BOUND FOR 2-EXPONENTS OF PRIMITIVE EXTREMAL TOURNAMENTS

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ABSTRACT. We consider a 2-coloring of arcs on the primitive extremal tournament with the largest exponent on n vertices. This 2-colored digraph is a 2-primitive tournament. Then we think of the 2-exponent of a 2-primitive tournament. In this paper we give an upper bound for the 2-exponent of the primitive extremal tournament.

1. INTRODUCTION

We use the notation and terminology for digraphs as in [1]. In this paper we let $D = (V, E)$ be a digraph and $\mathcal{D}$ be a 2-coloring of $D = (V, E)$. A nonnegative square matrix A is *primitive* provided there is a nonnegative integer k such that A^k is entrywise positive (some authors say *strictly positive*), denoted by $A^k \gg 0$. If A is primitive, the smallest integer k such that A^k has only positive entries is called the *exponent* of A, denoted by $\exp(A)$.

The following inequality was stated by Wielandt[10] and recently, Schneider[8] found his original manuscript with proof.

Proposition 1.1. [8, 10] *Let A be a primitive $(0,1)$-matrix of order $n \geq 2$. Then*

$$\exp(A) \leq (n-1)^2 + 1 \tag{1}$$

Equality holds in (1) if and only if $D(A)$ is isomorphic to Figure 1.

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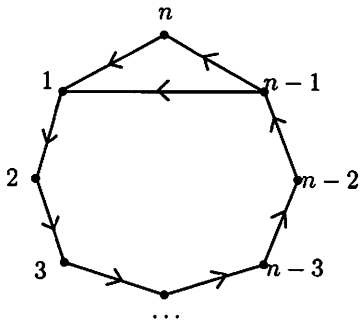

FIGURE 1. The extremal primitive digraph of Wielandt

As defined in [6], a positive discrete homogeneous 2D-system is described by the equation.

$$\mathbf{x}(h+1,k+1) = A\mathbf{x}(h,k+1) + B\mathbf{x}(h+1,k), h,k \in \mathbb{Z}, h+k \geq 0, \quad (2)$$

where A and B are n by n nonnegative matrices and the initial conditions $\mathbf{x}(h,-h)$ ($h \in \mathbb{Z}$ integers) are nonnegative n by 1 vectors. System (2) is called the 2D-system associated with the nonnegative matrix pair (A,B). Positive discrete homogeneous 2D-dynamical systems are used to model diffusion processes discretely(see [5]). A component of the vector $\mathbf{x}(h,k)$ typically represents a quantity such as pressure, concentration or density at a particular *site* along a stream. We can view this stream as flowing left-to-right along the line $y = -x$. The points $(h,-h)$ ($h \in \mathbb{Z}$) correspond to the discrete sites $h(h \in \mathbb{Z})$ along the stream. The vector $\mathbf{x}(h,k)$ represents the conditions at site h after $h+k$ time-steps. Thus, $\mathbf{x}(h,-h)$ describes the initial conditions at site h, the vector $\mathbf{x}(h,-h+1)$ describes the conditions at site h after 1 time-step, etc. Note that by setting $t = h+k+1$, the equation (2) indicates that conditions at site $h+1$ after $t+1$ time-steps are determined in a linear, time and location autonomous fashion from the conditions at site $h+1$ after t time-steps and the conditions at site h after t time-steps. Thus, at each time-step the conditions of a site are determined by its previous conditions and the conditions of the site directly upstream from it [9].

Definition 1.2. [9] *For nonnegative integers h and k, the (h,k)-*Hurwitz product, *$(A,B)^{(h,k)}$, of A and B is the sum of all matrices that are a product of h A's and k B's.*

For example, $(A,B)^{(1,0)} = A$ and

$$(A,B)^{(2,2)} = A^2B^2 + ABAB + AB^2A + BA^2B + BABA + B^2A^2.$$

We say the pair (A, B) of nonnegative matrices is *2-primitive* provided there exist nonnegative integers h and k such that $h+k > 0$ and $(A,B)^{(h,k)} \gg 0$. The *2-exponent* of the primitive pair (A, B) of matrices is defined to be the minimum value of $h + k$ taken over all pairs (h, k) such that $(A,B)^{(h,k)} \gg 0$. We write $\exp(A, B)$ for the 2-exponent of the pair (A, B).

In [6] it is shown that the nonnegative matrix pair (A, B) has both A and B nonzero and is 2-primitive if and only if the solutions to (2) are eventually strictly positive. This shows that the definition of 2-primitivity of matrix pairs truly generalizes the concept of primitivity for nonnegative matrices.

A *two-colored digraph* is a digraph whose arcs are colored red or blue. We allow loops and multiple colored arcs from i to j. There is a natural correspondence between two-colored digraphs and nonnegative matrix pairs. With each two-colored digraph $\mathcal{D}$ we associate a pair (A, B) of (0,1)-matrices where the (i, j)-entry of A is 1 if and only if there is a red arc from i to j, and the (i, j)-entry of B is 1 if and only if there is a blue arc from i to j. For each pair (A, B) of nonnegative n by n matrices, we associate the 2-colored digraph, $\mathcal{D}(A, B)$, with vertices $1, 2, \ldots, n$, a red arc from i to j if $a_{ij} > 0$ and a blue arc from i to j if $b_{ij} > 0$. An *(h, k)-walk* from i to j in $\mathcal{D}$ is a walk from i to j consisting of h red arcs and k blue arcs. One can easily show that the (i, j)-entry of $(A,B)^{(h,k)}$ is strictly positive if and only if there is an (h, k)-walk in $\mathcal{D}(A, B)$ from i to j. Given a walk w in $\mathcal{D}$, we write $r(w)$ and $b(w)$ for the numbers of red and blue arcs that w has in it, and we call the column vector

$$\begin{bmatrix} r(w) \\ b(w) \end{bmatrix} = (r(w), b(w)) \tag{3}$$

the *composition* of w. The two-colored digraph $\mathcal{D}$ is *strongly connected* provided for each pair (i, j) of vertices there is a walk in $\mathcal{D}$ from i to j. We say the 2-colored digraph $\mathcal{D}$ is *2-primitive* provided the associated matrix pair (A, B) is 2-primitive and the *2-exponent* of $\mathcal{D}$ is defined to be the 2-exponent of (A, B). The matrix pair (A, B) is 2-primitive if and only if there exist nonnegative integers h and k with $h+k>0$ such that for each pair (i,j) of vertices there exists an (h,k)-walk in $\mathcal{D}(A,B)$ from i to j.

Definition 1.3. [9] *Let $\mathcal{D}$ be a 2-colored digraph and let $C=\{\gamma_1,\gamma_2,\dots,\gamma_c\}$ be the set of all cycles of $\mathcal{D}$. Set M be the 2 by c matrix whose ith column is the composition of γ_i, and $\langle M\rangle$ be the additive subgroup of $\mathbb{Z}^2$ generated by the columns of M. We call M the* cycle matrix *of $\mathcal{D}$.*

In Definition 1.3, $\langle M\rangle=\mathbb{Z}^2$ implies that M has at least two linearly independent columns. Moreover, since each closed walk of $\mathcal{D}$ can be decomposed into cycles of $\mathcal{D}$, the composition of each closed walk of $\mathcal{D}$ belongs to $\langle M\rangle$.

Let $n\le c$. The *content* of the n by c matrix M, denoted by content(M), is defined to be 0 if the rank of M is less than n and to be the greatest common divisor of the determinants of the n by n submatrices of M, otherwise.

A *tournament* of order n is a digraph which can be obtained from the complete graph K_n by assigning a direction to each of its edges. Let A be the adjacency matrix of a tournament. Then A is a $(0,1)$-matrix satisfying the equation

$$A+A^T=J-I$$

where J is the matrix of all ones and is called a *tournament matrix* [1]. In this paper, we consider the 2-primitivity and the bound for 2-exponents of 2-colored tournaments.

In Section 2, we introduce some known results on tournaments and 2-primitivity of 2-colored digraphs.

In Section 3, we consider a primitive tournament with the largest exponent $n+2$, and we find an upper bound for the 2-exponent of $\mathcal{D}$ and (h,k)-walks from i to j for all $i,j\in V(\mathcal{D})$ satisfying our upper bound for 2-exponent.

Many applications of 2-primitivity can be found in [5, 9].

2. Preliminaries

In [7], Moon and Pullman showed the following interesting facts about tournaments.

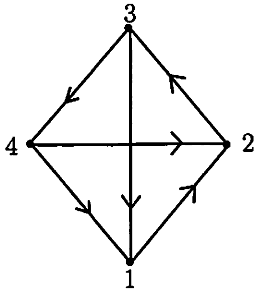

FIGURE 2. The irreducible tournament T_4

Proposition 2.1. [7] *Each vertex of an irreducible tournament T_n, when $n \geq 3$, is contained in at least one simple cycle of each length ℓ, $3 \leq \ell \leq n$.*

Proposition 2.2. [7] *A tournament T_n is primitive if and only if $n \geq 4$ and T_n is irreducible.*

Proposition 2.3. [7] *If T_n, where $n \geq 5$, is an irreducible tournament with exponent denoted by $\exp(T_n)$, then $3 \leq \exp(T_n) \leq n + 2$.*

Dulmage and Mendelsohn [3] have shown that there are gaps in the exponent set of primitive matrices, i.e., that not every integer between 1 and $(n-1)^2 + 1$ can be realized as the exponent of a primitive matrix in general. Moon and Pullman showed that there is essentially only one irreducible tournament T_4 (see Figure 2) and it has exponent 9 since

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \ldots, T^8 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, T^9 = J.$$

In fact, there are six structurally different irreducible tournaments T_5 (cf. Davis [2]) and they realize the exponents 4, 6 and 7. Dulmage and Mendelsohn showed that there are no gaps, however, in the exponent set of larger primitive tournaments.

Proposition 2.4. *If $n \geq 6$ and $3 \leq k \leq n + 2$ then there is a primitive tournament on n vertices with exponent k.*

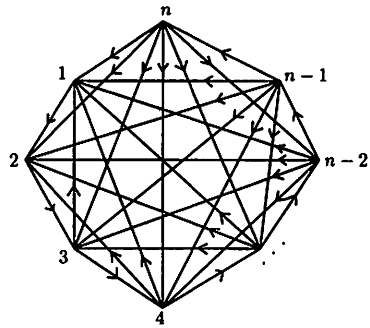

FIGURE 3. A primitive tournament D of order n with largest exponent $n+2$

3. EXPONENTS OF EXTREMAL PRIMITIVE TOURNAMENTS

In this section we consider a primitive tournament D with exponent $n+2$ which is the largest possible exponent by Proposition 2.3. Then we consider a 2-coloring of arcs on D, so the 2-colored $\mathcal{D}$ is 2-primitive, and we show $\exp(\mathcal{D}) \leq 4n+5$.

The following lemma shows that the digraph in Figure 3 is a primitive extremal tournament having the largest exponent $n+2$.

Lemma 3.1. [7, 11] *Let* $D = (V, E)$ *with* $V = \{1, 2, \ldots, n\}$, $(n \geq 3)$ *and* $E = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(i, j) \mid 3 \leq j+2 \leq i \leq n\}$ *be a digraph as is in* Figure 3. *Then* D *is a strongly connected tournament of order* n *and* $\exp(D) = n+2$ *when* $n \geq 5$.

An upper bound for 2-exponents of 2-primitive digraphs is given in Lemma 3.2.

Lemma 3.2. [9] *Let* $\mathcal{D}$ *be a strongly connected, 2-colored digraph with cycle matrix* M. *For each pair* (i, j) *of vertices let* p_{ij} *be a path from* i *to* j *and let* w *be a closed walk that goes through all of the vertices of* $\mathcal{D}$. *Suppose that* $\mathbf{z} = [\, z_1 \quad z_2 \quad \cdots \quad z_c \,]^T$ *is a nonnegative, integer vector such that each system*

$$M\mathbf{x} = \begin{bmatrix} r(p_{ij}) \\ b(p_{ij}) \end{bmatrix}$$

has an integer solution $\mathbf{x}_{ij}$ *with* $\mathbf{z} \geq \mathbf{x}_{ij}$. *Then* $\mathcal{D}$ *is 2-primitive and* $\exp(\mathcal{D}) \leq h + k$, *where* h *and* k *are defined by*

$$\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} r(w) \\ b(w) \end{bmatrix} + M\mathbf{z}.$$

Now we take an extremal primitive tournament D with the largest exponent and consider a 2-colored digraph $\mathcal{D}$ of it.

From now on, we only consider a 2-colored digraph $\mathcal{D}$ which is a 2-primitive digraph on the underlying digraph D in Figure 3 with the given cycle matrix M as all 3-cycles, 4-cycles and the n-cycle in $\mathcal{D}$ have only one blue arc. As we can see, each vertex in $\mathcal{D}$ is contained in a 3-cycle and 4-cycle and there are $(n-2)$ 3-cycles, $(n-3)$ 4-cycles, $(n-4)$ 5-cycles, ... , two $(n-1)$-cycles, and one n-cycle. When we give this 2-coloring to the underlying digraph, we still have remaining uncolored arcs in large cycles. So we can find a bound on the exponent in the following theorem.

Theorem 3.3. *Let* $\mathcal{D}$ *be a 2-primitive tournament on the underlying digraph* D *in Figure 3 with a cycle matrix* M *as all 3-cycles, 4-cycles and the* n*-cycle in* $\mathcal{D}$ *have only one blue arc. Then*

$$\exp(\mathcal{D}) \leq 4n + 5. \tag{4}$$

Moreover, there are $(3n+3, n+2)$*-walks from* i *to* j *for all* $i, j \in V$.

Proof Since each vertex in $\mathcal{D}$ is contained in a $(2,1)$-cycle and a $(3,1)$-cycle, each walk from i to j consists of a path p_{ij} and a collection of $(2,1)$-cycles and $(3,1)$-cycles. So we can use the 2×2 submatrix of cycle matrix M, that is, $M_1 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$. Note that for each pair of vertices i and j of $\mathcal{D}$ there exists a path p_{ij} from i to j with $r(p_{ij}) \leq n-1$ and $b(p_{ij}) \leq 1$. In fact, there are $(k,0)$ and $(\ell,1)$-paths for $0 \leq k \leq n-1$ and $0 \leq \ell \leq n-2$. Let

$$\begin{bmatrix} u \\ v \end{bmatrix} = M_1^{-1} \begin{bmatrix} r(p_{ij}) \\ b(p_{ij}) \end{bmatrix}$$

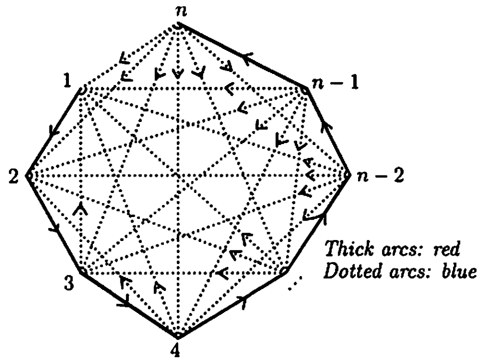

FIGURE 4. A 2-primitive tournament with $\exp(\mathcal{D}) = 4n + 5$

After we find integer solutions $\begin{bmatrix} u & v \end{bmatrix}^T$ for each path in case $b(p_{ij}) = 0$ or 1, we can easily see that $\begin{bmatrix} u \\ v \end{bmatrix} \leq \begin{bmatrix} n-1 \\ 3 \end{bmatrix}$. Hence, it follows that

$$\begin{aligned} \begin{bmatrix} h \\ k \end{bmatrix} &= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} n-1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 3n+3 \\ n+2 \end{bmatrix} \end{aligned}$$

Therefore, $\exp(\mathcal{D}) \leq h + k = 4n + 5$ by Lemma 3.2. Here we don't need to have a closed walk that goes through all of the vertices of $\mathcal{D}$ in Lemma 3.2 since we can find $(3n+3, n+2)$-walk of length $4n+5$ from i to j for all $i, j \in V(\mathcal{D})$ as in Remark 3.4. □

The following Remark 3.4 shows that for each pair of vertices i and j in $\mathcal{D}$ there exists a walk w_{ij} from i to j with composition $\begin{bmatrix} 3n+3 \\ n+2 \end{bmatrix}$.

Remark 3.4. For the equality in (4) we can find a $(3n+3, n+2)$-walk from i to j for all $i, j \in V$ in Figure 4. For the case of $i = j$, without loss of generality, we take $(i, j) = (1, 1)$. Our walk that starts at vertex $i = 1$ goes one time around $(2, 1)$-cycle and goes to vertex 2 along a red arc, then goes one time around $(2, 1)$-cycle and goes to vertex 3 along a red arc. Similarly repeat up to vertex $n - 1$ and go to n along a red arc. Now we go two times around $(3, 1)$-cycle and finally go to $j = 1$ along a blue arc. Then we have a composition

$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} n-1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3n+3 \\ n+2 \end{bmatrix}.$$

Therefore if $i = j$, since (h, k)-walk is a closed walk, it is just made of arcs on $(n-1)$ $(3,1)$-cycles and three $(2,1)$-cycles.

For the case of $i \neq j$, let p_{ij} be a path from i to j. Note that all (h, k)-walks from i to j in $\mathcal{D}$ are composed of the path p_{ij} and cycles. Our walk that starts at vertex i goes to vertex j along the path p_{ij}, then we find a closed walk from vertex j to vertex j by the similar way for the case $i = j$. This closed walk which is made of $(2,1)$-cycles and $(3,1)$-cycles has $(3n + 3 - r(p_{ij}))$-red arcs and $(n + 2 - b(p_{ij}))$-blue arcs. We now have a composition

$$\begin{bmatrix} r(p_{ij}) \\ b(p_{ij}) \end{bmatrix} + \begin{bmatrix} 3n+3-r(p_{ij}) \\ n+2-b(p_{ij}) \end{bmatrix} = \begin{bmatrix} 3n+3 \\ n+2 \end{bmatrix}.$$

As we have seen, we analyzed the case of 2-primitive tournament which is dense[1], and we now have a linear order $O(n^1)$ upper bound as expected.

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[1] Density of graph means the ratio of the number of edges and vertices of graph. [4]

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