The crossing number of flower snarks and related graphs *

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Abstract. For odd $n \geq 5$, the Flower Snark $F_n = (V, E)$ is a simple undirected cubic graph with $4n$ vertices, where $V = \{a_i : 0 \leq i \leq n-1\} \cup \{b_i : 0 \leq i \leq n-1\} \cup \{c_i : 0 \leq i \leq 2n-1\}$ and $E = \{b_i b_{(i+1) \bmod n} : 0 \leq i \leq n-1\} \cup \{c_i c_{(i+1) \bmod 2n} : 0 \leq i \leq 2n-1\} \cup \{a_i b_i, a_i c_i, a_i c_{n+i} : 0 \leq i \leq n-1\}$. For $n = 3$ or even $n \geq 4$, F_n is called the related graph of Flower Snark. We show that the crossing number of F_n equals $n-2$ if $3 \leq n \leq 5$, and n if $n \geq 6$.

Keywords. crossing number, cubic graph, flower snark

1 Introduction

We consider only *good drawing* of a simple graph, i.e. a drawing satisfies (i) no edge crosses itself; (ii) adjacent edges do not cross; (iii) crossing edges do so only once; (iv) edges do not cross vertices and (v) no more than two edges cross at a common point. Let G be a graph with vertex set V and edge set E. We denote the crossing number of G for the plane by $cr(G)$. If $D(G)$ is a good drawing of G, then $\nu(D(G))$ denotes the number of the crossings in $D(G)$. It is clear that $cr(G) \leq \nu(D(G))$. Crossing number of a graph represents a fundamental measure of non-planarity of graphs.

The crossing number of cubic graphs has been widely studied in recent years. Hliněný[7] proved that it is NP-hard to determine the crossing number of a simple 3-connected cubic graph. The generalized Petersen graph $P(n, k)$ is an important class of cubic graphs. Its crossing number has been

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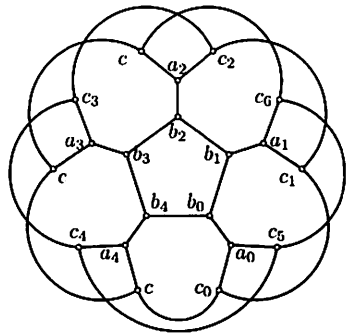

Fig. 1.1 Flower snark F_5

studied in [4, 5, 9, 10]. More details about crossing number can refer to [6, 8, 12, 13, 14, 15, 16].

Snarks are simple nontrivial connected cubic graphs of chromatic index 4 (i.e. without a 3-edge-coloring). The well-known Four color theorem is equivalent to the statement that no snark is planar. The *flower snarks*[1] are an important class of snarks. Fiorini proved that the flower snarks are all hypo-Hamiltonian. More details on flower snark can refer to [2, 3, 11].

For odd $n \geq 5$, the flower snark $F_n = (V, E)$ is a simple undirected cubic graph with $4n$ vertices, where $V = \{a_i : 0 \leq i \leq n-1\} \cup \{b_i : 0 \leq i \leq n-1\} \cup \{c_i : 0 \leq i \leq 2n-1\}$ and $E = \{b_i b_{(i+1) \bmod n} : 0 \leq i \leq n-1\} \cup \{c_i c_{(i+1) \bmod 2n} : 0 \leq i \leq 2n-1\} \cup \{a_i b_i, a_i c_i, a_i c_{n+i} : 0 \leq i \leq n-1\}$. For $n = 3$ or even $n \geq 4$, F_n is called the related graph of flower snark. Fig. 1.1 shows the graph F_5.

In this paper, we study the crossing number of F_n $(n \geq 3)$ to give a better understanding of their topological structure. We show that the crossing number of F_n equals $n-2$ if $3 \leq n \leq 5$, and n if $n \geq 6$.

2 Basic lemmas

Let A, B be two disjoint subsets of $E(G)$. In a drawing D, the number of the crossings crossed by an edge in A and another edge in B is denoted by $\nu_D(A, B)$. The number of the crossings that involve a pair of edges in A is denoted by $\nu_D(A)$. So $\nu(D) = \nu_D(E(G))$. In a drawing D, an edge is said to be *clean* in D if it is not crossed by any other edge, otherwise, it is *crossed* in D. An edge set is *clean* in D if all its edges are clean, otherwise, it is *crossed* in D. A crossed edge (set) *involves* the crossing on it, and vice versa. A pair of edge sets (A, B) is a *crossed-pair* if $\nu_D(A, B) > 0$. Let X be a subset of $V(G)$ or of $E(G)$ for a graph G. Then $G[X]$ denotes the subgraph of G induced by X.

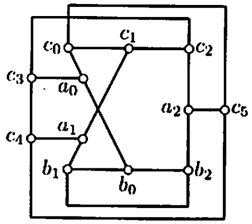

Fig. 3.1 A good drawing of F_3

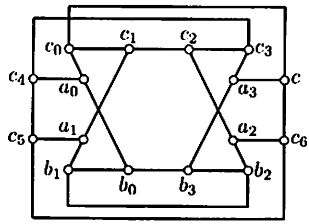

Fig. 3.2 A good drawing of F_4

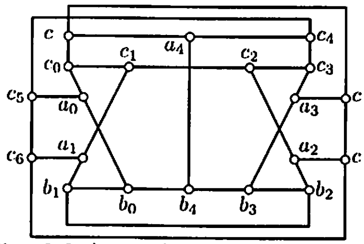

Fig. 3.3 A good drawing of F_5

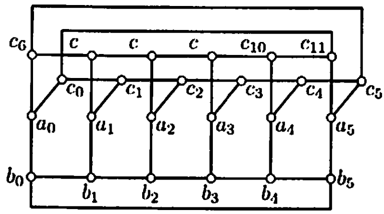

Fig. 3.4 A good drawing of F_6

The following are crucial, but elementary, observation.

Lemma 2.1. Let A, B, C be mutually disjoint subsets of $E(G)$. Then,

$$\begin{aligned} \nu_D(C, A \cup B) &= \nu_D(C, A) + \nu_D(C, B), \\ \nu_D(A \cup B) &= \nu_D(A) + \nu_D(B) + \nu_D(A, B). \end{aligned}$$

3 Upper bounds for F_n $(n \geq 3)$

Lemma 3.1. $cr(F_n) \leq n - 1$ for $3 \leq n \leq 5$ and $cr(F_n) \leq n$ for $n \geq 6$.

Proof. We show good drawings of F_n $(n = 3, 4, 5)$ with $n - 1$ crossings in Fig. 3.1-3.3, hence, $cr(F_n) \leq n - 1$ for $3 \leq n \leq 5$. Fig. 3.4 shows a good drawing of F_6 with 6 crossings. This drawing can be extended to produce a good drawing of F_n $(n \geq 6)$ with n crossings. Hence, $cr(F_n) \leq n$ for $n \geq 6$. □

4 Crossing number of F_n $(n \geq 3)$

There are one $2n$-cycle, one n-cycle and n claws in F_n by the definition of the graph F_n. For $0 \leq i \leq n - 1$, let

$$\begin{aligned} S_i &= F_n[\{a_i, b_i, c_i, c_{i+n}\}], \\ T_i &= E(S_i) = \{a_i b_i, a_i c_i, a_i c_{n+i}\}, \\ L_i &= \{b_i b_{i+1}, c_i c_{i+1}, c_{n+i} c_{n+i+1}\}, \\ E_i &= T_i \cup L_i. \end{aligned}$$

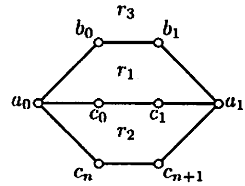

Fig. 3.5 X_0

In this paper, the subscripts of the vertices in $2n$-cycle are read modulo $2n$ and all the other subscripts are read modulo n unless specified otherwise. Now, we divide the edge set of F_n into n mutually disjoint subsets $E_0, E_1, \ldots, E_{n-1}$, i.e., $E = \bigcup_{i=0}^{n-1} E_i$ and $E_i \cap E_j = \emptyset$ $(0 \le i \ne j \le n-1)$.

Lemma 4.1. For $n \ge 3$, let D be a drawing of F_n. If all T_i $(0 \le i \le n-1)$ are clean in D, then $\nu(D) \ge n$.

Proof. For $0 \le i \le n-1$, we define function $f_D(i)$ counting the number of crossings in the drawing D as $f_D(i) = \nu_D(E_i) + \nu_D(E_i, E - E_i)/2$. Then, by Lemma 2.1, we have $\nu(D) = \sum_{i=0}^{k} f_D(i)$.

Suppose that there exists an i $(0 \le i \le n-1)$ satisfying $f_D(i) < 1$, say $f_D(0) < 1$. Let $X_0 = T_0 \cup T_1 \cup L_0$. Since $T_0 \cup T_1$ is clean and $f_D(0) < 1$, X_0 is uniquely drawn as shown in Fig. 3.5. The plane is divided into 3 regions r_1, r_2 and r_3.

If the vertex a_2 lies in r_1, then since $T_0 \cup T_1$ is clean, the paths $a_2c_{n+2}c_{n+1}$ and $a_2c_2c_3 \ldots c_{n-1}c_n$ would both cross the edges of $L_0 \subset E_0$. A contradiction to $f_D(0) < 1$.

If the vertex a_2 lies in r_2, then since the edge $a_2b_2 \in T_2$ is clean, the vertex b_2 has to lie in f_2. Since $T_0 \cup T_1$ is clean, the edge b_2b_1 and the path $b_2b_3 \ldots b_{n-1}b_0$ would both cross L_0. A contradiction to $f_D(0) < 1$.

If the vertex a_2 lies in r_3, then since $T_0 \cup T_1$ is clean, the paths $a_2c_2c_1$ and $a_2c_{n+2}c_{n+3} \cdots c_{2n-1}c_0$ would both cross L_0. A contradiction to $f_D(0) < 1$.

Hence, $f_D(i) \ge 1$ for all $0 \le i \le n-1$. Then $\nu(D) = \sum_{i=0}^{k} f_D(i) \ge n$. □

Lemma 4.2. $cr(F_3) = 2$.

Proof. By Lemma 3.1, $cr(F_3) \le 2$. We need only prove that $cr(F_3) \ge 2$. By contradiction, suppose that there is an optimal drawing D of F_3 satisfying $\nu(D) \le 1$.

Since the subgraph $F_3[E_0 \cup E_1 \cup T_2]$ is a subdivision of $K_{3,3}$, $\nu(D) \ge cr(K_{3,3}) = 1$. From the hypothesis $\nu(D) \le 1$, $\nu(D) = 1$. Since $\nu(D) = 1 < 3$, by Lemma 4.1, there is a crossed T_i $(0 \le i \le 2)$ in D, say $T_1 = \{a_1b_1, a_1c_1, a_1c_4\}$. By symmetry, we need only consider two cases.

Case 1. Suppose that the edge a_1b_1 is crossed. Deleting the edges a_1b_1, b_0b_2 and c_0c_5 from D results in a new drawing D^* with $\nu(D^*) = \nu(D) - 1 = 0$. However, since the graph corresponding to D^* is a subdivision of $K_{3,3}$, we would have $\nu(D^*) \geq 1$, a contradiction to $\nu(D^*) = 0$.

Case 2. Suppose that the edge a_1c_1 is crossed. Deleting the edges a_1c_1, b_0b_2 and a_2c_5 from D results in a new drawing D^* with $\nu(D^*) = \nu(D)-1 = 0$. However, since the graph corresponding to D^* is a subdivision of $K_{3,3}$, we would have $\nu(D^*) \geq 1$, a contradiction to $\nu(D^*) = 0$. □

Lemma 4.3. $cr(F_4) = 3$, $cr(F_5) = 4$.

Proof. Let D be an optimal drawing of F_4, then by Lemma 3.1, $\nu(D) \leq 3$. By Lemma 4.1, there is a crossed T_i $(0 \leq i \leq 3)$ in D. Deleting all the edges of T_i from D results in a new drawing D^* with $\nu(D) \geq \nu(D^*) + 1$. Since the graph corresponding to D^* is a subdivision of F_3, we have $\nu(D) \geq cr(F_3) + 1$. By Lemma 4.2, $\nu(D) \geq 3$. Hence $cr(F_4) = 3$.

Similarly, we have $cr(F_5) = 4$. □

Lemma 4.4. Let $n \geq 3$. Let D_n be a good drawing of F_n with $\nu(D_n) \leq n-1$. Let $X = \{x_1, x_2, \ldots, x_{n-2}\}$ be a set containing $n-2$ crossings in D_n. Then, there is an injection that maps each $x_j \in X$ to a T_{i_j}, such that x_j involves an edge of T_{i_j}.

Proof. By induction on n.

(1) For $n = 3$, if $\nu(D_3) \leq 2$, then by Lemma 4.1, there exists a crossed T_i $(0 \leq i \leq 2)$ in D_3. Hence, the lemma holds for $n = 3$.

(2) Suppose that for $3 \leq n \leq k-1$, if $\nu(D_n) \leq n-1$, then there exist $n-2$ crossings involve $n-2$ different edge sets of claw in D_n. For $n = k$, if $\nu(D_k) \leq k-1$, by Lemma 4.1, there exists a crossed T_i $(0 \leq i \leq k-1)$. Deleting all the edges of T_i from D_k results in a new drawing D^* whose corresponding graph is a subdivision of F_{k-1} with $\nu(D^*) \leq \nu(D_k) - 1 \leq k-2$. From the hypothesis of induction, there exist $k-3$ crossings involve $k-3$ different edge sets of claw in D^*. Hence, there exist $k-2$ crossings involve $k-2$ different edge sets of claw in D_k.

From (1) and (2), we have, for $n \geq 3$, if $\nu(D_n) \leq n-1$, then there exist $n-2$ crossings involve $n-2$ different edge sets of claw in D_n. □

Lemma 4.5. For $n \geq 4$, let D_n be a good drawing of F_n. If $\nu_{D_n}(T_s, T_t) = 0$ for every $0 \leq s < t \leq n-1$, then $\nu(D_n) \geq n$.

Proof. For $0 \leq i \leq n-1$, let $H_i = T_i \cup L_i \cup T_{i+1} \cup L_{i+1} \cup T_{i+2}$. It is easy to verify that $F_n[H_i]$ is a subdivision of $K_{3,3}$. In the drawing of H_i in D_n, the crossing that involves a pair of edges of $T_i \cup L_i$ (or a pair of $T_{i+2} \cup L_{i+1}$) results in a crossing between two edges incident with one common vertex in the corresponding drawing of $K_{3,3}$, hence we have $\nu_{D_n}(H_i) - \nu_{D_n}(T_i \cup L_i) - \nu_{D_n}(T_{i+2} \cup L_{i+1}) \geq cr(K_{3,3})$. Since $\nu_{D_n}(T_i, T_j) = 0$, by Lemma 2.1, we have

$$\begin{aligned} cr(K_{3,3}) &\leq \nu_{D_n}(H_i) - \nu_{D_n}(T_i \cup L_i) - \nu_{D_n}(T_{i+2} \cup L_{i+1}) \\ &= \nu_{D_n}(T_i \cup L_i, T_{i+1}) + \nu_{D_n}(T_i \cup L_i, T_{i+2} \cup L_{i+1}) + \\ &\quad \nu_{D_n}(T_{i+1}, T_{i+2} \cup L_{i+1}) \end{aligned}$$

$$\begin{aligned}
&= \nu_{D_n}(T_i,T_{i+1}) + \nu_{D_n}(T_i,T_{i+2}) + \nu_{D_n}(T_{i+1},T_{i+2}) + \\
&\quad \nu_{D_n}(L_i,T_{i+1}) + \nu_{D_n}(L_i,T_{i+2}) + \nu_{D_n}(L_{i+1},T_i) + \\
&\quad \nu_{D_n}(L_{i+1},T_{i+1}) + \nu_{D_n}(L_i,L_{i+1}) \\
&= \nu_{D_n}(L_i,T_{i+1}) + \nu_{D_n}(L_i,T_{i+2}) + \nu_{D_n}(L_{i+1},T_i) + \\
&\quad \nu_{D_n}(L_{i+1},T_{i+1}) + \nu_{D_n}(L_i,L_{i+1}). \qquad (4.1)
\end{aligned}$$

Since $E(F_n) = \bigcup_{i=0}^{n-1}(T_i \cup L_i)$, $\nu_{D_n}(T_i) = 0$ and $\nu_{D_n}(T_i,T_j) = 0$, we have

$$\nu(D_n) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\nu_{D_n}(T_i,L_j) + \sum_{i=0}^{n-1}\nu_{D_n}(L_i) + \sum_{0\le i<j\le n-1}\nu_{D_n}(L_i,L_j). \qquad (4.2)$$

From (4.1), for $n \geq 4$, we have

$$\begin{aligned}
n &= n\, cr(K_{3,3}) \\
&\leq \sum_{i=0}^{n-1}[\nu_{D_n}(L_i,T_{i+1}) + \nu_{D_n}(L_i,T_{i+2}) + \nu_{D_n}(L_{i+1},T_i) + \\
&\qquad \nu_{D_n}(L_{i+1},T_{i+1}) + \nu_{D_n}(L_i,L_{i+1})] \\
&= \sum_{i=0}^{n-1}\nu_{D_n}(L_i,T_{i+1}) + \sum_{i=0}^{n-1}\nu_{D_n}(L_i,T_{i+2}) + \sum_{i=0}^{n-1}\nu_{D_n}(L_{i+1},T_i) + \\
&\quad \sum_{i=0}^{n-1}\nu_{D_n}(L_{i+1},T_{i+1}) + \sum_{i=0}^{n-1}\nu_{D_n}(L_i,L_{i+1}) \\
&= \sum_{i=0}^{n-1}\nu_{D_n}(L_i,T_{i+1}) + \sum_{i=0}^{n-1}\nu_{D_n}(L_i,T_{i+2}) + \sum_{i=0}^{n-1}\nu_{D_n}(L_i,T_{i-1}) + \\
&\quad \sum_{i=0}^{n-1}\nu_{D_n}(L_i,T_i) + \sum_{i=0}^{n-1}\nu_{D_n}(L_i,L_{i+1}) \\
&= \sum_{i=0}^{n-1}\sum_{j=i-1}^{i+2}\nu_{D_n}(L_i,T_j) + \sum_{i=0}^{n-1}\nu_{D_n}(L_i,L_{i+1}) \\
&\leq \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\nu_{D_n}(T_i,L_j) + \sum_{i=0}^{n-1}\nu_{D_n}(L_i) + \sum_{0\le i<j\le n-1}\nu_{D_n}(L_i,L_j).
\end{aligned}$$

From (4.2), $\nu(D_n) \geq n$. □

Lemma 4.6. For $n \geq 4$, let D_n be a good drawing of F_n. If $\nu(D_n) \geq n-1$, then there exist at least $n-3$ different crossed-pairs (T_s,T_t) $(0 \leq s < t \leq n-1)$ in D_n.

Proof. By induction on n.

(1) For $n = 4$, if $\nu(D_4) \leq 3$, then by Lemma 4.5, there exists at least one crossed-pair (T_i,T_j) $(0 \leq i < j \leq 3)$ in D_4. Hence, the lemma holds for $n = 4$.

(2) Suppose that for $4 \le n \le k-1$, if $\nu(D_n) \le n-1$, then there exist at least $n-3$ different crossed-pairs of edge sets of claw in D_n. For $n=k$, if $\nu(D_k) \le k-1$, then by Lemma 4.5, there exists a crossed-pair (T_i, T_j) $(0 \le i < j \le k-1)$ in D_k. Deleting all the edges of T_i from D_k results in a new drawing D^* whose corresponding graph is a subdivision of F_{k-1}. Then, $\nu(D^*) \le \nu(D_k)-1 \le k-2$. From the hypothesis of induction, there exist at least $k-4$ different crossed-pairs of edge sets of claw in D^*. Hence, there exist at least $k-3$ different crossed-pairs of edge sets of claw in D_k.

From (1) and (2), we have, for $n \ge 4$, if $\nu(D_n) \le n-1$, then there exist $n-3$ different crossed-pairs of edge sets of claw in D_n. □

Lemma 4.7. $cr(F_6) = 6$.

Proof. By Lemma 3.1, $cr(F_6) \le 6$. We need only to prove that $cr(F_6) \ge 6$. By contradiction, suppose that there is an optimal drawing D of F_6 such that $\nu(D) \le 5$. By Lemma 4.4, there are 4 crossings involve 4 different edge sets of claw in D. By Lemma 4.6, there are 3 different crossed-pairs of edge sets of claw in D. Thus, there exists a T_i $(0 \le i \le 5)$ involve at least 2 crossings. Deleting all the edges of T_i from D results in a new drawing of a subdivision of F_5 with at most $\nu(D)-2 \le 3$ crossings, contradicting $cr(F_5) = 4$. □

Lemma 4.8. $cr(F_n) = n$ for $n \ge 6$.

Proof. By induction on n.

(1) By Lemma 4.7, $cr(F_6) = 6$.

(2) Suppose that for $6 \le n \le k-1$, $cr(F_n) = n$. Consider the case $n = k$. By Lemma 3.1, $cr(F_k) \le k$. We need only to prove that $cr(F_k) \ge k$. By contradiction, suppose that there is an optimal drawing D of F_k such that $\nu(D) \le k-1$. By Lemma 4.2, there is a crossed T_i $(0 \le i \le k-1)$ in D. Deleting all the edges of T_i from D results in a new drawing of D^* whose corresponding graph is a subdivision of F_{k-1}. Then by the hypothesis of induction, $\nu(D) \ge \nu(D^*)+1 \ge k$. A contradiction to $\nu(D) \le k-1$.

From (1) and (2), $cr(F_n) \ge n$ for $n \ge 6$. □

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