Note on Coloring the Square of an Outerplanar Graph

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Abstract

A new proof is given to the following result of ours. Let G be an outerplanar graph with maximum degree $\Delta \geq 3$. The chromatic number $\chi(G^2)$ of the square of G is at most $\Delta+2$, and $\chi(G^2) = \Delta+1$ if $\Delta \geq 7$.

1 Introduction

Only simple graphs are considered in this paper. For a plane graph $G = (V, E, F)$, let $\Delta(G)$ and $\delta(G)$ (Δ and δ for short) denote its maximum vertex degree and minimum vertex degree, respectively. A vertex (or a face) of degree k is called a *k-vertex* (or *k-face*). The degree of a face is defined to be the length of its boundary walk. Note that a cut edge is counted twice. A 3-face f with x, y, z as boundary vertices is expressed as $f = [xyz]$. The *distance*, denoted by $d_G(u, v)$, between two vertices u and v is defined to be the length of a shortest path connecting them in G. For a vertex $v \in V(G)$, let $N_i(v) = \{u \in V(G) \mid d_G(u, v) = i\}$ for $i = 1, 2$, and put $\beta(v) = |N_1(v) \cup N_2(v)|$.

A *k-coloring* of a graph G is a mapping ϕ from $V(G)$ to the set of colors $\{0, 1, \ldots, k-1\}$ such that $\phi(x) \neq \phi(y)$ for every edge xy of G. The *chromatic number* $\chi(G)$ is the smallest integer k such that G has a k-coloring. The *square* G^2 of a graph G is the graph defined by $V(G^2) = V(G)$

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and two vertices u and v are adjacent in G^2 if and only if $1 \le d_G(u,v) \le 2$. Obviously, a mapping ϕ is a k-coloring of G^2 if and only if $\phi(u)$ differs from $\phi(v)$ whenever u and v satisfy $1 \le d_G(u,v) \le 2$. Thus a k-coloring of G^2 is usually called a *square-k-coloring* of G.

It is obvious that $\chi(G^2) \ge \Delta + 1$ for any graph G. This lower bound is sharp. In fact, we have $\chi(T^2) = \Delta + 1$ for every tree T with $|V(T)| \ge 2$. On the other hand, it is easy to see that $\chi(G^2) \le \Delta^2 + 1$ for any graph G. This upper bound is also sharp. The 5-cycle and the Petersen's graph are two examples.

Wegner [8] first investigated the chromatic number of the square of a planar graph. He proved that $\chi(G^2) \le 8$ for every planar graph G with $\Delta = 3$ and conjectured that the upper bound could be reduced to 7. Recently, Thomassen [5] has established Wegner's conjecture. In [8], Wegner also proposed the following conjecture. The upper bounds are sharp if the conjecture is true.

Conjecture 1 *Let G be a planar graph of maximum degree Δ. Then*

$$\chi(G^2) \le \begin{cases} \Delta + 5 & if\ 4 \le \Delta \le 7; \\ \lfloor \frac{3}{2}\Delta \rfloor + 1 & if\ \Delta \ge 8. \end{cases}$$

This conjecture still remains open. van den Heuvel and McGuinness [1] proved $\chi(G^2) \le 2\Delta + 25$ for any planar graph G. The best known result so far is $\chi(G^2) \le \lceil 5\Delta/3 \rceil + 78$ [4]. Lih, Wang and Zhu [3] established the conjecture for a K_4-minor free graph. It is shown [7] that $\chi(G^2) \le \Delta + 16$ for a planar graph G of girth at least 5. In this paper, we study the chromatic number of the square of an outerplanar graph G. It is established that $\chi(G^2) \le \Delta(G) + 2$, and $\chi(G^2) = \Delta(G) + 1$ if $\Delta(G) \ge 7$. A different proof of this result will appear elsewhere [2].

2 A Structural Lemma

A planar graph is called *outerplanar* if it has a plane embedding such that all vertices lie on the boundary of some face. An *outerplane* graph G is a particular fixed embedding of an outerplanar graph. We choose one face of G that contains all vertices to be named the *outer* face of G, and we call the other faces *inner* faces. The edges belonging to the boundary of the outer face are called *outer* edges and other edges *inner* edges. An inner face f of G is called an *endface* if the boundary of f contains exactly one inner edge, i.e., the boundary of f contains exactly two vertices of degree 3 or more. When the vertex corresponding to the outer face is deleted, the dual graph of G is a forest of order at least 2. Thus there exist at least two leaves which determine two endfaces of G. We use $T(v)$ to denote the set of endfaces of degree 3 each of which is incident to the vertex v. For $n \geq 3$, let Q_n denote the outerplane graph obtained by adding n edges u_1u_2, u_2u_3, $\cdots$, u_nu_1 inside the $2n$-cycle $u_1v_1u_2v_2u_3v_3\ldots u_nv_nu_1$.

The following lemma first appeared in Wang and Zhang [6]. Its frequent use will be implied in the proof of Lemma 2.

Lemma 1 *Let G be a 2-connected outerplanar graph with at least 5 vertices. Then*

(1) *G contains at least two 2-vertices;*

(2) *every vertex is adjacent to at most two 2-vertices;*

(3) *for any two 2-vertices u and v, $N_1(u) \neq N_1(v)$.*

Lemma 2 *Let G be a 2-connected outerplane graph with $\Delta \geq 3$ and $G \ncong Q_n$ for all $n \geq 3$. Then there exists a 2-vertex v such that*

(i) *$\beta(v) \leq \Delta$ if $\Delta \geq 7$; and*

(ii) *$\beta(v) \leq \Delta + 1$ if $\Delta \leq 6$.*

Proof. First assume that $\Delta \geq 7$. To prove (i) by contradiction, we let G be a counterexample outerplane graph. Thus

$$\beta(v) \geq \Delta + 1 \geq 8 \text{ for every 2-vertex } v \text{ of } G. \qquad (*)$$

Claim 1. If a 2-vertex v is adjacent to two vertices x and y such that $d_G(x) \leq d_G(y)$, then $d_G(y) \geq 4$ if $xy \notin E(G)$, and $d_G(y) \geq 5$ if $xy \in E(G)$.

Suppose that $xy \notin E(G)$ and $d_G(y) \leq 3$. Then $\beta(v) = |N_1(v)| + |N_2(v)| \leq 2 + (d_G(x) - 1) + (d_G(y) - 1) \leq 6 < \Delta$, contradicting (*). If $xy \in E(G)$ and $d_G(y) \leq 4$, we have $\beta(v) = |N_1(v)| + |N_2(v)| \leq 2 + (4 - 2) + (4 - 2) = 6$, again contradicting (*).

Claim 1 implies that every 2-vertex of G is adjacent to at most one other 2-vertex. Furthermore, using (*) repeatedly and discussing case by case, we can show the following Claim 2.

Claim 2. Every 2-vertex v occurs in one of the following configurations (B1)-(B6):

(B1) a path $xvuy$ such that $d_G(u) = 2$ and $\min\{d_G(x), d_G(y)\} \geq \Delta - 1$;

(B2) a path xvy with $d_G(y) \geq d_G(x)$ such that the following hold:

If $xy \in E(G)$, then either $d_G(x) \geq 5$, or $d_G(x) = 4$ and $T(x) = \{[xvy]\}$;

If $xy \notin E(G)$, then either $d_G(x) \geq 4$, or $d_G(x) = 3$ and $T(x) = \emptyset$;

(B3) two vertex-disjoint endfaces $[xvy]$ and $[wuz]$ joined by an outer edge yw such that $d_G(u) = 2$, $d_G(y) = d_G(w) = 3$, and $d_G(x) = d_G(z) = \Delta$;

(B4) two edge-disjoint endfaces $[xvy]$ and $[yuz]$ with a common vertex y such that $d_G(u) = 2$, $d_G(y) = 4$, and $\min\{d_G(x), d_G(z)\} \geq \Delta - 1$;

(B5) a subgraph obtained from (B4) by removing the edge yz;

(B6) a subgraph obtained from (B4) by removing the vertex u, and furthermore by removing the edge xz (if $xz \in E(G)$), such that either $d_G(z) \geq 4$, or $d_G(z) = 3$ and $T(z) = \emptyset$.

For a 2-vertex v, we define the following operations (τ_1) to (τ_3).

(τ_1) If (B1) or (B2) holds, we remove v (and u for (B1)) then add the edge xy to G (provided $xy \notin E(G)$.)

(τ_2) If (B3) or (B4) holds, we remove u, v, y (and w for (B3)) then add the edge xz to G (provided $xz \notin E(G)$.)

(τ_3) If (B5) or (B6) holds, we remove v, y (and u for (B5)) then add the edge xz to G (provided $xz \notin E(G)$ in (B5).)

Let $v_1, v_2, \cdots, v_m$ be all the 2-vertices of G which are arranged in the boundary of the outer face in clockwise direction. We first carry out (τ_1) to (τ_3) for v_1, then for $v_2, v_3, \cdots, v_m$ in their order. Let H be the resultant graph. It is easy to see that H is a 2-connected outerplane graph. Let t be an arbitrary vertex of H. Then $t \in V(G)$ and $d_G(t) \geq 3$. We want to show that $d_H(t) \geq 3$, i.e., the operations (τ_1) to (τ_3) do not lead to new 2-vertices.

However, Lemma 1 asserts that $\delta(H) = 2$. A contradiction is produced.

Since t is adjacent to at most two 2-vertices in G by Lemma 1, it lies on the common boundaries of at most two endfaces of G. Moreover, the operations (τ_1)-(τ_3) and the structures of (B1)-(B6) imply that at most one 3-vertex or 4-vertex is removed in the meantime when some 2-vertex adjacent to t is deleted. Hence at most four neighbors of t in G are removed. This implies that $d_H(t) \geq d_G(t) - 4 \geq 3$ if $d_G(t) \geq 7$.

Assume that $5 \leq d_G(t) \leq 6$. We first notice that t is not incident to any endface $[uvt]$ such that $d_G(v) = 2$ and $d_G(u) = 3$ by $(*)$. If $d_H(t) \leq 2$, then t must belong to at least one configuration (B4) in G with $xz \in E(G)$ by (τ_1) to (τ_3). Thus some 2-vertex v in (B4) satisfies $\beta(v) \leq 7$, contradicting $(*)$. Therefore $d_H(t) \geq 3$.

Assume that $d_G(t) = 4$. Then $|T(t)| \leq 1$ (otherwise, t would be removed by (τ_2)), and t does not occur on any configuration (B4) with $xz \in E(G)$ by $(*)$. If $|T(t)| = 0$, then $d_H(t) = d_G(t) = 4$. If $|T(t)| = 1$, then $d_H(t) \geq 4 - 1 = 3$ by (τ_1) and (τ_3).

Assume that $d_G(t) = 3$. We claim that $|T(t)| = 0$, thus $d_H(t) = d_G(t) = 3$. Suppose on the contrary that t is incident to some endface $[tux]$ in G with $d_G(u) = 2$. Let y denote the neighbor of t in G that differs from u and x. Then it is easy to see that $d_G(x) = \Delta$ and $xy \notin E(G)$ by $(*)$. In this case, t should be removed from G by (τ_2) or (τ_3). This is a contradiction.

Now we prove (ii). If G contains a 2-vertex v adjacent to two other 2-vertices, then $\beta(v) \leq 2 + 2 = 4 \leq \Delta + 1$. If G contains a 2-vertex v lying on the boundary of some endface of degree 4, then $\beta(v) \le 2+(\Delta-1)+1-1 = \Delta+1$. So we assume that every endface of G is of degree 3. When $\Delta = 3$, there exists a 2-vertex v in any endface to make $\beta(v) \le 4 = \Delta+1$. Assume $\Delta = 4$. Let $[xyz]$ be an endface with $d_G(y) = 2$. If $\min\{d_G(x), d_G(z)\} \le 3$, then $\beta(y) \le 5$. Assume $d_G(x) = d_G(z) = 4$. If x lies on the boundary of some other endface $[uvx]$ such that $d_G(v) = 2$ and $uz \in E(G)$, then $\beta(y) \le 5$. Otherwise, we repeat the previous argument to show eventually G is isomorphic to Q_n, where $n = |G|/2$. This contradicts the definition of G. A similar argument works for z. Finally assume that $5 \le \Delta \le 6$ and suppose that the lemma is false. Then G possesses the following properties:

(a) G has no endface $[xyz]$ with $d_G(y) = 2$ such that either $d_G(x) = d_G(z) = 4$ or $\min\{d_G(x), d_G(z)\} = 3$;
(b) G has no two edge-disjoint endfaces $[xvy]$ and $[yuz]$ such that $d_G(v) = d_G(u) = 2$, $d_G(y) = 4$, and $xz \in E(G)$.

Let H denote the outerplane graph obtained from G by doing the following:
(1) removing all 2-vertices;
(2) if v is a 4-vertex incident to two endfaces $[vux]$ and $[vwy]$ with $d_G(u) = d_G(w) = 2$, then we remove v and afterward add the edge xy.

Analogously to the proof for (i), we can show $\delta(H) \ge 3$, which contradicts the fact that $\delta(H) = 2$ by Lemma 1. The proof of the lemma is complete. □

3 Coloring the Square

Let G be a connected graph of maximum degree Δ. It is straightforward to verify the following facts. If $\Delta = 0$, then $\chi(G^2) = 1$. If $\Delta = 1$, then $\chi(G^2) = 2$. If $\Delta = 2$ and G is a path, then $\chi(G^2) = 3 = \Delta+1$. If $\Delta = 2$ and G is a cycle, then $3 \le \chi(G^2) \le 5$. Moreover, $\chi(G^2) = 3 = \Delta+1$ if and only if $|V(G)| \equiv 0 \pmod 3$; $\chi(G^2) = 5 = \Delta+3$ if and only if $|V(G)| = 5$. Thus we always assume $\Delta \ge 3$ in the sequel.

The following lemma is an easy observation.

Lemma 3 *Let x be a cut vertex of the graph G. Let the vertex sets of the components of $G - x$ be $V_1, V_2, \ldots, V_m$. Let G_i be the subgraph induced by $V_i \cup \{x\}$ for $i = 1, 2, \cdots, m$. Then* $\chi(G^2) = \max\{d_G(x)+1, \chi(G_1^2), \chi(G_2^2), \ldots, \chi(G_m^2)\}$.

Lemma 4 *For any $n \ge 3$, $\chi(Q_n^2) = 5$ except $\chi(Q_3^2) = \chi(Q_4^2) = \chi(Q_7^2) = 6$.*

Proof. For every $n \ge 3$, it is easy to show that $5 \le \chi(Q_n^2) \le 6$. Since Q_3^2 is isomorphic to K_6 and Q_4^2 contains K_6 as a subgraph, we derive

$\chi(Q_3^2) = \chi(Q_4^2) = 6$. Note that, for any square-k-coloring of Q_7, every color class V_i, $0 \le i \le k-1$, contains at most three vertices. If $|V_i| = 3$, then V_i contains at least two 2-vertices. Since Q_7 has seven 2-vertices, there are at most three i's such that $|V_i| = 3$. This implies that the number of colors that are assigned to at most two vertices is at least 3. Thus $k \ge 5$, i.e., $\chi(Q_7^2) = 6$.

Now assume $n \ge 5$ and $n \ne 7$. It suffices to construct a square-5-coloring of Q_n in every possible case.

If $n \equiv 0 \pmod 5$, we color the sequence of vertices $u_1, v_1, u_2, v_2, \cdots, u_n, v_n$ with repeated uses of the color sequence 0, 1, 2, 3, 4.

If $n \equiv 1 \pmod 5$, we first color u_1 and u_4 with 0, u_2 and u_5 with 1, u_3 and u_6 with 2, v_1, v_3, v_5 with 3, and v_2, v_4, v_6 with 4. Then we color the sequence of vertices $u_7, v_7, u_8, v_8, \cdots, u_n, v_n$ with repeated uses of the color sequence 0, 3, 1, 2, 4.

If $n \equiv 2 \pmod 5$ and $n \ge 12$, we first color u_1, u_4, u_7, u_{10} with 0, u_2, u_5, u_8, u_{11} with 1, u_3, u_6, u_9, u_{12} with 2, $v_1, v_3, v_5, v_7, v_9, v_{11}$ with 3, and $v_2, v_4, v_6, v_8, v_{10}, v_{12}$ with 4. Then we color the sequence of vertices u_{13}, v_{13}, u_{14}, v_{14}, $\cdots, u_n$, v_n with repeated uses of the color sequence 0, 3, 1, 2, 4.

If $n \equiv 3 \pmod 5$, we first color v_1, v_3, v_5, v_7 with 0, u_1, u_4, v_6 with 1, u_2, v_4, u_7 with 2, v_2, u_5, u_8 with 3, and u_3, u_6, v_8 with 4. Then we color the sequence of vertices u_9, v_9, u_{10}, v_{10}, $\cdots, u_n$, v_n with repeated uses of the color sequence 1, 2, 0, 3, 4.

If $n \equiv 4 \pmod 5$, we first color u_1, u_4, u_7 with 0, u_2, v_4, v_6, v_8 with 1, v_2, u_5, v_7, v_9 with 2, v_1, v_3, v_5, u_8 with 3, and u_3, u_6, u_9 with 4. Then color the sequence of vertices u_{10}, v_{10}, u_{11}, v_{11}, $\cdots, u_n$, v_n with repeated uses of the color sequence 0, 3, 1, 4, 2. □

Theorem 5 *If G is an outerplanar graph with $\Delta \ge 3$, then $\chi(G^2) \le \Delta + 2$.*

Proof. We proceed by induction on the order $|V(G)|$. We may suppose the connectedness of G. If $|V(G)| \le 4$, the theorem holds trivially. Let G be an outerplanar graph with $\Delta \ge 3$ and $|V(G)| \ge 5$. Suppose that u is a cut vertex of G, i.e., $G = G_1 \cup G_2 \cup \cdots \cup G_m$ such that $V(G_i) \cap V(G_j) = \{u\}$ for all $i \ne j$. If $\Delta(G_i) \ge 3$, then, by the induction hypothesis, $\chi(G_i^2) \le \Delta(G_i) + 2 \le \Delta + 2$. If $\Delta(G_i) \le 2$, then $\chi(G_i^2) \le 5 \le \Delta + 2$ as remarked in the beginning of this section. Thus $\chi(G^2) \le \Delta + 2$ by Lemma 3.

Now suppose that G is 2-connected. If G is isomorphic to Q_n for some $n \ge 3$, then $\chi(G^2) \le 6 = \Delta + 2$ by Lemma 4. Otherwise, there is a 2-vertex $v \in V(G)$ such that $\beta(v) \le \Delta + 1$ by Lemma 2. Let x and y be the neighbors of v. If $xy \notin E(G)$, let $H = G - v + xy$; otherwise let $H = G - v$. By the induction hypothesis, H has a square-$(\Delta + 2)$-coloring. We can extend this coloring to the graph G since the vertex v has at most $\Delta + 1$ forbidden colors whereas the number of colors used is $\Delta + 2$. □

Using Lemma 2, and similarly to the proof of Theorem 5, we have the following.

Theorem 6 *If G is an outerplanar graph with $\Delta \geq 7$, then $\chi(G^2) = \Delta + 1$.*

For an outerplanar graph G with $\Delta = 3$ and containing a 5-cycle C_5, we have $\chi(G^2) \geq \chi(C_5^2) = 5$. Thus $\chi(G^2) = 5 = \Delta + 2$ by Theorem 5. Lemma 4 asserts that there exist outerplanar graphs G with $\Delta = 4$ and $\chi(G^2) = 6 = \Delta + 2$. In contrast, we would like to pose the following.

Conjecture 2 *Every outerplanar graph G with $5 \leq \Delta \leq 6$ has $\chi(G^2) = \Delta + 1$.*

Acknowledgment. This work was done while the first author was visiting the Institute of Mathematics, Academia Sinica, Taipei. The financial support provided by the Institute is greatly appreciated.

The research is also partially supported by NSFC (Grant No. 10471131) and NSFZJ (Grant No. M103094).

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