Notes on factor-criticality, extendibility and independence number *

Zan-Bo Zhang[12]†, Dingjun Lou[2], Xiaoyan Zhang[3]‡

[1]Department of Computer Engineering, Guangdong Industry Technical College, Guangzhou 510300, China

[2]Department of Computer Science, Sun Yat-sen University, Guangzhou 510275, China

[3]School of Mathematics and Computer Science & Institute of Mathematics, Nanjing Normal University, Nanjing 210097, China

Abstract

In this paper, we give a sufficient and necessary condition for a k-extendable graph to be $2k$-factor-critical when $k = \nu/4$, and prove some results on independence numbers in n-factor-critical graphs and $k\frac{1}{2}$-extendable graphs.
Key words: k-extendable, $k\frac{1}{2}$-extendable, n-factor-critical, independence number

1 Introduction and preliminary results

We consider undirected, simple, finite and connected graphs in this paper. All terminologies and notations undefined follow that of [2] and [6].

Let G be a graph, vertex set and edge set of G are denoted by $V(G)$ and $E(G)$. The number of vertices of G, the number of odd components

*Corresponding email address: eltonzhang2001@yahoo.com.cn(Zan-Bo Zhang).
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of G, the independence number of G, the edge independence number of G, the connectivity of G and the minimal degree of vertices of G are denoted by $\nu(G)$, $o(G)$, $\alpha(G)$, $\alpha'(G)$, $\kappa(G)$ and $\delta(G)$, respectively. Let G_1 and G_2 be two disjoint graphs. The *union* $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *join* $G_1 \vee G_2$ is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2. Let X and Y be two disjoint subsets of $V(G)$, the number of edges of G from X to Y is denoted by $e(X, Y)$.

A connected graph G is said to be *k-extendable* for $0 \leq k \leq (\nu - 2)/2$, if it contains a matching of size k and any matching in G of size k is contained in a perfect matching of G. The concept of k-extendable graphs was introduced by Plummer in [8]. In [9], Yu generalized the idea of k-extendibility to $k\frac{1}{2}$-extendibility for graph of odd order. A connected graph G is said to be *$k\frac{1}{2}$-extendable* if (1) for any vertex v of G there exists a matching of size k in $G - v$, and (2) for every vertex v of G, every matching of size k in $G - v$ is contained in a perfect matching of $G - v$.

A graph G is said to be *n-factor-critical*, for $0 \leq n \leq \nu - 2$, if $G - S$ has a perfect matching for any $S \subseteq V(G)$ with $|S| = n$. For $n = 1$, 2, that is *factor-critical* and *bicritical*. The concept of n-factor-critical graphs was introduced by Favaron [3] and Yu [9], independently.

In [8], Plummer showed a result on connectivity in k-extendable graphs.

Theorem 1.1. *If G is k-extendable, then $\kappa(G) \geq k + 1$.*

Favaron obtained a similar result for n-factor-critical graphs in [3].

Theorem 1.2. *Every n-factor-critical graph is n-connected.*

In [5], Lou and Yu improved the lower bound of connectivity for k-extendable graph with large k.

Theorem 1.3. *If G is a k-extendable graph on ν vertices with $k \geq \nu/4$, then either G is bipartite or $\kappa(G) \geq 2k$.*

It is easy to verify that a $2k$-factor-critical graph is always k-extendable, while the converse does not hold generally. However, taking $n = 2k$ in Theorem 1.2 and comparing it with Theorem 1.3 and Theorem 1.1, we find that the connectivity of a non-bipartitie k-extendable graph increases greatly, when $k \geq \nu(G)/4$, and becomes comparable to that of a $2k$-factor-critical graph. This fact has motivated the authors to study the relation between non-bipartite k-extendable graphs and $2k$-factor-critical graphs when $k \geq \nu(G)/4$, and find the following theorem.

Theorem 1.4. *(Zhang et al. [10]). If $k \geq (\nu(G) + 2)/4$, then a non-bipartite graph G is k-extendable if and only if it is $2k$-factor-critical.*

In this paper, we handle the unsettled case that $k = \nu(G)/4$. Precisely, we give a sufficient and necessary condition for a k-extendable graph with $k = \nu(G)/4$ to be $2k$-factor-critical.

In the rest of the paper we study the relationships between independence number, factor-criticality and extendibility. Some existing results are summarized in the following theorems.

Theorem 1.5. *(Maschlanka and Volkmann [7]). Let G be an k-extendable non-bipartite graph. Then $\alpha(G) \leq \nu/2 - k$. Moreover, the upper bound for $\alpha(G)$ is sharp for all k and ν.*

Theorem 1.6. *(Ananchuen and Caccetta [1]). Let G be a graph of even order ν and k a positive integer such that $\nu/4 \leq k \leq \nu/2 - 2$, $\nu/2 - k$ is even and $\delta(G) \geq \nu/2 + k - 1$. Then G is k-extendable if and only if $\alpha(G) \leq \nu/2 - k$.*

Some known results that will be used in our proofs are listed below.

Let G be any graph. Denote by $D(G)$ the set of vertices in G which are not covered by at least one maximum matching of G. Let $A(G)$ be the set of vertices in $V(G) - D(G)$ adjacent to at least one vertex in $D(G)$, and $C(G) = V(G) - A(G) - D(G)$.

Lemma 1.7. *(The Gallai-Edmonds Structure Theorem [6]). If G is a graph and $D(G)$, $A(G)$ and $C(G)$ are defined as above, then*
(a) the components of the subgraph induced by $D(G)$ are factor-critical,
(b) if M is any maximum matching of G, it contains a near-perfect matching of each component of $D(G)$, a perfect matching of each component of $C(G)$ and matches all vertices of $A(G)$ with vertices in distinct components of $D(G)$,
(c) $\alpha'(G) = (\nu(G) - o(D(G)) + |A(G)|)/2$, where $o(D(G))$ denotes the number of components of the subgraph induced by $D(G)$.

Lemma 1.8. *(Yu [9]). A graph G of odd order is $k\frac{1}{2}$-extendable if and only if $G \vee K_1$ is $(k+1)$-extendable.*

Lemma 1.9. *(Yu [9], Favaron [3]). A graph G is n-factor-critical if and only if $\nu(G) \equiv n \pmod 2$ and for any vertex set $S \subseteq V(G)$ with $|S| \geq n$, $o(G - S) \leq |S| - n$.*

Lemma 1.10. *(Zhang et al. [10]). If G is a k-extendable graph, then G is also m-extendable for all integers $0 \leq m \leq k$.*

2 Extendibility and factor-criticality

Theorem 2.1. *Let G be a non-bipartite k-extendable graph with $\nu(G) = 4k$, then*

(1) if $\delta(G) \geq 3k$, G is $2k$-factor-critical,
(2) if $\delta(G) = 2k$, G is not $2k$-factor-critical,
(3) if $2k+1 \leq \delta(G) \leq 3k-1$, then G is not $2k$-factor-critical if and only if there exists a partition of $V(G)$ into V_1 and V_2, where $|V_1| = |V_2| = 2k$. Each of $G[V_1]$ and $G[V_2]$ is composed of two factor-critical components of size no less than 3.

Proof. Let G be a non-bipartite k-extendable graph with $\nu(G) = 4k$. By Theorem 1.3, $\delta(G) \geq \kappa(G) \geq 2k$. We will, one by one, discuss the three cases above.

(1) $\delta(G) \geq 3k$. If G is not $2k$-factor-critical, then there exists a set $S \subseteq V(G)$ of size $2k$, such that $G - S$ does not have a perfect matching. Let $M_S = \{u_1v_1, u_2v_2, \ldots, u_rv_r\}$ be a maximum matching of $G[S]$ of size r. Clearly $r \leq k-1$. Hence there are at least two vertices w_1, $w_2 \in S$ that are not covered by M_S. If w_1u_i, $w_2v_i \in E(G)$ for any $1 \leq i \leq r$, then $(M_S \backslash \{u_iv_i\}) \cup \{w_1u_i, w_2v_i\}$ is a matching of $G[S]$ of size $r+1$, contradicting the maximality of M_S. So $|\{w_1u_i, w_2v_i\} \cap E(G)| \leq 1$. Similarly $|\{w_1v_i, w_2u_i\} \cap E(G)| \leq 1$. Therefore $e(\{w_1, w_2\}, \{u_i, v_i\}) \leq 2$ for $1 \leq i \leq r$. Then we have

$$6k \leq d(w_1) + d(w_2) \leq 2r + 2k + 2k = 4k + 2r \leq 6k - 2,$$

a contradiction.

(2) $\delta(G) = 2k$. Let v be a vertex of degree $2k$. Then v is an isolated vertex in $G - N(v)$, where $N(v)$ denotes the set of the neighbors of v in G. So $G - N(v)$ does not have a perfect matching and G is not $2k$-factor-critical.

(3) $2k+1 \leq \delta(G) \leq 3k-1$. If there exists a partition of $V(G)$ as stated, then $G[V_2] = G - V_1$ does not have a perfect matching and hence G is not $2k$-factor-critical.

Conversely, suppose that G is not $2k$-factor-critical. Then there exists a vertex set $S \subseteq V(G)$ of order $2k$, such that $G - S$ does not have a perfect matching. We choose S so that $\alpha'(G[S])$ has the maximum value. Clearly, $\alpha'(G[S]) \leq k-1$.

Let M_S be a maximum matching of $G[S]$, then there exist two vertices u_1 and u_2 in $G[S]$ that are not covered by M_S. By Lemma 1.10, M_S is contained in a perfect matching M of G. Let $u_iv_i \in M$, where $v_i \in V(G-S)$, $i = 1, 2$. Let $S' = (S \backslash \{u_2\}) \cup \{v_1\}$. Then $M_S \cup \{u_1v_1\}$ is a matching of $G[S']$ of size $\alpha'(G[S]) + 1$. By the choice of S, $G - S'$ has a perfect matching $M_{\bar{S'}}$ of size k. Then $M_{\bar{S'}}$ is contained in a perfect matching M' of G. Clearly, $M' \cap E(G[S'])$ is a perfect matching of $G[S']$ and $M' \cap E(G[S])$ is a matching of $G[S]$ of size $k-1$. Therefore, $|M_S| = \alpha'(G[S]) = k-1$. Furthermore, $M \cap E(G-S)$ is a matching of $G-S$ of size $k-1$, hence $\alpha'(G-S) = k-1$.

Apply Lemma 1.7 on $G[S]$ and $G-S$. Let $C_S = C(G[S])$, $A_S = A(G[S])$, $D_S = D(G[S])$, $C_{\bar{S}} = C(G-S)$, $A_{\bar{S}} = A(G-S)$ and $D_{\bar{S}} = D(G-S)$.

Firstly, we have D_S, $D_{\bar{S}} \neq \emptyset$. Let $v \in D_S$. Then v is missed by a maximum matching, say M_S, of $G[S]$. Since $\delta(G) \geq 2k+1$, v has at least one neighbor, say u, in $V(G-S)$. By the extendibility of G, $M_S \cup \{uv\}$ is contained in a perfect matching M_0 of G. Moreover $M_0 \cap E(G-S)$ is a maximum matching of $G-S$, which misses u. So $u \in D_{\bar{S}}$. Therefore, $e(D_S, A_{\bar{S}} \cup C_{\bar{S}}) = e(D_{\bar{S}}, A_S \cup C_S) = 0$.

Now we prove that $A_S \cup C_S = A_{\bar{S}} \cup C_{\bar{S}} = \emptyset$. By contradiction, suppose that at least one of the equalities does not hold, say $A_S \cup C_S \neq \emptyset$. If $A_{\bar{S}} \cup C_{\bar{S}} = \emptyset$, then D_S is a cut set of G of size less than $2k$, contradicting $\kappa(G) \geq 2k$. Hence we can assume that $A_{\bar{S}} \cup C_{\bar{S}} \neq \emptyset$. Then both $D_S \cup A_{\bar{S}}$ and $D_{\bar{S}} \cup A_S$ are cut sets of G. Thus we have $|D_S \cup A_{\bar{S}}| \geq 2k$ and $|D_{\bar{S}} \cup A_S| \geq 2k$. However $|D_S \cup A_{\bar{S}}| + |D_{\bar{S}} \cup A_S| = \nu(G) - |C_S| - |C_{\bar{S}}| \leq 4k$. So all equalities must hold, that is, $|D_S| + |A_{\bar{S}}| = |D_{\bar{S}}| + |A_S| = 2k$ and $C_S = C_{\bar{S}} = \emptyset$.

By our assumption, we must have $A_S, A_{\bar{S}} \neq \emptyset$. Suppose that there is an edge $e \in E(G)$ connecting two vertices in A_S. Take a maximum matching $M_{\bar{S}}$ of $G-S$, by the extendibility of G, $M_{\bar{S}} \cup \{e\}$ is contained in a perfect matching M_1 of G. But then $M_1 \cap E(G[S])$ is a maximum matching of $G[S]$ containing e, contradicting Lemma 1.7 (b). Hence A_S spans no edge of G. Then for any $w \in A_S$, $d(w) \leq |A_{\bar{S}}| + |D_S| = 2k$, contradicting $\delta(G) \geq 2k+1$. Therefore $A_S = \emptyset$ and similarly $A_{\bar{S}} = \emptyset$.

Now we have $D_S = S$ and $D_{\bar{S}} = V(G) \backslash S$. By Lemma 1.7 (a), each component of $G[S]$ and $G-S$ is factor-critical. By Lemma 1.7 (c), $o(D_S) = o(D_{\bar{S}}) = 2$, hence each of $G[S]$ and $G-S$ consists of two factor-critical components. Finally, since $\delta(G) \geq 2k+1$, all the components must have size at least 3. □

3 Factor-criticality and independence number

The lower bound in the theorem below has been proved in a remark in [4]. Note that since every $2k$-factor-critical graph is k-extendable, it is a straight consequence of Theorem 1.5 when n is even. The sharpness can be verified by the graph $G = K_{(\nu+n)/2} \vee ((\nu-n)/2)K_1$.

Theorem 3.1. *Let G be a n-factor-critical graph of order ν. Then $\alpha(G) \leq (\nu-n)/2$. The bound for $\alpha(G)$ is sharp.*

Again in a remark in [4], Favaron pointed out that the conditions in Theorem 1.6 yield $2k$-factor-criticality rather than k-extendibility. Here we prove a general version for all n-factor-critical graphs.

Theorem 3.2. *Let G be a graph on ν vertices, n a positive integer satisfying $\nu \equiv n \pmod 2$, $\delta(G) \geq (\nu+n)/2-1$ and $\alpha(G) \leq (\nu-n)/2$. Then G is not n-factor-critical if and only if $(\nu-n)/2$ is odd, $G = G_0 \vee (G_1 \cup G_2)$, where $\nu(G_0) = n$ and $G_1 = G_2 = K_{(\nu-n)/2}$. The bounds for $\delta(G)$ and $\alpha(G)$ are sharp.*

Proof. Suppose that G is not n-factor-critical. By Lemma 1.9, there exists $S \subseteq V(G)$ of size at least n, such that $o(G-S) > |S|-n$. By parity we have $o(G-S) \geq |S|-n+2$.

Let G_1 be an odd component of $G-S$ of the minimum size, and $v \in V(G_1)$. Then $\nu(G_1) \leq (\nu-|S|)/(|S|-n+2)$. Hence

$$\frac{\nu+n}{2}-1 \leq \delta(G) \leq d(v) \leq \nu(G_1)+|S|-1 \leq \frac{\nu-|S|}{|S|-n+2}+|S|-1. \quad (1)$$

Solving $|S|$ from (1) we have $|S| \leq n$ or $|S| \geq (\nu+n)/2-1$. If $|S| \geq (\nu+n)/2-1$, then $o(G-S) \geq (\nu+n)/2-1-n+2 = (\nu-n)/2+1$. Selecting one vertex from each odd component of $G-S$ we form an independent set of G of order no less than $(\nu-n)/2+1$, contradicting $\alpha(G) \leq (\nu-n)/2$. Therefore we can assume that $|S| \leq n$. However $|S| \geq n$ by our selection of $|S|$, so we have $|S| = n$ and all equalities in (1) must hold. Hence $G-S$ has exactly two components G_1 and G_2 of odd size $(\nu-n)/2$, G_1 and G_2 must be $K_{(\nu-n)/2}$ and all vertices in $V(G_1) \cup V(G_2)$ are adjacent to all vertices in S. Thus $G = G_0 \vee (G_1 \cup G_2)$, where $G_0 = G[S]$ is of order n and $G_1 = G_2 = K_{(\nu-n)/2}$.

On the contrary if G satisfies the conditions stated, then $G - V(G_0)$ does not have a perfect matching, so G is not n-factor-critical.

The graph $G = ((\nu-n)/2+1)K_1 \vee K_{(\nu+n)/2-1}$ shows that the bound for $\alpha(G)$ is sharp, while the sharpness of the bound for $\delta(G)$ can be verified by the graph $G = (K_3 \cup ((\nu-n)/2)-1)K_1) \vee K_{(\nu+n)/2-2}$. □

Combining Theorem 3.1 and Theorem 3.2 we have the following result.

Theorem 3.3. *Let G be a graph on ν vertices, n a positive integer satisfying $\nu \equiv n \pmod 2$, $\delta(G) \geq (\nu+n)/2-1$. Suppose that G can not be expressed as $G = G_0 \vee (G_1 \cup G_2)$, where $(\nu-n)/2$ is odd, $\nu(G_0) = n$ and $G_1 = G_2 = K_{(\nu-n)/2}$. Then G is n-factor-critical if and only if $\alpha(G) \leq (\nu-n)/2$.*

By Theorem 1.4 and Theorem 2.1, when $k \geq \nu(G)/4$ and $\delta(G) \geq \nu(G)/2+k-1$, G is k-extendable if and only if G is $2k$-factor-critical, only except that when $k = \nu(G)/4$ is odd and $G = (K_k \cup K_k) \vee (K_k \cup K_k)$, G is k-extendable but not $2k$-factor-critical. But then G is exactly the exceptional graph in Theorem 3.3. Hence we have the following corollary which has Theorem 1.6 as part of it.

Corollary 3.4. *Let G be a graph with even order ν, k a positive integer such that $\nu/4 \leq k \leq \nu/2 - 1$ and $\delta(G) \geq \nu/2 + k - 1$. Suppose that G can not be expressed as $G = G_0 \vee (G_1 \cup G_2)$, where $\nu/2 - k$ is odd, $\nu(G_0) = 2k$ and $G_1 = G_2 = K_{\nu/2-k}$. Then the following are equivalent.*
(1) G is k-extendable,
(2) G is $2k$-factor-critical,
(3) $\alpha(G) \leq \nu/2 - k$.

4 Extendibility and independence number

In this section we generalize Theorem 1.5 for $k\frac{1}{2}$-extendable graphs.

Theorem 4.1. *Let G be an $k\frac{1}{2}$-extendable graph on ν vertices. Then $\alpha(G) \leq (\nu - 1)/2 - k$. Moreover, the upper bound for $\alpha(G)$ is sharp for all k and ν.*

Proof. By Lemma 1.8, $G \vee K_1$ is $(k+1)$-extendable. By Theorem 1.5, $\alpha(G \vee K_1) \leq (\nu + 1)/2 - (k + 1) = (\nu - 1)/2 - k$. Furthermore, any independent set S of $G \vee K_1$ with $|S| > 1$ can not contain the vertex in K_1. Hence S is also an independent set of G. So $\alpha(G) = \alpha(G \vee K_1) \leq (\nu - 1)/2 - k$.

To see that the bound is sharp, consider the graph $G = ((\nu - 1)/2 - k)K_1 \vee K_{(\nu+1)/2+k}$. □

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