Acyclic Domination Number and Minimum Degree in 2-Diameter-Critical Graphs*

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Abstract: A subset S of the vertex set of a graph G is called *acyclic* if the subgraph it induces in G contains no cycles. We call S an *acyclic dominating set* if it is both acyclic and dominating. The minimum cardinality of an acyclic dominating set, denoted by $\gamma_a(G)$, is called the *acyclic domination number* of G. A graph G is *2-diameter-critical* if it has diameter 2 and the deletion of any edge increases its diameter. In this paper, we show that for any positive integers k and $d \geq 3$, there is a 2-diameter-critical graph G such that $\delta(G) = d$ and $\gamma_a(G) - \delta(G) \geq k$, and our result answers a question posed by Cheng et al. in negative.

Keywords: Acyclic Domination Number, Diameter Two Critical

1. Introduction

Let $G = (V(G), E(G))$ be a finite simple graph without loops. The neighborhood $N(v)$ of a vertex v is the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The minimum degree of G is denoted by $\delta(G)$. For

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$S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G. If $G[S]$ contains no edge (cycle, respectively), then we call S an *independent set* (*acyclic set*, respectively). The *distance* of two distinct vertices u, v, denoted by $d(u, v)$, is the length of a shortest path connecting u and v. The *diameter* of G, denoted by $d(G)$, is defined as: $d(G) = \max\{d(u, v) \mid u, v \in V(G)\}$. A graph G is *2-diameter-critical* if $d(G) = 2$ and $d(G-e) > 2$ for any edge $e \in E(G)$. A set $S \subseteq V(G)$ is called a *dominating set* if every vertex u in $V(G) - S$ is adjacent to at least one vertex v in S. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. A set $S \subseteq V(G)$ is called an *acyclic dominating* set of G if it is both acyclic and dominating. The minimum cardinality of an acyclic dominating set in a graph G is called the *acyclic domination number* of G, denoted by $\gamma_a(G)$. For $X, Y \subseteq V(G)$, we say X dominates Y(or Y is dominated by X) if $N(y) \cap X \neq \emptyset$ for every vertex $y \in Y$. There are many types of domination and domination-related parameters, say for instance the lower and upper irredundance numbers $ir(G)$ and $IR(G)$, the lower and upper domination numbers $\gamma(G)$ and $\Gamma(G)$ and the independent domination number and vertex independence number $i(G)$ and $\alpha(G)$, which has been studied in the literature, (see [1, 2]). For more domination-related parameters, see [6]. The following well-known inequality chain was first defined by Cockayne et al. [5] on domination-related parameters of a graph G:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G). \tag{1}$$

Since then more than 100 research papers have been published in which this inequality chain is the focus of study. The concept of acyclic domination was introduced by S.M. Hedetniemi et al. in [7]. This invariant is particularly interesting in that it is a fundamental type of domination and it provides more examples of parameters whose values lie between two consecutive parameters in (1):

$$\gamma(G) \leq \gamma_a(G) \leq i(G).$$

In the same paper, they posed some open questions on acyclic domination and the following is one of them.

Question 1 (Hedetniemi et al. [7]). Let G be a graph with $d(G) = 2$. Is $\gamma_a(G) \leq \delta(G)$?

It is shown in [3] that $\gamma_a(G) \leq \delta(G)$ does not hold when $\delta(G) = 3$. In [4], Cheng et al. answer the question above in negative by showing that for any positive integers k and $d \geq 3$, there is a graph G of diameter two with $\delta(G) = d$ such that $\gamma_a(G) - \delta(G) \geq k$. Obviously, any graph G of diameter two has a spanning subgraph G' which is 2-diameter-critical. In general, the size of G' is less than that of G. Thus one may ask whether the answer to Question 1 is affirmative if G is 2-diameter-critical. Since the examples given in [3] and [4] are not 2-diameter-critical, Cheng et al. posed in [4] the following.

Question 2 (Cheng et al. [4]). Let G be a 2-diameter-critical graph. Is $\gamma_a(G) \leq \delta(G)$?

In this paper, we answer Question 2 in negative by showing that for any positive integers k and $d \geq 3$, there is a 2-diameter-critical graph G such that $\delta(G) = d$ and $\gamma_a(G) - \delta(G) \geq k$.

2. Example

In this section, we will give a 2-diameter-critical graph $G(d,n)$ and show the graph is a counterexample to Question 2.

Let $d \geq 3$ be an integer. Take $n = (d-1)t + 1$, where t is a positive integer. In order to define the graph $G(d,n)$, we first define a graph $H(d,n)$ as follows:

- $V(H(d,n)) = \{a_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq n\}$ and
- $E(H(d,n)) = \cup_{1 \leq i \leq 6} E_i$ if $d = 3$, and
  $E(H(d,n)) = \cup_{7 \leq i \leq 9} E_i$ if $d \geq 4$,

where

$E_1 = \{a_{i,t+1}a_{ij} \mid 1 \leq i \leq n,\ 2 \leq j \leq 2t \textit{ and } j \neq t+1\}$,
$E_2 = \{a_{ik}a_{j,k+1} \mid 1 \leq i,j \leq n \textit{ and } k = 1, 2t\}$,
$E_3 = \{a_{ik}a_{jk} \mid 1 \leq i < j \leq n,\ 2 \leq k \leq 2t \textit{ and } k \neq t+1\}$,
$E_4 = \{a_{i2}a_{ij} \mid 1 \leq i \leq n,\ t+2 \leq j \leq 2t+1\}$,
$E_5 = \{a_{i,2t}a_{ij} \mid 1 \leq i \leq n \textit{ and } 3 \leq j \leq t\}$,
$E_6 = \{a_{ij}a_{ik} \mid 2 \leq i \leq n,\ 3 \leq j \leq t \textit{ and } t+2 \leq k \leq 2t-1\}$,
$E_7 = \{a_{ik}a_{jk} \mid 1 \leq k \leq n-1 \textit{ and } i \neq j\}$,
$E_8 = \{a_{in}a_{ij} \mid 1 \leq i \leq n \textit{ and } 1 \leq j \leq n-1\}$ and
$E_9 = \{a_{ij}a_{ik} \mid 2 \leq i \leq n, lt+1 \leq j \leq (l+1)t, mt+1 \leq k \leq (m+1)t$ *and* $l \neq m\}$.

Let $B = \{v_i \mid 0 \leq i \leq d\}$. Now, we define the graph $G(d,n)$ as below:

- $V(G(d,n)) = V(H(d,n)) \cup B$ and
- $E(G(d,n)) = E(H(d,n)) \cup E_{10} \cup E_{11}$ if $d = 3$, and

$E(G(d,n)) = E(H(d,n)) \cup E_{12} \cup E_{13}$ if $d \geq 4$,

where

$E_{10} = \{v_i v_j \mid 0 \leq i < j \leq 3\}$,
$E_{11} = \{v_1 a_{ij}, v_2 a_{i,t+1}, v_3 a_{ik} \mid 1 \leq i \leq n, 1 \leq j \leq t \text{ and } t+2 \leq k \leq 2t+1\}$,
$E_{12} = \{v_i v_j \mid 0 \leq i < j \leq d-1\} \cup \{v_0 v_d\}$ and
$E_{13} = \{v_k a_{ij}, v_d a_{in} \mid 1 \leq k \leq d-1, 1 \leq i \leq n \text{ and } (k-1)t+1 \leq j \leq kt\}$.

Lemma 1. $G(d,n)$ is 2-diameter-critical.

Proof. We distinguish the following two cases separately.

Case 1. $d = 3$

We first show $d(G(3,n)) = 2$. Let u, v be any two distinct vertices of $G(3,n)$. If $u \in B$ or $v \in B$, then it is easy to see that $d(u,v) \leq 2$. Thus we may assume $u, v \in V(H(3,n))$, say $u = a_{ij}$ and $v = a_{kl}$. If $i = k$, then $d(u,v) \leq 2$ and hence we may assume $i \neq k$. If $j = t+1$ or $l = t+1$ or $j, l \leq t$ or $j, l \geq t+2$, then noting that the definition of E_{11}, we can easily get that $d(u,v) \leq 2$. Thus we may assume $j \leq t$ and $l \geq t+2$. In this case, we have $d(u,v) = 2$. Therefore, we have $d(G(3,n)) = 2$.

Next we show that $d(G(3,n) - e) > 2$ for any edge $e \in E(G)$. Let e be a given edge and $G = G(3,n) - e$. If $e \in E_1$, say $e = a_{i,t+1} a_{ij}$, then $d_G(a_{i,t+1}, a_{kj}) > 2$ for all k with $k \neq i$. If $e \in E_2$, then $d_G(a_{i1}, a_{jn}) > 2$ if $e = a_{i1} a_{j2}$ and $d_G(a_{i3}, a_{jn}) > 2$ if $e = a_{i,n-1} a_{jn}$. If $e \in E_3$, say $e = a_{ik} a_{jk}$, then $d_G(a_{i,t+1}, a_{jk}) > 2$. If $e \in E_4$, say $e = a_{i2} a_{ij}$, then $d_G(a_{i1}, a_{ij}) > 2$. If $e \in E_5$, say $e = a_{ij} a_{i,n-1}$, then $d_G(a_{ij}, a_{1n}) > 2$. If $e \in E_6$, say $e = a_{ij} a_{ik}$, then $d_G(a_{1j}, a_{ik}) > 2$. If $e \in E_{10}$, then $d(v_0, u) > 2$ for any $u \in N(v_i) \cap V(H(3,n))$ if $e = v_0 v_i$, $d_G(a_{11}, v_2) > 2$ if $e = v_1 v_2$, $d_G(a_{1n}, v_2) > 2$ if $e = v_2 v_3$ and $d_G(a_{11}, v_3) > 2$ if $e = v_1 v_3$. If $e \in E_{11}$, say $e = v_i a_{jk}$, then $d_G(v_0, a_{jk}) > 2$. Thus, $G(3,n)$ is 2-diameter-critical.

Case 2. $d \geq 4$

It is easy to check that $d(G(d,n)) = 2$ for $d \geq 4$. Let e be any given edge of $G(d,n)$ and $G = G(d,n) - e$. We now show that $d(G) > 2$. If $e \in E_7$, say $e = a_{ik} a_{jk}$, then we have $d_G(a_{in}, a_{jk}) > 2$. If $e \in E_8$, say $e = a_{in} a_{ij}$, then $d_G(a_{in}, a_{kj}) > 2$ for all $1 \leq k \leq n$ with $k \neq i$. If $e \in E_9$, say $e = a_{ij} a_{ik}$, then we have $d_G(a_{1j}, a_{ik}) > 2$. If $e \in E_{12}$, then we have $d_G(a_{(i-1)t+1,j}, v_0) > 2$ if $e = v_0 v_i$ and $d_G(v_i, a_{1,jt}) > 2$ if $e = v_i v_j$ with $1 \leq i < j \leq d-1$. If $e \in E_{13}$, say $e = v_k a_{ij}$, then we have $d_G(v_0, a_{ij}) > 2$. Thus, $G(d,n)$ is 2-diameter-critical for $d \geq 4$. ∎

Lemma 2. $\gamma_a(G(3,n)) = (n+1)/2$.

Proof. Let S be an acyclic dominating set of $G(3,n)$ and $|S \cap N(v_0)| = l$. Obviously, $0 \le l \le 2$. Set $H = H(3,n)$. If $v_2 \notin S$, then since $N_H[a_{i,t+1}] \cap N_H[a_{j,t+1}] = \emptyset$ for $i \ne j$, S contains at least n vertices of H in order to dominate the set $\{a_{i,t+1} \mid 1 \le i \le n\}$, which implies $|S| \ge n+1$. Assume now $v_2 \in S$. Set $I_1 = \{a_{12}, a_{23}, \ldots, a_{t-1,t}\}$, $I_2 = \{a_{i,t+1} \mid 1 \le i \le n\}$ and $I_3 = \{a_{t+1,t+2}, a_{t+2,t+3}, \ldots, a_{n-2,n-1}\}$. For any $u, v \in I_i$ with $1 \le i \le 3$, it is easy to see that $N_H[u] \cap N_H[v] = \emptyset$. Thus, if $S' \subseteq V(H)$ and S' dominates I_i, then $|S'| \ge |I_i|$. If $l = 1$, then in order to dominate $I_1 \cup I_3$, it is not difficult to show that S contains at least $n-3$ vertices of H and hence $|S| \ge (n-3)+1 = n-2$. If $l = 2$, then in order to dominate I_1 or I_3, S contains at least $t-1$ vertices of H, which implies $|S| \ge (t-1)+2 = t+1$. Thus we have $|S| \ge t+1$ in all the cases. Since $I_1 \cup \{v_2, v_3\}$ is an acyclic domination set of $G(3,n)$, we have $\gamma_a(G(3,n)) = t+1 = (n+1)/2$. ∎

Lemma 3. $\gamma_a(G(d,n)) = 3 + (d-3)(n-1)/(d-1)$ for $d \ge 4$.

Proof. Let S be an acyclic dominating set of $G(d,n)$ and $|S \cap N(v_0)| = l$. Obviously, $0 \le l \le 3$. Set $H = H(d,n)$ and $I = \{a_{in} \mid 1 \le i \le n\}$. If $v_d \notin S$, then since $N_H[a_{in}] \cap N_H[a_{jn}] = \emptyset$ for all $1 \le i < j \le n$, S contains at least n vertices of H in order to dominate I, which implies $|S| \ge n+1$. Assume now $v_d \in S$. Since $l \le 3$, by symmetry we may assume $S \cap N(v_0) \subseteq \{v_{d-2}, v_{d-1}, v_d\}$. Let $I_l = \{a_{ii} \mid 1 \le i \le (d-l)t\}$. Obviously, $S \cap N(v_0)$ cannot dominate any vertex a_{ii} of I_l. Thus, in order to dominate I_l, S contains at least $(d-l)t$ vertices of H, which implies $|S| \ge l + (d-l)t \ge 3 + (d-3)t = 3 + (d-3)(n-1)/(d-1)$. This implies $|S| \ge 3 + (d-3)(n-1)/(d-1)$ in all the cases. On the other hand, $I_3 \cup \{v_{d-2}, v_{d-1}, v_d\}$ is an acyclic dominating set of $G(d,n)$, we have $\gamma_a(G(d,n)) = 3 + (d-3)(n-1)/(d-1)$. ∎

Theorem 1. For any positive integers k and $d \ge 3$, there is a 2-diameter-critical graph G such that $\delta(G) = d$ and $\gamma_a(G) - \delta(G) \ge k$.

Proof. Take $G = G(d,n)$. It is easy to see $\delta(G) = d$. By Lemma 1, G is 2-diameter-critical. If $d = 3$, then by Lemma 2 we have $\gamma_a(G) = (n+1)/2$. Since $\gamma_a(G) - \delta(G) = (n-5)/2 \to \infty$ as $n \to \infty$, we see the conclusion holds. If $d \ge 4$, then by Lemma 3 we have $\gamma_a(G) = 3+(d-3)(n-1)/(d-1)$ and hence $\gamma_a(G) - \delta(G) = 3 - d + (d-3)(n-1)/(d-1)$. Since $3 - d + (d-3)(n-1)/(d-1) \to \infty$ as $n \to \infty$ for fixed d, we see that the conclusion also holds. ∎

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