Some designs related to group actions

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Abstract

Some designs using the action of the linear fractional groups $L_2(q)$, $q = 11, 13, 16, 17, 19, 23$ are constructed. We will show that $L_2(q)$ or its automorphism group acts as the full automorphism group of each of the constructed designs except in the case $q = 16$. For designs constructed from $L_2(16)$, we will show that $L_2(16)$, $L_2(16):2$, $L_2(16):4$ or S_{17} can arise as the full automorphism group of the design.

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1 Introduction

Considering the primitive actions of the Janko groups J_1 and J_2, Key and Moori in [4] constructed designs, codes and graphs that have the above groups acting as automorphism group. Observing the results of [4] the authors conjectured that in certain cases the automorphism group of the design obtained by a primitive representation of a simple group G will have

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the automorphism group $Aut(G)$ as its full automorphism group. But later in [5] they found counter examples to the above conjecture. We also found another counter example for the conjecture in this paper. However it is interesting to apply the above method to other groups and their primitive representations to construct designs and groups and verify the above conjecture. Our intention in this paper is to consider some linear fractional groups $L_2(q)$ for certain q. Primitive permutation representations of these groups are given in [3] and our computations use the GAP system described in [6].

2 Definitions

Let F_q denote the Galois field with $q = p^n$ elements, where p is a prime number and $n \in \mathbb{N}$. For a detailed definition of the linear fractional group $L_2(q)$ we may refer the reader to [2]. The group of all the invertible 2 by 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with entries in F_q is denoted by $GL_2(q)$. The projective linear transformation associated with $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad - bc \neq 0$, is defined as follows. Adjoin the symbol ∞ to F_q and define the following permutation $f_{a,b,c,d}$ of $F_q \cup \{\infty\}$:

$$f_{a,b,c,d} : F_q \cup \{\infty\} \longrightarrow F_q \cup \{\infty\}$$

$$f_{a,b,c,d}(x) = \begin{cases} \frac{ax+b}{cx+d} \text{ if } x \in F_q - \{-\frac{d}{c}\} \text{ and } c \neq 0, \\ \infty \text{ if } x = -\frac{d}{c} \text{ and } c \neq 0, \\ \frac{a}{c} \text{ if } x = \infty \text{ and } c \neq 0, \\ \infty \text{ if } x = \infty \text{ and } c = 0. \end{cases}$$

Then the set of all such mappings is denoted by $PGL_2(q)$, i.e.

$$PGL_2(q) = \{f_{a,b,c,d} \mid a, b, c, d \in F_q,\ ad - bc \neq 0\}.$$

The projective special linear group in dimension 2 is also called the linear fractional group and it is denoted by either $PSL_2(q)$ or $L_2(q)$ and consists of the following elements

$$L_2(q) = \{f_{a,b,c,d} \mid a, b, c, d \in F_q,\ ad - bc \in F_q^{\square}\}$$

where $F_q^\square$ denotes the set of non-zero squares in F_q.

For the structure of the maximal subgroups of the groups $L_2(q)$ that we are concerned with we refer the reader to [3]. The notation for designs is standard, and as in [1]. Let $D = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an incidence structure with point set $\mathcal{P}$ and block set $\mathcal{B}$ and incidence relation $\mathcal{I}$. For $p \in \mathcal{P}$ and $B \in \mathcal{B}$ we will write $p\mathcal{I}B$ if and only if $(p, B) \in \mathcal{I}$. Then D is called a $t-(v,k,\lambda)$ design if $|\mathcal{P}| = v$, and $|B| = k$ for each $B \in \mathcal{B}$, and every t points of $\mathcal{P}$ is incident with precisely λ blocks of $\mathcal{B}$. The design is called symmetric if the number of points v is equal to the number of blocks b. The number of blocks through a set of s points is denoted by λ_s and is independent of the set if $s \leq t$. It is easy to deduce that

$$\lambda_s = \lambda_t \binom{v-s}{t-s} / \binom{k-s}{t-s}$$

where $\lambda = \lambda_t$. Here we remark that D is also an $s-(v,k,\lambda_s)$ design as well.

A $t-(v,k,\lambda)$ design is called trivial if every subset of $\mathcal{P}$ with cardinality k is a block of $\mathcal{B}$, i.e. $b = \binom{v}{k}$. The dual of the incidence structure $D = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is $D^t = (\mathcal{B}, \mathcal{P}, \mathcal{I})$; hence if D is a $t-(v,k,\lambda)$ design, then D^t is a design with b points such that the size of every block is λ_1. The incidence matrix of a structure $D = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a $|\mathcal{P}| \times |\mathcal{B}|$ matrix A whose rows are labeled by points in $\mathcal{P}$ and whose columns are labeled by blocks in $\mathcal{B}$ and the entry $(p, B) \in \mathcal{P} \times \mathcal{B}$ is equal to 1 if and only if p is incident with B, otherwise it is zero. Therefore A is a matrix with entries 0 or 1 and the incidence matrix of D^t is A^t which is the transpose of A. Two structures $D = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ and $D' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ are called isomorphic if there is a one to one correspondence $\theta : \mathcal{P} \longrightarrow \mathcal{P}'$ with the following property:

$$p\mathcal{I}B \Longleftrightarrow \theta(p)\mathcal{I}'\theta(B), p \in \mathcal{P}, B \in \mathcal{B}.$$

In this case we will write $D \cong D'$. The structure D is called self-dual if $D \cong D^t$.

An isomorphism of D onto itself is called an automorphism of D. The set of all the automorphisms of D is a group and it is denoted by $Aut(D)$. If the incidence matrix of the structure D is A, then $Aut(D)$ consists of precisely the pairs (P, Q), where P is a permutation of the rows of A and

Q is a permutation of the columns of A such that $PAQ = A$. Note that P as a permutation of rows (columns) of A can be viewed as a permutation matrix.

3 Method

The method of construction is in accordance with Proposition 1 of [4]. But we can describe the general case as mentioned in page 155 of [2]. Let G act on a set Ω of size n. Let $B \subseteq \Omega$ with $|B| \geq 2$. Then $D = (\Omega, B^G, \in)$ is an incidence structure, where $B^G = \{B^g \mid g \in G\}$, where G_B denotes the global stabilizer of B under G. If the action of G on Ω is t-homogeneous and $|B| \geq t$, then $D = (\Omega, B^G, \in)$ is a $t-(v,k,\lambda)$ design with parameters, $v = n$,

$$\lambda = \lambda_t = b\binom{k}{t} / \binom{v}{t} = (|G|\binom{k}{t})/(|G_B|\binom{v}{t}).$$

Now in the special case that the action of G on Ω is transitive we have $t = 1$ and will obtain a 1-design D with parameters $1-(n,k,r)$, where $r = [G : G_B]\frac{k}{n}$. If for a point $\omega \in \Omega$ we have $|G_B| = |G_\omega|$, which happens if the action of G on Ω is primitive and $B \neq \{\omega\}$ is an orbit of G_ω on Ω, then the design D has parameters $1-(n,k,k)$.

The construction of the 1-designs continues as follows: Consider a group which acts primitively on a set Ω of size n. Take $\omega \in \Omega$ and let Δ be an orbit with $|\Delta| = k > 1$ of the stabilizer G_ω on Ω. Then Δ^G is the block set of a symmetric design with parameters $1-(n,k,k)$. If the action of G on Ω is 2-transitive, then G_ω, $\omega \in \Omega$, has only two orbits $\{\omega\}$ and $\Omega - \{\omega\}$ on Ω and the design obtained in this way is trivial. In our investigations we will not consider trivial designs.

Note that according to [4] the above construction gives a self-dual block design and G acts as an automorphism group of this design, i.e. $G \leq Aut(D)$ and as a permutation group on the set of blocks is primitive as well.

4 Primitive actions of the groups $L_2(q)$, $q = 11, 13, 16, 17, 19$ and 23

Before starting to explain the designs obtained from each group, we will give some information about the tables below. The maximal subgroups of $L_2(q)$ up to conjugacy are given in [3], and the shapes of these are given in the second column of the table. Degree denotes the index of the maximal subgroup in the group. # indicates the number of orbits of the stabilizer of a point in the action of the group on the set of right cosets of a maximal subgroup. The rest of the entries after # are devoted to the length of the orbits of a point stabilizer, where an entry of the form $m(n)$ indicates n orbits of length m. The number beneath each orbit length is the order of the automorphism group of the design. Note that if the entry under # is 2, then this means that the action of the group on the set of the cosets of the subgroup is 2-transitive and hence the design obtained is trivial and will not be considered. All the calculations have been carried out using GAP.

Tables 1-6 give information we need about the groups $L_2(q)$, $q = 11$, 13, 16, 17, 19 and 23. According to these tables we have the following results.

1. $L_2(11)$ is the full automorphism group of the design with parameters $1-(55,3,3)$ and $1-(55,6,6)$. But $Aut(L_2(11)) = L_2(11):2$ is the full automorphism group of the design with parameters $1-(55,12,12)$.

2. $L_2(13)$ is the full automorphism group of the designs with parameters $1-(91,12,12)$, $1-(91,6,6)$ and $1-(78,7,7)$, but $Aut(L_2(13)) = L_2(13):2$ is the full automorphism group of the designs with parameters $1-(91,4,4)$, $1-(91,6,6)$, $1-(91,3,3)$ and $1-(78,14,14)$.

3. $L_2(16)$ is the full automorphism group of the designs with parameters $1-(136,15,15)$, $1-(120,17,17)$. The group $L_2(16):2$ is the full automorphism group of the designs with parameters $1-(136,15,15)$, $1-(120,17,17)$, $1-(68,12,12)$ and $1-(68,15,15)$. The group $Aut(L_2(16)) = L_2(16):4$ is the full automorphism group of the designs with parameters $1-(120,17,17)$ and $1-(68,20,20)$. For the full automorphism group of the design with parameters $1-(136,30,30)$ we have the following proposition. But first we need a Lemma.

Lemma 1 *If G is a simple group and $|G| = |A_{17}|$, then $G \cong A_{17}$.*

Proof. According to the classification of finite simple groups, G is isomorphic to one of the groups: An alternating group A_n, $n \geq 5$, a sporadic group or a simple group of Lie type. If $G \cong A_n$, then it is clear that $n = 17$ and we are done. From the list of orders of the sporadic groups in [3] we see that G cannot be a sporadic group. Therefore we examine the possibility of G being isomorphic to a simple group of Lie type defined over a finite field of characteristic p. Since we have assumed $|G| = |A_{17}|$, where $\pi(G) = \{2, 3, 5, 7, 11, 13 \text{ or } 17\}$, where $\pi(G)$ denotes the set of primes dividing $|G|$, and therefore p may be one of the numbers 2, 3, 5, 7, 11, 13 or 17. From the list of orders of simple groups of Lie type given in [3] we observe that the order of G is divisible by numbers of the form $p^k \pm 1$, $k \in \mathbb{N}$. We consider the possible cases.

Case 1 $p = 3, 5, 7$ or 11. In these cases the smallest k for which $17 \mid p^k + 1$ is 8 and the smallest l for which $17 \mid p^l - 1$ is 16. Examinations of the above p's and considering the numbers $p^r + 1$, $r \leq 8$ and $p^r - 1$, $r \leq 16$, we obtain primes other than primes in $\pi(G)$ which is a contradiction.

Case 2 $p = 2$ or 13. In this case $11 \mid n^5 + 1$ and $11 \mid n^{10} - 1$ for both $n = 2$ and 13 and similarly we obtain a contradiction.

Case 3 $p = 17$. From [3] we see that the order of a simple group of Lie type is divisible by p^k for some $k \geq 1$, hence in this case we must have $k = 1$ and $G \cong PSL_2(17)$, a contradiction.

Therefore $G \cong A_{17}$ and the Lemma is proved. ■

Proposition 1 *Let Ω denote the set of right cosets of a subgroup of $G = L_2(16)$ isomorphic to D_{30}. Then for $\omega \in \Omega$, the point stablizer G_ω has an orbit Δ of length 30 and orbiting Δ under G we obtain a design with parameters $1 - (136, 30, 30)$ whose full automorphism group is isomorphic to S_{17}.*

Proof. It is clear that $|\Omega| = [L_2(16) : 30] = 136$. By Table 3 there is an orbit of length 30 under the action of G_ω, $\omega \in \Omega$, on Ω. Using a program in GAP we have obtained $|Aut(D)| = 355687428096000 = 2^{15}3^6 5^3 7^2 11.13.17$. Using GAP we found that $Aut\ D$ has only one non-trivial normal proper subgroup N whose order is $\frac{1}{2}|Aut(D)|$. But again using GAP we found out

N is a simple group. Since $|N| = |A_{17}|$ by the previous Lemma we deduce that $N \cong A_{17}$. But $\frac{Aut(D)}{N} \cong \mathbb{Z}_2$ and $Aut(D)$ does not contain elements of order 34, hence $Aut(D) = S_{17}$. ∎

Remark 1 *The design D constructed in the above proposition is another counter example to the conjecture made in* [4].

4. $L_2(17)$ is the full automorphism group of the designs with parameters $1-(153,8,8)$, $1-(153,16,16)$, $1-(136,9,9)$, $1-(102,3,3)$, $1-(102,6,6)$, $1-(102,8,8)$, $1-(102,12,12)$ and $1-(102,24,24)$. But $Aut(L_2(17)) = L_2(17):2$ is the full automorphism group of the designs with parameters $1-(153,16,16)$, $1-(153,4,4)$ and $1-(136,18,18)$.

5. $L_2(19)$ is the full automorphism group of the designs with parameters $1-(190,9,9)$, $1-(171,5,5)$, $1-(171,10,10)$, $1-(57,6,6)$, $1-(57,20,20)$ and $1-(57,30,30)$. But $Aut(L_2(19)) = L_2(19):2$ is the full automorphism group of the designs with parameters $1-(190,18,18)$ and $1-(171,20,20)$.

6. $L_2(23)$ is the full automorphism of the designs with parameters $1-(276,11,11)$, $1-(253,4,4)$, $1-(253,6,6)$, $1-(253,8,8)$, $1-(253,12,12)$ and $1-(253,24,24)$. But $Aut(L_2(23)) = L_2(23):2$ is the full automorphism group of the designs with parameters $1-(276,22,22)$ and $1-(253,24,24)$.

After examination of the automorphism groups of the designs obtained from the primitive action of the groups $L_2(q)$ for $q = 11$, 13, 16, 17, 19 and 23 we put forward the following conjecture.

Conjecture The full automorphism group of the designs obtained in the manner described so far, is either $L_2(q)$ or $Aut(L_2(q)) = L_2(q):2$, provided q is a prime number.

Full details of the results obtained can be found at the web site:

http://www.fos.ut.ac.ir/~darafsheh

Those who are interested may contact the authors for the source files of the program.

Table 1: Orbits of the point-stabilizer of $L_2(11)$

no.	*maximal subgroup*	*degree*	#	*length and* $\|Aut\|$		
1	D_{12}	55	9	3(2) 660	6(4) 660	12(2) 1320
2	$11:5$	12	2	11(1)		
3	A_5	11	2	10(1)		
4	A_5	11	2	10(1)		

Table 2: Orbits of the point-stabilizer of $L_2(13)$

no.	*maximal subgroup*	*degree*	#	*length and* $\|Aut\|$		
1	A_4	91	11	4(3) 2184	6(1) 2184	12(6) 1092
2	D_{12}	91	12	3(2) 2184	6(4) 1092	12(5) 1092
3	D_{14}	78	9	7(5) 1092	14(3) 2184	
4	A_5	11	2	13(1)		

Table 3: Orbits of the point-stabilizer of $L_2(16)$

no.	*maximal subgroup*	*degree*	#	*length*	*and*	$\lvert Aut \rvert$
1	D_{30}	136	9	15(7)		30(1)
				4080 *or*	8160	355687428096000
2	D_{34}	120	8	17(7)		
				4080,	8160 *or*	16320
3	A_5	68	5	12(1)	15(1)	20(2)
				8160	8160	16320
4	$2^4 : 15$	17	2		16(1)	

Table 4: Orbits of the point-stabilizer of $L_2(17)$

no.	*maximal subgroup*	*degree*	#		*length*	*and*	$\lvert Aut \rvert$	
1	D_{16}	153	15		4(2)	8(6)	16(6)	
					4896	2448	2448 *or*	4896
2	D_{14}	136	12		9(7)	18(4)		
					2448	4896		
3	S_4	102	8	3(1)	6(1)	8(1)	12(1)	24(3)
				2448	2448	2448	2448	2448
4	S_4	102	8	3(1)	16(1)	8(1)	12(1)	24(3)
				2448	2448	2448	2448	2448
5	$17 : 8$	18	2			17(1)		

Table 5: Orbits of the point-stabilizer of $L_2(19)$

no.	*maximal subgroup*	*degree*	#	*length*	*and*	$\|Aut\|$
1	D_{18}	190	16		9(9)	18(6)
					3420	6840
2	D_{20}	171	15	5(2)	10(8)	20(4)
				3420	3420	6840
3	A_5	57	4	6(1)	20(1)	30(1)
				3420	3420	3420
4	A_5	57	4	6(1)	20(1)	30(1)
				3420	3420	3420
5	$19:9$	20	2		19(1)	

Table 6: Orbits of the point-stabilizer of $L_2(23)$

no.	*maximal subgroup*	*degree*	#		*length*	*and*	$\|Aut\|$	
1	D_{22}	276	19		11(11)	22(7)		
					6072	12144		
2	D_{24}	253	16	4(1)	6(2)	8(1)	12(3)	24(8)
				6072	6072	6072	6072	6072
3	S_4	253	16	4(1)	6(2)	8(1)	12(3)	24(8)
				6072	6072	6072	6072	6072
4	S_4	253	18		6(2)	12(10)	24(5)	
					6072	6072	12144	
5	$23:11$	24	12			23(1)		

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