On Variations of Graceful Labelings

M.A. Seoud and E.A. El Sakhawi

Faculty of Science, Ain Shams University

Abbassia, Cairo, Egypt

Abstract

In this paper, we show that some families of graphs are arbitrarily graceful or almost graceful.

Introduction

Recall that a (p,q) graph G is called graceful if there exists an injective function $f : V(G) \to \{0,1,2,...,q\}$ such that the induced function $f^* : E(G) \to \{1,2,...,q\}$ defined by $f^*(xy) = |f(x) - f(y)|$, for all edge $xy \in E(G)$ is an injection. The notion of graceful labeling was introduced by Rosa [6] in 1967.

A natural generalization of graceful graphs is the notion of *k-graceful graphs* introduced independently by Slater [8] in 1982 and by Maheo and Thuillier [4] in 1982. A graph G with q edges is k-graceful if there is labeling f from the vertices of G to {0, 1, 2, . . ., q+k−1} such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is {k, k+1, k+2, . . ., q+k−1}. Obviously, 1-graceful is graceful. Graphs that are k-graceful for all k are sometimes called arbitrarily graceful.

While proving a conjecture of Rosa [7], Moulton [5] introduced the concept of *almost graceful* labeling by permitting the vertex labels to come from

$\{0, 1, 2, ..., q-1, q+1\}$ while the edge labels are $\{1, 2, ..., q-1, q\}$ or $\{1, 2, ..., q-1, q+1\}$. Youssef [9] proved that $C_n, n \equiv 0 or 3 \pmod 4$ are pseudograceful, and hence they are almost graceful [2]

Cahit [1] has introduced a variation of both graceful and harmonious labelings. Let f be a function from the vertices of G to {0, 1}, and for each edge xy assign the label $|f(x)-f(y)|$. Call f a *cordial labeling* if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. Cahit [1]showed that an Eulerian graph is not cordial if its size is congruent to $2 \pmod 4$. This necessary condition is called the cordial parity condition. All the notions here can one find in Gallian's survey [2].

Results of Maheo and Thuillier [4] together with those of Slater [8] show that : C_n is k-graceful if and only if either $n \equiv 0$ or 1 (mod 4) with k even and $k \leq \frac{(n-1)}{2}$, or $n \equiv 3$ (mod 4) with k odd and $k \leq \frac{n^2-1}{2}$. Maheo and Thuiller also proved that the wheel W_{2k+1} is k-graceful, while Liang, Sun and Xu [3] proved that W_{2k} is k-graceful when $k \neq 3$ or $k \neq 4$. Here we prove that the following graphs are arbitrarily graceful; all paths P_n, $n \geq 2$; all ladders L_n, $n \geq 2$ and the symmetric product of the path P_n with the null graph $\overline{K_2}$. For $n \geq 3$, K_n is k-graceful if and only if $k = 1$ and $n \leq 4$. Also we prove that $K_5 \cup K_{1,n}$, $K_6 \cup K_{1,n}$ and the ladder L_n are almost graceful. Finally, we show that the Möbius ladder M_n, $n \geq 2$, is cordial if and only if $n \not\equiv 2$ (mod 4).

Theorem 1

All paths P_n , $n \geq 2$ are arbitrarily graceful.

Proof :

Let $V(P_n) = \{u_1, u_2, u_3, \ldots, u_n\}$, $|E(P_n)| = q$, and let us define

$f : V(P_n) \longrightarrow \{0, 1, 2, \ldots, q+k-1\}$ as follows

$$f(u_{2i+1}) = i \qquad 0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$f(u_{2i}) = (q+k) - i \qquad 1 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil$$

and $f^* : E(P_n) \rightarrow \{k, k+1, \ldots, q+k-1\}$

Observe that f is injective. Then f^* is injective as required and it satisfies that the path P_n , $n \geq 2$ is arbitrarily graceful.

The ladder L_n is defined as the graph $P_n \times P_2$

Theorem 2

All ladders L_n are arbitrarily graceful.

Proof :

Let $V(L_n) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\}$

$E(L_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$

and put $q = |E(L_n)|$. For $n \geq 2$, define the function $f : V(L_n) \rightarrow \{0, 1, \ldots, q+k-1\}$

as follows $f(u_{2i-1}) = q + k - 4(i-1)$ $\quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil$

$f(u_{2i}) = 2i$ $\quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

$f(v_{2i-1}) = 2i - 1$ $\quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil$

$f(v_{2i}) = q + k - 2(2i - 1)$ $\quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

This gives an arbitrarily graceful labeling for the ladder L_n.

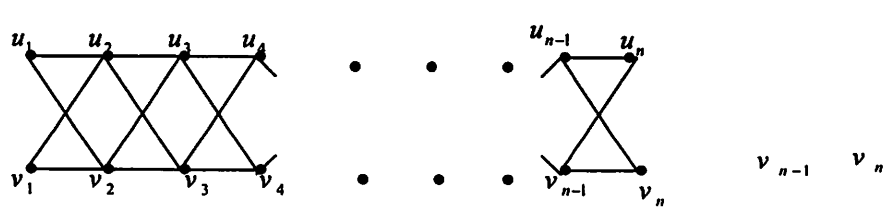
The symmetric product $G_1 \oplus G_2$ of G_1 and G_2 is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set { $(u_1, v_1)(u_2, v_2)$ such that $u_1 u_2 \in E(G_1)$ or $v_1 v_2 \in E(G_2)$ but not both }.

Theorem 3

The symmetric product of the path P_n, $n \ge 2$ with the null graph $\overline{K}_2$ is arbitrarily graceful.

Proof :

Let $P_n \oplus \overline{K}_2$ be described as in Figure



Define a labeling function

$$f : V(P_n \oplus \overline{K_2}) \rightarrow \{0, 1, \ldots, q+k-1\}$$

as follows

$$f(u_{2i+1}) = 4(n-i-1) + (k-1) \quad , \quad 0 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$f(u_{2i}) = 4(i-1) \quad , \quad 1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil$$

$$f(v_{2i+1}) = 4(n-i-1) - 2 + (k-1) \quad , \quad 0 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$f(v_{2i}) = 4(i-1) + 1 \quad , \quad 1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil$$

The path $u_1 u_2 u_3 \ldots u_n$ induces the edge labels $4 + (k-1)$, $8 + (k-1), \ldots, 4(n-1) + (k-1)$, the path $v_1 v_2 \ldots v_n$ induces the edge labels $1 + (k-1)$, $5 + (k-1), \ldots, 4(n-1) - 3 + (k-1)$, the path $u_1 v_2 u_3 v_4 \ldots u_n$ (n odd) or $u_1 v_2 u_3 v_4 \ldots v_n$ (n even) induces the edge labels $3 + (k-1)$, $7 + (k-1), \ldots, 4(n-1) - 1 + (k-1)$ and the path $v_1 u_2 v_3 u_4 \ldots v_n$ (n odd) or $v_1 u_2 v_3 u_4 \ldots u_n$ (n even) induces the edge labels $2 + (k-1)$, $6 + (k-1), \ldots, 4(n-1) - 2 + (k-1)$. So we obtain all the edge labels from k to $k+q-1$. Hence the graph is arbitrarily graceful.

Theorem 4

For $n \ge 3$, K_n is k-graceful $\Leftrightarrow$ $k = 1$ and $n \le 4$.

Proof :

$\Leftarrow$ If $k = 1$ and $n \le 4$, then clearly that K_n is graceful and so is 1-graceful.

$\Rightarrow$

We prove that if $k \ne 1$ or $n \ge 5$, then K_n is not k-graceful. We know that K_n is not graceful for all $n \ge 5$, so if K_n is k-graceful, $n \ge 5$, then $k \ge 2$. Now suppose that K_n is k-graceful, $k \ge 2, n \ge 5$ with a k-graceful labeling f, then we must have $f(x) = 0$ and $f(y) = q+k-1$ for some $xy \in E(K_n)$, where $q = | E(K_n) |$.

Now, we have to obtain the edge labeled q+k-2 This can be done by two ways only :

If $f(z)= 1$ for some vertex $z \in V(K_n)$, then we will obtain the edge labeled 1 which contradicts that K_n is k-graceful with $k \ge 2$.

If $f(z)= q+k-2$ for some vertex $z \in V(K_n)$, then we will also obtain the edge labeled 1 which is also contradiction. Hence K_n is not k-graceful if $k \ge 2$ or $n \ge 5$.

Now let $k \ge 2$ and $n \le 4$.

Case (1) : $n = 3$.

The edge label k+2 is induced by the vertex labels 0 and k+2. The edge label k is induced from : either : (i) The vertex labels 0 and k. Now the edge label 2 is induced by the vertex labels k and k+2, but this is not equal to k+1, since $k \geq 2$: a contradiction, or : (ii) the vertex labels 2 and k+2. But we have now the edge label 2, and we obtain the same contradiction as in (i).

Case (2) : n = 4.

The edge label k+5 is obtained from the vertex labels 0 and k+5. Now the edge label k+4 could appear from either : (i) the vertex labels 0 and k+4, so we obtain the edge label 1, while $k \geq 2$: a contradiction, or (ii) the vertex labels 1 and k+5, and we then have the same contradiction as in (i).

Theorem 5

(a) $K_5 \cup K_{1,n}$ is almost graceful for all n.

(b) $K_6 \cup K_{1,n}$ is almost graceful if $n \notin \{1, 3\}$.

Proof :

Let v be the center vertex of $K_{1,n}$.

(a) For $n \geq 1$ define

$$f : V(K_5 \cup K_{1,n}) \rightarrow \{0, 1, 2, \ldots, n+9, n+11\}$$

such that

$f(V(K_5)) = \{1, 2, 5, n+9, n+11\}$

$f(v) \quad = 3$

$$f(V_{K_{1,n}}) = \begin{cases} \{3,\ 8,\ 9,\ 10,\ .\ .\ .,\ n+6,\ n+8\} & n > 2 \\ \{3,\ 9\} & n = 1 \\ \{3,\ 8,\ 10\} & n = 2 \end{cases}$$

Then it is easily seen to be almost graceful labeling of $K_5 \cup K_{1,n}$.

(b) Define f : $V(K_6 \cup K_{1,n}) \rightarrow \{0, 1, 2, \ldots, n+14, n+16\}$

such that

$f(V(K_6)) = \{1, 2, 5, n+9, n+14, n+16\}$

$f(v) \quad = n + 13$

$$f(V_{K_{1,n}}) = \begin{cases} \{3,\ 7,\ 8,\ 10,\ 11,\ .\ .\ .,\ n+5,\ n+7,\ n+13\} & n \geq \\ \{3,\ 7,\ 8,\ n+7,\ n+13\} & n = \\ \{3,\ 7,\ n+13\} & n = 2 \end{cases}$$

Then it is easily seen to be almost graceful labeling of $K_6 \cup K_{1,n}$. Hence the result.

Theorem 6

L_n is almost graceful for all $n \geq 2$.

Proof:

Let $V(L_n) = \{ u_i : 1 \le i \le n \} \cup \{v_i : 1 \le i \le n\}$ and $E(L_n) = \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{u_i v_i : 1 \le i \le n\}$ and put $q = | E(L_n) | = 3n-2$.

For $n \ge 2$, define the function

$$f : V(L_n) \to \{0, 1, \ldots, q-1, q+1\}$$

as follows

$$f(u_{2i-1}) = q + 5 - 4i \quad , \quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$f(u_{2i}) = 2i \quad , \quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(v_{2i-1}) = 2i - 1 \quad , \quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$f(v_{2i}) = q + 3 - 4i \quad , \quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

Observe that f is injective since

$$\min \{ f(u_{2i-1}) : 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, \ f(v_{2i}) : 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \} \ge n+1$$

and $$\max \{ f(u_{2i}) : 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor, \ f(v_{2i-1}) : 1 \le i \le \left\lceil \frac{n}{2} \right\rceil \} \le n$$

This vertex labeling induces the edge labeling $\{1, 2, \ldots, q\}$.

The Möbius ladder M_n, $n \ge 2$ is the graph obtained from the ladder $P_n \times P_2$ by joining the opposite end vertices of the two copies of P_n.

Theorem 7

The Möbius ladder M_n, $n \ge 2$ is cordial if and only if $n \not\equiv 2 \pmod 4$.

Proof :

Let $V(P_n) = \{ u_1 , u_2 , \ldots, u_n \}$

and $V(P_2) = \{ v_1 , v_2 \}$. It is clear that the size of M_n is 3n.

Necessity: Suppose that M_n is cordial when $n \equiv 2 \pmod 4$. Then $M_n + K_1$ is also cordial but $M_n + K_1$ is an Eulerian graph of size $5n \equiv 2 \pmod 4$ which contradicts the cordial parity condition given by Cahit [1]. Hence M_n is not cordial if $n \equiv 2 \pmod 4$.

For sufficiency: Let the number of vertices labeled "0" and "1" be $N_v(0)$ and $N_v(1)$ respectively, and the number of edges lebeled "0" and "1" be $N_e(0)$ and $N_e(1)$ respectively. We consider the following three cases : Define the binary labeling

$$f : V(M_n) \longrightarrow \{0, 1\} \text{ as follows}$$

Case 1 : $n \equiv 0 \pmod 4$

$$f(u_i, v_1) = \begin{cases} 0 & 1 \le i \le \frac{n}{2} \\ 1 & \frac{n}{2}+1 \le i \le n \end{cases}$$

$$f(u_i, v_2) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

It is clear that $N_v(0) = n = N_v(1)$, and

$$N_e(0) = \frac{3n}{2} = N_e(1) .$$

Case 2 : $n \equiv 1 \pmod 4$

$$f(u_i, v_1) = \begin{cases} 0 & 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ 1 & \lfloor \frac{n}{2} \rfloor < i \le n \end{cases}$$

$$f(u_i, v_2) = \begin{cases} 1 & \text{if } i \text{ is odd} < n \\ 0 & \text{if } i \text{ is even} \\ 0 & i = n \end{cases}$$

Also we have $N_v(0) = n = N_v(1)$, and

$$N_e(0) = \frac{3n+1}{2}$$

$$N_e(1) = \frac{3n-1}{2}.$$

Case 3 : $n \equiv 3 \pmod 4$

$$f(u_i, v_1) = \begin{cases} 0 & 1 \le i \le \lceil \frac{n}{2} \rceil \\ 1 & \lceil \frac{n}{2} \rceil < i \le n \end{cases}$$

$$f(u_i, v_2) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

It is clear that $N_v(0) = n = N_v(1)$, and

$$N_e(0) = \frac{3n-1}{2} \quad \text{while}$$

$$N_e(1) = \frac{3n+1}{2}.$$

References

[1] I. Cahit, Cordial graphs : a weaker version of graceful and harmonious graphs, Ars Combin., 23 (1987) 201-207.

[2] J.A .Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, # DS6 (2005).

[3] Z.H.Liang ,D.Q.Sun and R.J.Xu, k-graceful labelings of the wheel graph W_{2k} , J.Hebei Normal College,1 (1993)33-44.

[4] M.Maheo, and M.Thuillier, On d-graceful graphs, Ars Combin., 13 (1982) 181-192.

[5] D.Moulton, Graceful labelings of triangular snakes, Ars Combin., 28 (1989) 3-13.

[6] A.Rosa , On certain valuations of the vertices of a graph , Theory of Graphs (Internet. Symposium , Rome, July 1966), Gordan and Breach, N.Y. and Dunod Paris (1967)

[7] A.Rosa , cyclic Steiner triple systems and labelings of triangular cacti, Scientia, 1(1988) 87-95.

[8] P.J.Slater, On k-graceful graphs, Proc. of the 13th S.E. Conf. on Combinatorics, Graph Theory and Computing, (1982) 53-57.

[9] M.Z.Youssef , Pseudograceful labeling of cycles , Ars Combin., 76(2005)241-247.