} \ : \\
&\qquad\qquad 1 \leq i \leq k, 2 \leq j \leq n_i - 2\}.
\end{aligned}$$

Without loss of generality, we may assume that

$$n_i (\text{mod } 4) \leq n_{i+1} (\text{mod } 4), \qquad (1 \leq i < k).$$

Let

$$k_t = |\{n_i \ : \ n_i (\text{mod } 4) = t\}|,$$

then $k_0 + k_1 + k_2 + k_3 = k$ and

$$n_i (\text{mod } 4) \ = \ \begin{cases} 0, & 1 \leq i \leq k_0, \\ 1, & k_0 + 1 \leq i \leq k_0 + k_1, \\ 2, & k_0 + k_1 + 1 \leq i \leq k_0 + k_1 + k_2, \\ 3, & k_0 + k_1 + k_2 + 1 \leq i \leq k. \end{cases}$$

Theorem 2.1. For $1 \le i \le k$ and for $n_i \ge 4$, the graph $G_k = G(n_1, n_2, \dots, n_k)$ is cordial.

Proof. We define a vertex labeling f as follows:

$$\begin{aligned} f(u) &= 0, \\ f(v) &= 1. \end{aligned}$$

For $1 \le i \le k_0$ and $1 \le j \le n_i - 2$,

$$f(u_{i,j}) = \begin{cases} 0, & j \pmod 4 = 0, 1, \\ 1, & j \pmod 4 = 2, 3. \end{cases}$$

For $k_0 + 1 \le i \le k_0 + k_1, (i - k_0) \pmod 2 = 1$ and $1 \le j \le n_i - 2$,

$$f(u_{i,j}) = \begin{cases} 0, & j \pmod 4 = 2, 3, \\ 1, & j \pmod 4 = 0, 1. \end{cases}$$

For $k_0 + 1 \le i \le k_0 + k_1, (i - k_0) \pmod 2 = 0$ and $1 \le j \le n_i - 2$,

$$f(u_{i,j}) = \begin{cases} 0, & j \pmod 4 = 0, 1, \\ 1, & j \pmod 4 = 2, 3. \end{cases}$$

For $k_0 + k_1 + 1 \le i \le k_0 + k_1 + k_2$ and $1 \le j \le n_i - 2$,

$$f(u_{i,j}) = \begin{cases} 0, & j \pmod 4 = 0, 3, \\ 1, & j \pmod 4 = 1, 2. \end{cases}$$

For $k_0 + k_1 + k_2 + 1 \le i \le k, (i - k_0 - k_2) \pmod 2 = 1$ and $1 \le j \le n_i - 2$,

$$f(u_{i,j}) = \begin{cases} 0, & j \pmod 4 = 0, 1, \\ 1, & j \pmod 4 = 2, 3. \end{cases}$$

For $k_0 + k_1 + k_2 + 1 \le i \le k, (i - k_0 - k_2) \pmod 2 = 0$ and $1 \le j \le n_i - 2$,

$$f(u_{i,j}) = \begin{cases} 0, & j \pmod 4 = 2, 3, \\ 1, & j \pmod 4 = 0, 1. \end{cases}$$

For $1 \le i \le k$, let

$$\begin{aligned} &p_i = \lfloor n_i/4 \rfloor, \\ &v_{f_i}(0) = |\{v \ : \ v \in (V(H_i) - \{u, v\}), f(v) = 0\}|, \\ &v_{f_i}(1) = |\{v \ : \ v \in (V(H_i) - \{u, v\}), f(v) = 1\}|, \\ &e_{f_i^*}(0) = |\{e \ : \ e \in (E(H_i) - \{uv\}), f^*(e) = 0\}|, \\ &e_{f_i^*}(1) = |\{e \ : \ e \in (E(H_i) - \{uv\}), f^*(e) = 1\}|. \end{aligned}$$

Now, we count $v_{f_i}(0) - v_{f_i}(1)$ and $e_{f_i^*}(0) - e_{f_i^*}(1)$ according to the following four cases.

Case 1. For $1 \le i \le k_0$, we have $n_i = 4p_i$. The labels of $V(H_i)$ are $0(0110)^{p_i-1}011$, the labels of $E_{ci}(H_i)$ are $(0101)^{p_i-1}0101$ and the labels of $E_{ch}(H_i)$ are $(1100)^{p_i-1}1$. Hence,

$$v_{f_i}(0) - v_{f_i}(1) = (2p_i - 1) - (2p_i - 1) = 0,$$
$$e_{f_i^*}(0) - e_{f_i^*}(1) = 2p_i - (2p_i - 1) + (2p_i - 2) - (2p_i - 1) = 0.$$

Case 2. For $k_0 + 1 \le i \le k_0 + k_1$, we have $n_i = 4p_i + 1$.
Case 2.1. For $(i-k_0)(\bmod\ 2) = 1$, the labels of $V(H_i)$ are $0(1001)^{p_i-1}1001$, the labels of $E_{ci}(H_i)$ are $1(1010)^{p_i-1}1011$ and the labels of $E_{ch}(H_i)$ are $(0011)^{p_i-1}00$. Hence,

$$v_{f_i}(0) - v_{f_i}(1) = 2p_i - (2p_i - 1) = 1,$$
$$e_{f_i^*}(0) - e_{f_i^*}(1) = (2p_i - 1) - (2p_i + 1) + 2p_i - (2p_i - 2) = 0.$$

Case 2.2. For $(i-k_0)(\bmod\ 2) = 0$, the labels of $V(H_i)$ are $0(0110)^{p_i-1}0111$, the labels of $E_{ci}(H_i)$ are $0(1010)^{p_i-1}1001$ and the labels of $E_{ch}(H_i)$ are $(1100)^{p_i-1}11$. Hence,

$$v_{f_i}(0) - v_{f_i}(1) = (2p_i - 1) - 2p_i = -1,$$
$$e_{f_i^*}(0) - e_{f_i^*}(1) = (2p_i + 1) - (2p_i - 1) + (2p_i - 2) - 2p_i = 0.$$

Case 3. For $k_0 + k_1 + 1 \le i \le k_0 + k_1 + k_2$, we have $n_i = 4p_i + 2$. The labels of $V(H_i)$ are $0(1100)^{p_i}1$, the labels of $E_{ci}(H_i)$ are $(1010)^{p_i}11$ and the labels of $E_{ch}(H_i)$ are $(1001)^{p_i-1}100$. Hence,

$$v_{f_i}(0) - v_{f_i}(1) = 2p_i - 2p_i = 0,$$
$$e_{f_i^*}(0) - e_{f_i^*}(1) = 2p_i - (2p_i + 1) + 2p_i - (2p_i - 1) = 0.$$

Case 4. For $k_0 + k_1 + k_2 + 1 \le i \le k$, we have $n_i = 4p_i + 3$.
Case 4.1. For $(i-k_0-k_2)(\bmod\ 2) = 1$, the labels of $V(H_i)$ are $0(0110)^{p_i}01$, the labels of $E_{ci}(H_i)$ are $(0101)^{p_i}011$ and the labels of $E_{ch}(H_i)$ are $(1100)^{p_i}$. Hence,

$$v_{f_i}(0) - v_{f_i}(1) = (2p_i + 1) - 2p_i = 1,$$
$$e_{f_i^*}(0) - e_{f_i^*}(1) = (2p_i + 1) - (2p_i + 1) + 2p_i - 2p_i = 0.$$

Case 4.2. For $(i-k_0-k_2)(\bmod\ 2) = 0$, the labels of $V(H_i)$ are $0(1001)^{p_i}11$, the labels of $E_{ci}(H_i)$ are $1(1010)^{p_i}01$ and the labels of $E_{ch}(H_i)$ are $(0011)^{p_i}$. Hence,

$$v_{f_i}(0) - v_{f_i}(1) = 2p_i - (2p_i + 1) = -1,$$
$$e_{f_i^*}(0) - e_{f_i^*}(1) = (2p_i + 1) - (2p_i + 1) + 2p_i - 2p_i = 0.$$

According to the Cases 1-4, we have

$$|v_f(0) - v_f(1)| = |1 - 1 + \textstyle\sum_{i=1}^{k}(v_{f_i}(0) - v_{f_i}(1))| = (k_1 + k_3)(\bmod\ 2) \le 1,$$
$$|e_{f^*}(0) - e_{f^*}(1)| = |-1 + \textstyle\sum_{i=1}^{k}(e_{f_i^*}(0) - e_{f_i^*}(1))| = |-1 + 0| = 1.$$

By the definition of the cordial graph, we can conclude that G_k is cordial. □

In Figure 2.1, we illustrate our cordial labelings for one edge union k shell graphs $G(8,9,6,7), G(9,9,7,7), G(9,7,7,7)$ and $G(9,9,7,7,7)$.

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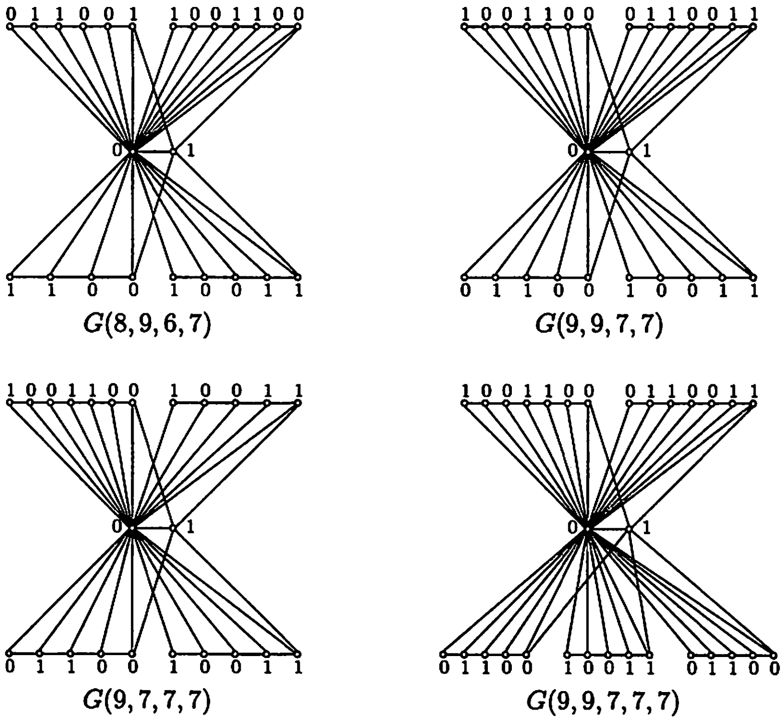

Figure 2.1 The cordial labelings of one edge union k shell graphs $G(8,9,6,7), G(9,9,7,7), G(9,7,7,7)$ and $G(9,9,7,7,7)$.