The spectral radius of tricyclic graphs with n vertices and k pendant edges

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Abstract

In this paper we determine unique graph with largest spectral radius among all tricyclic graphs with n vertices and k pendant edges.

1. Introduction

We consider only finite undirected graphs without loops or multiple edges in this paper. Let $G = (V, E)$ be a graph with n vertices, and $A(G)$ be a $(0,1)$–adjacency matrix of G. Since $A(G)$ is symmetric, its eigenvalues are real. Without loss of generality, we can write them as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and call them the eigenvalues of G. The characteristic polynomial of G is just $det(\lambda I - A(G))$, denoted by $\phi(G, \lambda)$. The largest eigenvalue $\rho = \lambda_1(G)$ is called the index of G. If G is connected, then $A(G)$ is irreducible and so it is well-known that $\lambda_1(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\lambda_1(G)$ by the Perron-Frobenius theory of nonnegative matrices. We shall refer to such an eigenvector as the Perron vector of G.

The investigation of the index of graphs is an important topic in the theory of graph spectra. The reference [6] is a wonderful survey which includes a large number of references on this topic. The recent developments on this topic also involve the problem concerning graphs with maximal index in a given class of graphs.

Let $\mathcal{H}(n, n+t)$ denote the set of all connected graphs having n vertices and $n+t$ edges ($t \geq -1$). The maximal index problem for $\mathcal{H}(n, n+t)$ is solved for certain values of t ([1, 2, 3, 5, 11, 12, 15, 16]).

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Now, let $\mathcal{H}(n, n+t, k)$ denote the set of all connected graphs having n vertices, $n+t$ edges and k pendant vertices. A pendant vertex of G is a vertex of degree 1. Obviously, $\mathcal{H}(n, n+t, k) \subseteq \mathcal{H}(n, n+t)$. The maximal index problem for this class has been solved by Wu et al. [17] for $t = -1$ and by Guo at al. [8, 13] for $0 \leq t \leq 1$. The solutions of this problem are presented in Figure 1. The graphs with maximal index in these classes are obtained from the graphs K_1, K_3 and $K_3 \cdot K_3$ (coalescence of graphs K_3 and K_3), respectively, by attaching k paths of almost equal length to a vertex of maximal degree in these graphs. Here k paths $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$ are said to have almost equal lengths if $l_1, l_2, \ldots, l_k$ satisfy $|l_i - l_j| \leq 1$ for $1 \leq i < j \leq k$.

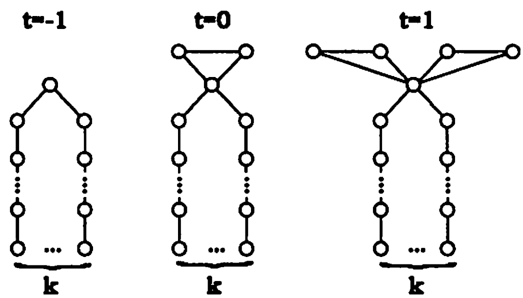

Fig.1

In this paper we give a solution of this problem for $t = 2$. The importance of the solution to a maximal index problem in these classes come from the fact that these graphs are the most irregular graphs in these classes. (Here the proposed measure of irregularity is $\delta = \rho - \overline{d}$, where ρ denotes index and $\overline{d}$ the average degree.)

2. Notation and lemmas

Denote by C_n and P_n the cycle and the path, respectively, each on n vertices. Let $G - x$ or $G - xy$ denote the graph that arises from G by deleting the vertex $x \in V(G)$ or the edge $xy \in E(G)$. Similarly, $G + xy$ is a graph that arises from G by adding an edge $xy \notin E(G)$, where $x, y \in V(G)$.

In order to complete the proof of our main result, we need the following lemmas. For $v \in V(G)$, $d(v)$ denotes the degree of vertex v and $N(v)$ denotes the set of all neighbors of the vertex v in G.

Lemma 1 ([17]) *Let G be a connected graph and let $\lambda_1(G)$ be the spectral radius of $A(G)$. Let u, v be two vertices of G and let $d(v)$ be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s \in N(v) \setminus N(u)$ $(1 \leq s \leq d(v))$ and $x = (x_1, x_2, \ldots, x_n)$ is the Perron vector of $A(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\lambda_1(G^*) > \lambda_1(G)$.*

Lemma 1 was first given by Wu, Xiao and Hong and it is a stronger version of a similar lemma in [15].

Let G be connected graph and let $uv \in E(G)$. The graph $G_{u,v}$ is obtained from G by subdividing the edge uv, i.e. adding a new vertex w and edges wu,wv in $G - uv$. Hoffman and Smith define an internal path of G as a walk $v_0v_1 \ldots v_s$ $(s \geq 1)$ such that the vertices $v_0, v_1, \ldots, v_s$ are distinct, $d(v_0) > 2$, $d(v_s) > 2$ and $d(v_i) = 2$, whenever $0 < i < s$. And s is called the length of the internal path. An internal path is closed if $v_0 = v_s$. They prove the following result.

Lemma 2 ([9]) *Let uv be an edge of the connected graph G on n vertices.*

1^0 *If uv does not belong to an internal path of G, and $G \neq C_n$, then $\lambda_1(G_{u,v}) > \lambda_1(G)$.*

2^0 *If uv belongs to an internal path of G, and $G \neq W_n$, where W_n is shown in Figure 2, then $\lambda_1(G_{u,v}) < \lambda_1(G)$.*

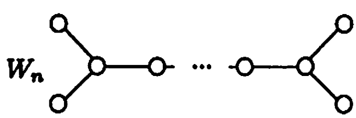

Fig.2

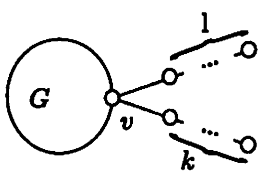

Fig.3

Lemma 3 ([7, 10]) *Let v be a vertex of the non-trivial connected graph G, and let $G(k, l)$ $(k \geq l \geq 1)$ denote the graph obtained from G by adding pendant paths of lengths k and l at v (Figure 3). Then $\lambda_1(G(k, l)) > \lambda_1(G(k + 1, l - 1))$.*

The following result is often used to calculate the characteristic polynomials of graphs.

Lemma 4 ([14]) *Let v be a vertex of G and $C(v)$ be the set of all cycles* *of G that contain v. Then*

$$\phi(G,\lambda) = \lambda\phi(G-v,\lambda) - \sum_{(u,v)\in E(G)} \phi(G-u-v,\lambda) - 2\sum_{Z\in C(v)} \phi(G-V(Z),\lambda),$$

where $G - V(Z)$ is the graph obtained by removing from G the vertices belonging to Z.

Lemma 5 ([4], p. 19) *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of a graph G and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ eigenvalues of an induced subgraph H. Then the inequalities*

$$\lambda_{n-m+i} \leq \mu_i \leq \lambda_i \quad (i = 1, \ldots, m)$$

hold.

Thus e.g. if $m = n-1$, we have $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$. Also $\lambda_1 > \mu_1$ if G is connected.

Lemma 6 ([4], p. 54) *If $G_1, G_2, \ldots, G_t$ are the components of a graph G, then we have*

$$\phi(G,\lambda) = \phi(G_1,\lambda)\cdot\phi(G_2,\lambda)\cdots\phi(G_t,\lambda).$$

Lemma 7 *If the graphs G and H have exactly one eigenvalue greater than some constant a and if $\phi(H,\lambda_1(G)) < 0$, then $\lambda_1(G) < \lambda_1(H)$.*

Denote by G_i^* $(i = 1, \ldots, 5)$ tricyclic graphs presented in Figure 4. Let $G_i^*(n,k)$ be the graph on n vertices obtained from G_i^* by attaching k paths of almost equal lengths to a vertex v of maximal degree $(i = 1, \ldots, 5)$. Then $G_i^*(n,k) \in \mathcal{H}(n, n+2, k)$.

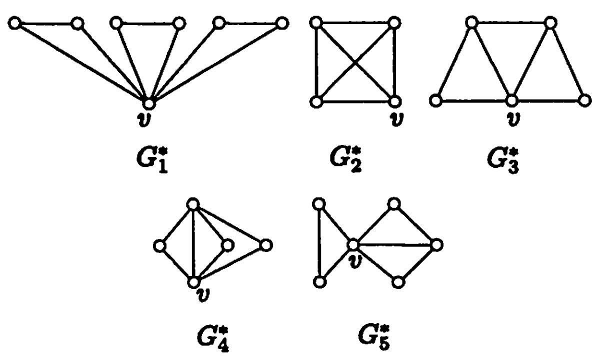

Fig.4

Lemma 8

$$
\begin{array}{lrcl}
(a) & n \geq k+7 & \Rightarrow & \lambda_1(G_1^*(n,k)) > \lambda_1(G_4^*(n,k)) \wedge \\
 & & & \lambda_1(G_1^*(n,k)) > \lambda_1(G_5^*(n,k)); \\
(b) & n \geq k+5 & \Rightarrow & \lambda_1(G_4^*(n,k)) > \lambda_1(G_2^*(n,k)) \wedge \\
 & & & \lambda_1(G_4^*(n,k)) > \lambda_1(G_3^*(n,k)); \\
(c) & n = k+6 \wedge k \leq 3 & \Rightarrow & \lambda_1(G_4^*(n,k)) > \lambda_1(G_5^*(n,k)); \\
 & n = k+6 \wedge k \geq 4 & \Rightarrow & \lambda_1(G_5^*(n,k)) > \lambda_1(G_4^*(n,k)).
\end{array}
$$

Proof. First, we prove

$$n \geq k+7 \Rightarrow \lambda_1(G_1^*(n,k)) > \lambda_1(G_5^*(n,k)).$$

The vertex of $G_1^*(n,k)$ that has degree $k+6$ is denoted by v. Also, the vertex of $G_5^*(n,k)$ that has degree $k+5$ is denoted by v. Denote by l the maximal number of vertices of a path attached to the vertex v of $G_5^*(n,k)$ and by m the minimal number of vertices of a path attached to the vertex v of $G_1^*(n,k)$. If so, then $m = l - 1$.

Let G be the graph analogous to $G_5^*(n,k)$ in which all paths attached to vertex v have l vertices. Also, let H be the graph analogous to $G_1^*(n,k)$ in which all paths attached to vertex v have m vertices.

Evidently, H is an induced subgraph of $G_1^*(n,k)$, whereas $G_5^*(n,k)$ is an induced subgraph of G. Therefore, by Lemma 5,

$$\lambda_1(H) \leq \lambda_1(G_1^*(n,k))$$

with equality if and only if $n = km + 7$. Also,

$$\lambda_1(G_5^*(n,k)) \leq \lambda_1(G)$$

with equality if and only if $n = kl + 6$.

Thus for the proof of the inequality $\lambda_1(G_5^*(n,k)) < \lambda_1(G_1^*(n,k))$ it is sufficient to show that $\lambda_1(G) < \lambda_1(H)$. We do this in the following.

Because of Lemmas 5 and 6, the graphs G and H have exactly one eigenvalue greater than 3. (This is because all components of the subgraphs $G - v$ and $H - v$ are paths, and the spectral radii of paths are less than 2. Therefore $\lambda_2(G) < 2$ and $\lambda_2(H) < 2$. By direct calculation we check that in the case $n = 7$, $k = 1$ the greatest eigenvalue of G is greater than 3. Also, in the case $n = 8$, $k = 1$ the greatest eigenvalue of H is greater than 3. Therefore the greatest eigenvalues of G and H are greater than 3 for all values of n and k ($n \geq k+7$).)

By applying Lemma 4 to the vertex v of G we obtain

$$
\begin{aligned}
\phi(G,\lambda) \;=\; & \lambda\phi(P_l,\lambda)^{k-1}[(\lambda^5 - 8\lambda^3 - 6\lambda^2 + 9\lambda + 8)\phi(P_l,\lambda) - \\
& k(\lambda^4 - 3\lambda^2 + 2)\phi(P_{l-1},\lambda)].
\end{aligned}
$$

In the analogous manner we obtain

$$\begin{aligned}\phi(H,\lambda) &= (\lambda^2-1)^2\phi(P_m,\lambda)^{k-1}[(\lambda^3-7\lambda-6)\phi(P_m,\lambda) - \\ &\quad k(\lambda^2-1)\phi(P_{m-1},\lambda)].\end{aligned}$$

Denote the greatest eigenvalue of G by r. Then $r > 3$ and from the above expression for $\phi(G,\lambda)$ it is seen that r satisfies the equation

$$(1) \quad (r^5 - 8r^3 - 6r^2 + 9r + 8)\phi(P_l, r) - k(r^4 - 3r^2 + 2)\phi(P_{l-1}, r) = 0\,.$$

The linear difference equation (1) of order 1 has solution

$$\phi(P_l,r) = \left(\frac{k(r^4-3r^2+2)}{r^5-8r^3-6r^2+9r+8}\right)^l \qquad (l = 0,1,2,\ldots)\,.$$

Now the inequality

$$\begin{aligned}\phi(H,r) &= k(r^2-1)^2\phi(P_m,r)^{k-1}\left(\frac{(k(r^4-3r^2+2))^{m-1}}{(r^5-8r^3-6r^2+9r+8)^m}\right) \\ &((r^3-7r-6)(r^4-3r^2+2)-(r^2-1)(r^5-8r^3-6r^2+9r+8)) < 0\end{aligned}$$

holds if and only if

$$Q(r) = ((r^3-7r-6)(r^4-3r^2+2)-(r^2-1)(r^5-8r^3-6r^2+9r+8)) < 0,$$

hence if and only if

$$Q(r) = -r^5 + 6r^3 + 4r^2 - 5r - 4 < 0$$

(the expressions $r^4 - 3r^2 + 2$ and $r^5 - 8r^3 - 6r^2 + 9r + 8$ are positive for $r > 3$).

Because the equation $Q(r) = 0$ has exactly two positive roots which belong to the intervals $(0,1)$, $(1,3)$ and $\lim_{r\to+\infty} Q(r) = -\infty$, we conclude that $Q(r) < 0$.

So, $\phi(H,r) < 0$ and by Lemma 7 we conclude that $\lambda_1(G) < \lambda_1(H)$. The proofs of the remaining inequalities are similar and we omit them. □

3. Main result

Theorem 1 *Let G be a graph in $\mathcal{H}(n, n+2, k)$, $k \geq 1$. Then $n \geq k+4$ and the following inequalities hold:*

(a) If $n \geq k+7$, then $\lambda_1(G) \leq \lambda_1(G_1^*(n,k))$ and the equality holds if and only if $G = G_1^*(n,k)$;

(*b*) *If* $n = k+6$ *and* $k \geq 4$, *then* $\lambda_1(G) \leq \lambda_1(G_5^*(n,k))$ *and the equality holds if and only if* $G = G_5^*(n,k)$;

(*c*) *If* $n = k+6$ *and* $k \leq 3$, *or* $n = k+5$, *then* $\lambda_1(G) \leq \lambda_1(G_4^*(n,k))$ *and the equality holds if and only if* $G = G_4^*(n,k)$;

(*d*) *If* $n = k+4$, *then* $\lambda_1(G) \leq \lambda_1(G_2^*(n,k))$ *and the equality holds if and only if* $G = G_2^*(n,k)$.

Proof. The smallest tricyclic graph without pendant vertices is the graph K_4 and the number n of vertices of any tricyclic graph with $k \geq 1$ pendant vertices is at least $k+4$.

Chose $G \in \mathcal{H}(n, n+2, k)$ such that the spectral radius of G is as large as possible. Denote the vertex set of G by $\{v_1, v_2, \ldots, v_n\}$ and the Perron vector of G by $x = (x_1, x_2, \ldots, x_n)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$.

We first prove that each two cycles C_p and C_q of G have at least one common vertex. Assume, on the contrary, that it is not true. Then there exists a path $v_1, v_2, \ldots, v_l$ which joins cycles C_p and C_q ($v_1 \in V(C_p)$, $v_l \in V(C_q)$, $l \geq 2$). Without loss of generality, we may assume that $x_1 \geq x_l$. Denote by v_{l+1} and v_{l+2} vertices of C_q which are adjacent to the vertex v_l. Then at least one of the vertices v_{l+1} and v_{l+2} is not adjacent to the vertex v_1, for example v_{l+1} (in the opposite case G is not tricyclic graph, a contradiction). Let

$$G^* = G - \{v_l v_{l+1}\} + \{v_1 v_{l+1}\}.$$

Then $G^* \in \mathcal{H}(n, n+2, k)$. By Lemma 1, we have $\lambda_1(G^*) > \lambda_1(G)$, a contradiction.

Now, we distinguish the following two cases:

Case 1. Each two cycles of G have exactly one common vertex.

Case 2. There exist two cycles of G which have more then one common vertex.

In the first case all cycles of G have exactly one common vertex, i.e. all three cycles C_p, C_q and C_r of G form a bundle which we denote by G_1^0 (Figure 5).

In the second case there exists a spanning subgraph H_0 of G which contains three vertex disjoint paths P_1,P_2 and P_3 and at most one of them is of length 1 (Figure 6). Denote by v_1 and v_2 the common vertices of the paths P_1,P_2 and P_3. The vertices v_1 and v_2 are of degree 3 in H_0, and other vertices in H_0 are of degree 2. Also, there exist either the fourth path P_4 which joins

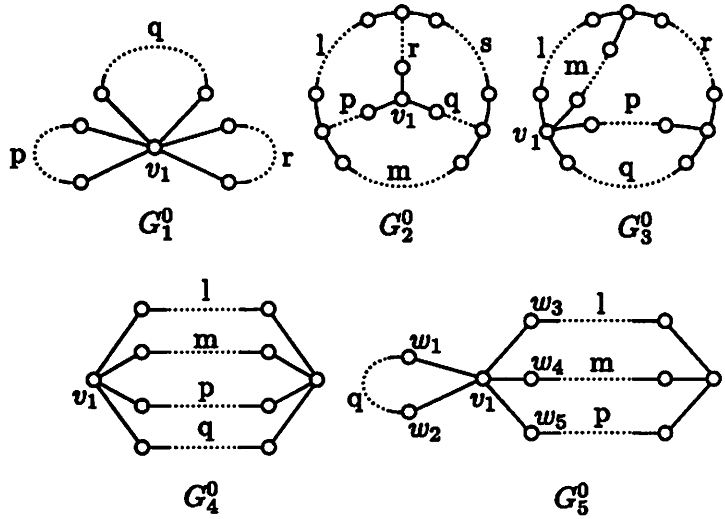

Fig.5

(2.1) two vertices of degree 2 of H_0 which do not belong to the same path P_i $(i = 1, 2, 3)$;

(2.2) one vertex of degree 3 and one vertex of degree 2 of H_0;

(2.3) two vertices of degree 3 of H_0;

or

(2.4) a cycle which has exactly one common vertex with H_0. (Their common vertex is of degree 3 in H_0.)

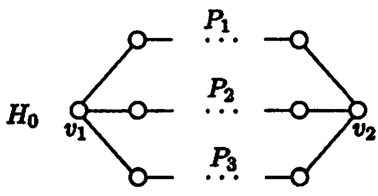

Fig.6

So, in the second case all cycles of G form the graph G_2^0 (subcase 2.1), G_3^0 (subcase 2.2), G_4^0 (subcase 2.3) or G_5^0 (subcase 2.4) (Figure 5).

We notice that the graphs G_i^0 are functions of the corresponding parameters. So, $G_1^0 = G_1^0(p, q, r)$, $G_2^0 = G_2^0(l, m, p, q, r, s)$, $G_3^0 = G_3^0(l, m, p, q, r)$, $G_4^0 = G_4^0(l, m, p, q)$ and $G_5^0 = G_5^0(l, m, p, q)$. These parameters are either lengths of the corresponding paths or lengths of the corresponding cycles of the graphs G_i^0 $(i = 1,\dots,5)$. Also, for the graphs G_i^* $(i = 1,\dots,5)$ from Figure 4 hold: $G_1^* = G_1^0(3,3,3)$ and $|G_1^0| \geq |G_1^*| = 7$, $G_2^* = G_2^0(1,1,1,1,1,1)$ and $|G_2^0| \geq |G_2^*| = 4$, $G_3^* = G_3^0(2,1,1,2,1)$ and $|G_3^0| \geq |G_3^*| = 5$, $G_4^* = G_4^0(2,2,2,1)$ and $|G_4^0| \geq |G_4^*| = 5$, $G_5^* = G_5^0(2,2,1,3)$ and $|G_5^0| \geq |G_5^*| = 6$.

Consequently, the graph G contains one of the graphs $G_1^0 - G_5^0$ as an induced subgraph. Denote by $\mathcal{H}_i(n,n+2,k) \subseteq \mathcal{H}(n,n+2,k)$ the set of all tricyclic graphs which contain the graph G_i^0 as an induced subgraph $(i = 1,\dots,5)$. Then $\mathcal{H}_i(n,n+2,k) \cap \mathcal{H}_j(n,n+2,k) = \emptyset$ $(i,j = 1,\dots,5;$ $i \neq j)$ and $G \in \cup_{i=1}^5 \mathcal{H}_i(n,n+2,k)$ for $n \geq k+7$, $G \in \cup_{i=2}^5 \mathcal{H}_i(n,n+2,k)$ for $n = k+6$, $G \in \cup_{i=2}^4 \mathcal{H}_i(n,n+2,k)$ for $n = k+5$ and $G \in \mathcal{H}_2(n,n+2,k)$ for $n = k+4$.

Suppose that G lies in $\mathcal{H}_5(n,n+2,k)$. Then $n \geq k+6$. We denote by v_1 the vertex of G_5^0 of degree 5 and prove that G consists of G_5^0 with a tree attached at v_1. Assume, on the contrary, that there exists a vertex v_i of G_5^0 such that $v_i \neq v_1$ and there exists a tree T attached to v_i. If $x_1 \geq x_i$, let $z_1,\dots,z_s$ $(s \geq 1)$ be the vertices of T which are adjacent to the vertex v_i and

$$G^* = G - \{v_iz_1,\dots,v_iz_s\} + \{v_1z_1,\dots,v_1z_s\}.$$

Now, let $x_1 < x_i$ and $N(v_1) = \{w_1,w_2,\dots,w_t\}$ $(t \geq 5)$ (Figure 5). If v_i is a vertex of C_q, let

$$G^* = G - \{v_1w_3,\dots,v_1w_t\} + \{v_iw_3,\dots,v_iw_t\}.$$

If v_i is a vertex of G_5^0 of degree 2, for example on the path of length l, let

$$G^* = G - \{v_1w_1,v_1w_2,v_1w_5,\dots,v_1w_t\} + \{v_iw_1,v_iw_2,v_iw_5\dots,v_iw_t\}.$$

If v_i is a vertex of G_5^0 of degree 3, let

$$G^* = G - \{v_1w_1,v_1w_2,v_1w_6,\dots,v_1w_t\} + \{v_iw_1,v_iw_2,v_iw_6\dots,v_iw_t\}.$$

In all cases $G^* \in \mathcal{H}_5(n,n+2,k)$. By Lemma 1, we have $\lambda_1(G^*) > \lambda_1(G)$, a contradiction. Hence G has a unique attached tree to the vertex v_1 of G_5^0.

Now we prove that each vertex v of T has degree $d(v) \leq 2$, i.e., G is a graph G_5^0 with k paths attached to v_1. Assume, on the contrary, that there exists one vertex v_i of T such that $d(v_i) > 2$. Denote $N(v_i) = \{z_1,\dots,z_s\}$, $N(v_1) = \{w_1,\dots,w_t\}$. Then $s \geq 3$ and $t \geq 6$. Assume that z_1 is the root v_1 of T or is joined to the root v_1 of T, w_1 and w_2 belong to C_q, and $w_6 = v_i$ or w_6 is joined to v_i. If $x_1 \geq x_i$, let

$$G^* = G - \{v_iz_3,\dots,v_iz_s\} + \{v_1z_3,\dots,v_1z_s\}.$$

If $x_1 < x_i$, let

$$\begin{aligned} G^* = G \quad & - \{v_1w_2,\dots,v_1w_5,v_1w_7,\dots,v_1w_t\} \\ & + \{v_iw_2,\dots,v_iw_5,v_iw_7,\dots,v_iw_t\}. \end{aligned}$$

Then in either case $G^* \in \mathcal{H}_5(n, n+2, k)$ and by Lemma 1, we have $\lambda_1(G^*) > \lambda_1(G)$, a contradiction.

Moreover, we claim that the k paths attached to v_1 have almost equal lengths. Assume that $P_{l_1}, \ldots, P_{l_k}$ are the k paths. We will prove that $|l_i - l_j| \leq 1$ for $1 \leq i < j \leq k$. Assume that there exist two paths P_{l_1} and P_{l_2} such that $l_1 - l_2 \geq 2$, say $P_{l_1} = v_1u_1u_2 \ldots u_{l_1}$, $P_{l_2} = v_1w_1w_2 \ldots w_{l_2}$. Let

$$G^* = G - \{u_{l_1-1}u_{l_1}\} + \{w_{l_2}u_{l_1}\}.$$

Then $G^* \in \mathcal{H}_5(n, n+2, k)$ and by Lemma 3, we have $\lambda_1(G^*) > \lambda_1(G)$, a contradiction.

By the definition of G_5^0, we have that $l, m, p \geq 1$ and at most one of them is 1. We claim that one of them is 1 and the other two are 2. Assume, on the contrary, that $l \geq 3$. Let $P_l = v_1v_2 \ldots v_{l+1}$ and $v_1u_1 \ldots u_m$ $(m \geq 1)$ be a path attached to v_1 of G_5^0. Obviously, $G \neq C_n$, $G \neq W_n$, $v_1v_2 \ldots v_{l+1}$ is an internal path, and $v_1u_1 \ldots u_m$ is not an internal path. Let

$$G^* = G - \{v_2v_3, v_3v_4\} + \{v_2v_4, u_mv_3\}.$$

Then $G^* \in \mathcal{H}_5(n, n+2, k)$. By Lemma 2, we have $\lambda_1(G^*) > \lambda_1(G)$, a contradiction. Hence $l \leq 2$. Similarly, we can verify that $m \leq 2$, $p \leq 2$, that one and only one of l, m, p is 1 and the cycle C_q has length 3. Thus $G_5^0 = G_5^*$ and $G = G_5^*(n, k)$.

Similarly to the previous proof, we can verify that if $G \in \mathcal{H}_1(n, n+2, k)$ then $n \geq k+7$ and $G = G_1^*(n, k)$, if $G \in \mathcal{H}_2(n, n+2, k)$ then $n \geq k+4$ and $G = G_2^*(n, k)$, if $G \in \mathcal{H}_3(n, n+2, k)$ then $n \geq k+5$ and $G = G_3^*(n, k)$, if $G \in \mathcal{H}_4(n, n+2, k)$ then $n \geq k+5$ and $G = G_4^*(n, k)$.

By Lemma 8 we conclude that $G = G_1^*(n, k)$ for $n \geq k+7$, $G = G_5^*(n, k)$ for $n = k+6$ and $k \geq 4$, $G = G_4^*(n, k)$ for $n = k+6$ and $k \leq 3$, or $n = k+5$ and $G = G_2^*(n, k)$ for $n = k+4$. This complete the proof. □

We conclude with the following conjecture. The graph presented in the Figure 7 is unique graph with maximal index in the class $\mathcal{H}(n, n+t, k)$ $(t \geq 0)$.

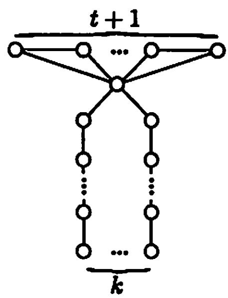

Fig.7

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