Meandric polygons

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Abstract

The notion of meandric polygons is introduced in this paper. A bijection exists between the set of meandric polygons and that of closed meanders. We use these polygons to enumerate the set of meanders which have fixed number of arcs of the meandric curves lying above and below the horizontal line at a given point.

1 Introduction

In this paper we introduce the meandric polygons in order to represent the closed meanders. This contributes to the study of a cutting problem.

We recall that a *closed meander of order* n is a closed self avoiding curve, crossing an infinite horizontal line $2n$ times [4]. From now on, when we refer to a meander we will mean a closed meander.

In Section 2 we present the meandric polygons, which are derived by the transformation of the nested sets of the meanders into two Dyck paths. There exists a bijection between the set of meanders and that of the meandric polygons. We study the properties of these polygons and we obtain their construction, using strict contractions and binary trees. This section is completed by presenting some criteria for meandric polygons.

Section 3 concerns the problem of the set of meanders which have fixed number of arcs of the meandric curves lying above and below the horizontal line at a given point. In order to enumerate these meanders, at first we obtain an upper bound for the number of the corresponding meandric polygons, which are constructed using binary trees. Then, by applying the conditions for the matching property, we arrive at the exact solution of the problem.

The following definitions and notation refer to notions that are necessary for the development of the paper.

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A word $u \in \{a, \bar{a}\}$ is called *Dyck word* if $|u|_a = |u|_{\bar{a}}$ and for every factorization $u = pq$ we have $|p|_a \geq |p|_{\bar{a}}$, where $|u|_a$, $|p|_a$ (resp. $|u|_{\bar{a}}$, $|p|_{\bar{a}}$) denote the number of occurrences of a (resp. $\bar{a}$) in the words u, p.

We denote the set of all Dyck words of length $2n$ by D_{2n}. It is well known that the cardinality of D_{2n} equals to the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Let $u = u_1 u_2 \cdots u_{2n}$ with $u \in D_{2n}$. Two indices i, j such that $i < j$, $u_i = a$, $u_j = \bar{a}$ are called *conjugates* with respect to u, if j is the smallest element of $\{i+1, i+2, \ldots, 2n\}$ for which the subword $u_i u_{i+1} \cdots u_j$ is a Dyck word.

A *Dyck path* of length $2n$ is a lattice path in the first quadrant, which begins at the origin $(0,0)$, ends at $(2n, 0)$ and consists of steps $(1,1)$ and $(1,-1)$.

A set S of disjoint pairs of $[2n]$, such that $\bigcup_{\{a,b\} \in S} \{a, b\} = [2n]$ and for any $\{a,b\}, \{c,d\} \in S$ we never have $a < c < b < d$, is called *nested set* of pairs on $[2n]$. Each pair of a nested set consists of an odd and an even number. We denote the set of all nested sets of pairs on $[2n]$ by N_{2n}. Sapounakis and Tsikouras [8] have presented a construction of N_{2n}.

We denote by dom$S$ all the elements of $\mathbb{N}^*$, that belong to any pair of a nested set of pairs S; we recall that two nested sets S_1, S_2 are called *matching* if domS_1 = domS_2 and dom$A$ = dom$B$, $A \subsetneq S_1$, $B \subsetneq S_2$ imply that either $A = B = \emptyset$ or dom$A$ = domS_1 [5].

Finally, a permutation with repetitions $\sigma = \sigma(1)\sigma(2)\cdots\sigma(2n+1)$ on $[n+1]$ is called *strict contraction* of size $2n+1$ when

(i) $\sigma(1) = \sigma(2n+1) = 1$

(ii) $|\sigma(i+1) - \sigma(i)| = 1$, $i = 1, 2, 3, \ldots, 2n$.

We easily notice that each meandric curve is separated by the horizontal line into two sets of arcs. The arcs above (resp. below) the horizontal line uniquely define an upper nested set U (resp. a lower nested set L) of N_{2n}. The sets U, L are matching [1]. Conversely, two matching nested sets S, S' of N_{2n} uniquely define a meander of order n with $U = S$ and $L = S'$ [5].

For example, for the meander of Figure 1 we have
$U = \{\{1,8\}, \{2,5\}, \{3,4\}, \{6,7\}, \{9,12\}, \{10,11\}\}$ and
$L = \{\{1,2\}, \{3,10\}, \{4,7\}, \{5,6\}, \{8,9\}, \{11,12\}\}$.

We denote the number of closed meanders of order n by M_n. Some important results on the enumeration of M_n can be found in [2] and [4].

It is well known [8] that there exists a bijection between D_{2n} and N_{2n}, according to which each $u \in \mathcal{D}_{2n}$ corresponds to an $S \in N_{2n}$ as follows : $\{i,j\} \in S$ iff i, j are conjugate indices with respect to u.

We denote by u (resp. u') the Dyck word corresponding to U (resp. L). Hence, the corresponding Dyck words of the meander of Figure 1 are $u = aaa\bar{a}\bar{a}a\bar{a}\bar{a}aa\bar{a}\bar{a}$ and $u' = a\bar{a}aaa\bar{a}\bar{a}a\bar{a}\bar{a}a\bar{a}$.

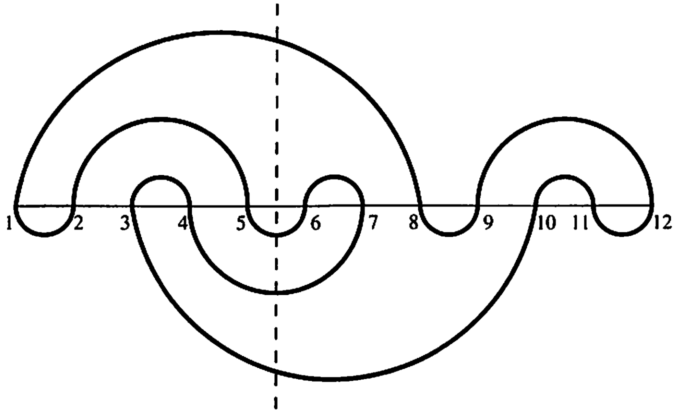

Figure 1: A meander of order 6

The same ideas have been applied in [6] where, for the presentation of meanders, Motzkin words which are based on the Dyck words u, u' have been used.

2 Meandric polygons

To every word u (resp. u') of D_{2n} corresponds a Dyck path P (resp. P') of length $2n$, which passes through the points $(i, f(i))$ (resp. $(i, f'(i))$, $i = 0, 1, \ldots, 2n$ of the first quadrant (resp. fourth quadrant) of the plane, where

$$f(0) = 0 \text{ and } f(i) = \begin{cases} f(i-1)+1, & \text{if } u_i = a, \quad i \in [2n] \\ f(i-1)-1, & \text{if } u_i = \bar{a}, \quad i \in [2n] \end{cases}$$

$$(\text{resp. } f'(0) = 0 \text{ and } f'(i) = \begin{cases} f'(i-1)-1, & \text{if } u'_i = a, \quad i \in [2n] \\ f'(i-1)+1, & \text{if } u'_i = \bar{a}, \quad i \in [2n] \end{cases}).$$

Notice that the function f' (resp. Dyck path P') can be considered as the "mirror image" (with respect to the x-axis) of some function f (resp. Dyck path P).

Those two Dyck paths P and P', corresponding to the matching nested sets U, L, create a polygon π, which characterizes every meander of order n and it will be called *meandric polygon*.

For example for the meander of Figure 1 we have the meandric polygon of Figure 2.

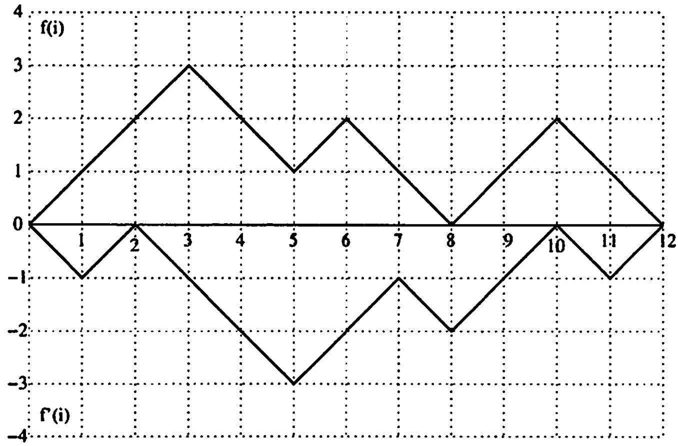

Figure 2: The meandric polygon of the meander of Figure 1

We notice that there exists a bijection between the set of meanders of order n and the set of meandric polygons.

Obviously, from the previous, every couple of nested sets $S, S' \in N_{2n}$ (matching or not), translated into Dyck paths, defines a polygon passing through the points $(0,0)$ and $(2n,0)$, which is inscribed in the square defined by the vertices $(0,0)$, (n,n), $(2n,0)$ and $(n,-n)$. The set of all those polygons is denoted by Π_n. Since the number of all Dyck paths of length $2n$ is equal to C_n, we deduce that the number of all polygons of Π_n is equal to C_n^2.

If $p_i = u_1u_2\cdots u_i$ (resp. $p'_i = u'_1u'_2\cdots u'_i$) is a subword of the Dyck word u (resp. u') for $i = 1, 2, \ldots, 2n-1$, then we have the following proposition.

Proposition 2.1 *Let π be a meandric polygon of Π_n. Then, $f(i) = |p_i|_a - |p_i|_{\bar{a}}$ (resp. $|f'(i)| = |p'_i|_a - |p'_i|_{\bar{a}}$), $i \in [2n-1]$.*

Proof. Indeed, the segment of the Dyck path which corresponds to the subword $p_i = u_1u_2\cdots u_i$ (resp. $p'_i = u'_1u'_2\cdots u'_i$) goes up $|p_i|_a$ (resp. $|p'_i|_a$) times and down $|p_i|_{\bar{a}}$ (resp. $|p'_i|_{\bar{a}}$) times, until it passes through the point $(i, f(i))$ (resp. $(i, f'(i))$). □

We introduce the sequences $f = (f(i))_{i\in[2n-1]}$ and $f' = (|f'(i)|)_{i\in[2n-1]}$, for every meandric polygon. We remark that if we include the terms $f(0) = f(2n) = f'(0) = f'(2n) = 0$, then they define a unique meandric polygon. Hence, we shall use the notation of (f, f') in order to define a meandric polygon π of Π_n.

For example, for the meandric polygon of Figure 2 we have

	1	2	3	4	5	6	7	8	9	10	11
f	1	2	3	2	1	2	1	0	1	2	1
f'	1	0	1	2	3	2	1	2	1	0	1

The terms of these sequences express the heights of the vertices of a meandric polygon of Π_n and their sum $\sum_{i=1}^{2n-1}(f(i)+|f'(i)|)$ expresses its area. This is an immediate application of the well-known formula by Pick [7] according to which the area of a simple lattice polygon equals to $I+\frac{B}{2}-1$, where I is the number of interior lattice points and B is the number of boundary lattice points. In the case of meandric polygons we have that the area is equal to $\sum_{i=1}^{2n-1} f(i)+\sum_{i=1}^{2n-1}(|f'(i)|-1)+\frac{4n}{2}-1=\sum_{i=1}^{2n-1}(f(i)+|f'(i)|)-(2n-1)+(2n-1)=\sum_{i=1}^{2n-1}(f(i)+|f(i)|)$.

From the sequences f and f', we obtain the sequences $\hat{f}=(f(2n-i))_{i\in[2n-1]}$, $\hat{f}'=(|f'(2n-i)|)_{i\in[2n-1]}$ and we deduce the following proposition.

Proposition 2.2 *If (f,f') defines a meandric polygon of Π_n, then (f',f), $(\hat{f},\hat{f}')$ and $(\hat{f}',\hat{f})$ define also meandric polygons of Π_n.*

Proof. The pairs (f,f'), (f',f), $(\hat{f},\hat{f}')$, $(\hat{f}',\hat{f})$ define meandric polygons corresponding to the basic symmetries of meanders. In fact, the meander corresponding to (f',f) is the reflection of the meander corresponding to (f,f') with respect to the horizontal line, $(\hat{f},\hat{f}')$ is the reflection of (f,f') with respect to the vertical line and finally $(\hat{f}',\hat{f})$ is the 180^o rotation of (f,f'). □

We remark that the above sequence f can be related to a strict contraction σ. In fact, from every strict contraction σ we can deduce the sequence f, through the relation $f(i)=\sigma(i+1)-1$.

For the sequence f and the strict constraction σ we have that $|f(i+1)-f(i)|=|\sigma(i+2)-\sigma(i+1)|=1$. Also, the relations $\sigma(2)=\sigma(2n)=2$ always hold and imply the relations $f(1)=f(2n-1)=1$. So, there exists a bijection between the set of the sequences f and that of the strict contractions σ.

We can see an example in the following table.

σ	1	2	1	2	3	4	3	4	3	2	1
f	0	1	0	1	2	3	2	3	2	1	0

Now, in order to construct the sequences f and f', we use the binary trees, which produce the strict contractions. More specifically, in [8], for the determination of the strict contractions of size $2n+1$ a binary tree Δ_{2n+1} is used, which satisfies the following properties:

(i) The vertices lie on $2n+1$ levels and correspond to the possible values of $\sigma(i)$, $i \in [2n+1]$.

(ii) Each vertex is labeled by an integer $\nu \in [2n+1]$, so that the root is labeled by 1 and each left (resp. right) child of a vertex ν is labeled by $\nu - 1$ (resp. $\nu + 1$).

(iii) If a vertex ν belongs to the k^{th} level, then $\nu \leq \min\{k, 2n+2-k\}$.

So, we have the following result.

Proposition 2.3 *A binary tree T_{2n-1} of height $2n-1$ and with each vertex labeled by an integer $\nu \in \{0, 1, 2, \ldots, n\}$, such that*

(i) the root is labeled by 1,

(ii) each left (resp. right) child of a vertex ν is labeled by $\nu-1$ (resp. $\nu+1$),

(iii) if a vertex ν belongs to the e^{th} level then $\nu \leq \min\{e, 2n-e\}$,

produces the C_n finite sequences f.

Proof. Indeed, if we reduce the numbers of the labels of the vertices of the tree Δ_{2n+1} by 1 and we omit the first and the last level, we obtain the tree T_{2n-1}. In this tree every path from the root to a leaf, corresponds to a sequence f. □

For example, for meanders of order 4 we have the following binary tree T_7, providing $C_4 = 14$ sequences f.

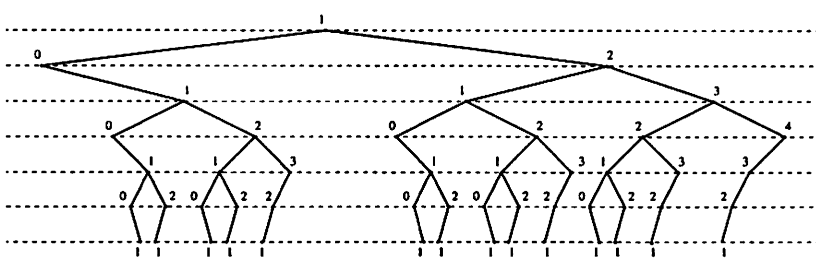

Figure 3: The binary tree T_7.

Proposition 2.4 *If π is a meandric polygon and*

$$j = \min\{k \in \{i+1, i+2, \ldots, 2n\} : f(i-1) = f(k) < f(k-1) = f(i)\},$$

then $(i, j) \in U$.

Proof. From the relations $f(i-1) < f(i)$ and $f(k) < f(k-1)$, we deduce that $u_i = a$ and $u_k = \bar{a}$. Since for these indices i, j we have that $i < j$, $u_i = a$, $u_j = \bar{a}$ and j is the smallest element of $\{i+1, i+2, \ldots, 2n\}$ for which the subword $u_i u_{i+1} \cdots u_j$ is a Dyck word, we conclude that i, j are conjugate indices. Thus, $\{i, j\} \in U$. □

In order to determine whether a polygon p of Π_n is meandric, we label the consecutive steps of the Dyck path P (resp. P') corresponding to the nested sets S, (resp. S') with the numbers $1, 2, \ldots, 2n$ (resp. $1', 2', \ldots, (2n)'$). Practically, according to Proposition 2.4 we remark that $\{i, j\} \in S$ if the i and j steps "are facing each other" and there is no other intermediate step between them.

With vertices the middle points of these steps, we create a hamiltonian cycle $h_\delta = [1, i_1, i'_1, i'_2, i_2, i_3, \ldots, i'_\delta, 1']$, with $(1, i_1) \in S$, $(i'_1, i'_2) \in S'$, $(i_2, i_3) \in S, \ldots, (i'_\delta, 1') \in S'$.

Proposition 2.5 *A polygon p of Π_n is meandric iff there exists one hamiltonian cycle h_δ of length $4n$.*

Proof. Let π be a meandric polygon of Π_n. Then, its corresponding nested sets U, L are matching. By the definition of matching sets, this means that U, L do not have proper subsets both using the same elements of $[2n]$; hence, a hamiltonian cycle of length less than $4n$ cannot be defined.

Conversely, if there exists a hamiltonian cycle of length $4n$, then all the elements of $[2n]$ "are used" by the nested sets S, S' and hence there does not exist any pair of proper subsets of S, S' using the same elements of $[2n]$. Thus, S, S' are matching. □

Obviously, when there exists a hamiltonian cycle of length less than $4n$, then the polygon is not meandric.

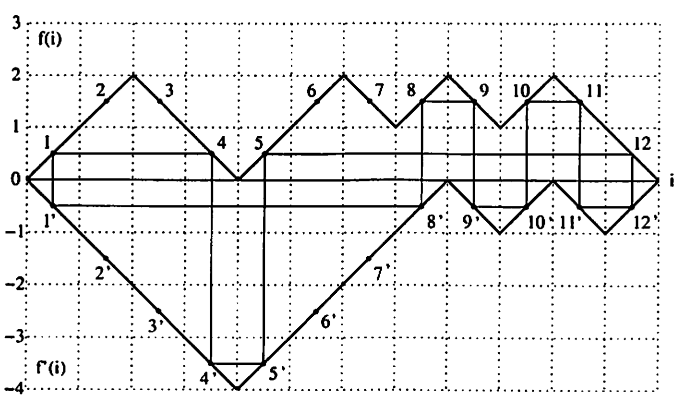

Figure 4: A non-meandric polygon

For example, for the polygon p produced by the nested sets

$$S = \{\{1,4\},\{2,3\},\{5,12\},\{6,7\},\{8,9\},\{10,11\}\}$$
$$S' = \{\{1',8'\},\{2',7'\},\{3',6'\},\{4',5'\},\{9',10'\},\{11',12'\}\}$$

we obtain the hamiltonian cycle

$$h_\delta = [1,4,4',5',5,12,12',11',11,10,10',9',9,8,8',1']$$

of length $16 < 24$. Thus p is not a meandric polygon (see Figure 4).

We can also apply the following criteria, before the application of the previous necessary and sufficient condition.

Proposition 2.6 *Let $p \in \Pi_n$. If any of the following conditions holds, then the polygon p is not meandric.*

(i) $f(i) + f'(i) = 0$, $f(i), f'(i) \neq 0$, *for every* $i \in [2n-1]$.

(ii) There exists at least one $i \in [2n-1]$, *such that* $f(i) = f'(i) = 0$.

(iii) There exists at least one $i \in [2n-1]$, *such that* $f(i) = f(i+2) = f(i+1)+1$ *and* $f'(i) = f'(i+2) = f'(i+1) - 1$.

(iv) There exists at least one pair $(i,j) \in [2n-1] \times [2n-1]$, *such that* $f(i) = f(j)$, $f'(i) = f'(j)$ *and* $f(k) > f(i)$, $f'(k) < f'(i)$, *for every* $k \in \{i+1, i+2, \ldots, j-1\}$.

Proof. Indeed, the polygon p is not meandric, since in every one of the above cases we can form respectively the following hamiltonian cycles, which are of length 4 (less than $4n$):

(i) $[1, 2n, (2n)', 1']$, (ii) $[1, k, k', 1']$, where k is the first element of $[2n-1]$ with $f(k) = f'(k) = 0$, (iii) $[i, i+1, (i+1)', i']$, (iv) $[i+1, j, j', (i+1)']$. □

3 Cutting problem

Let a meander of order n. For any $i \in [2n-1]$ we consider the vertical line, passing through the middle point of the segment $(i, i+1)$, which we call the *i-line*. We denote by $\theta(i)$ and $\theta'(i)$ respectively, the number of arcs of the meandric curve, which intersect the i-line and lie above and below the horizontal line, which we call *numbers of cuttings*.

It is obvious that $\theta(i)$ (resp. $\theta'(i)$) is the number of pairs of the nested set U (resp. L) of the meander, that contain i between their elements.

For example, for the meander of Figure 1 we have for $i = 5$, the numbers of cuttings $\theta(5) = 1$ and $\theta'(5) = 3$.

Jensen, in the well known transfer matrix algorithm [2] introduces the signature (h, S) where h is the number of loop-ends below the horizontal line and S is the integer whose binary representation corresponds to the configuration of loop-ends. Although, the direction is completely different, we would like to underline the relations between the numbers of cuttings $\theta(i), \theta'(i)$ and the signature (h, S): $\theta(i) = |S| - h$, $\theta'(i) = h$.

We denote the sequences of the numbers of cuttings by $\theta = (\theta(i))_{i \in [2n-1]}$ and $\theta' = (\theta'(i))_{i \in [2n-1]}$.

For the meander of Figure 1 we have the following sequences of numbers of cuttings.

	1	2	3	4	5	6	7	8	9	10	11
θ	1	2	3	2	1	2	1	0	1	2	1
θ'	1	0	1	2	3	2	1	2	1	0	1

Since the sum $\theta(i) + \theta'(i)$ is always even, the numbers $\theta(i)$, $\theta'(i)$ are of the same parity.

From the left side of the i-line of the meander there can be at most i arcs that have traces on it; the same argument holds for the right side, with $2n-i$ arcs. So, we have that $\theta(i) \leq \min\{i, 2n-i\}$ and $\theta'(i) \leq \min\{i, 2n-i\}$.

We observe that $\theta(i) < n$, for every $i \neq n$ and $\theta(n) \leq n$. The case $\theta(n) = n$ is obtained for the meanders of order n, with nested set $U = \{\{i, 2n+1-i\} : i = 1, 2, \ldots, n\}$. Finally we have that $\theta(i)$, $\theta'(i) \in \{0, 1, \ldots, n\}$ and obviously $\theta(1) = \theta(2n-1) = \theta'(1) = \theta'(2n-1) = 1$. Thus, we obtain the following proposition.

Proposition 3.1 *For the sequences θ and θ' we have that:*

(i) i, $\theta(i)$ *and* $\theta'(i)$ *have the same parity.*

(ii) $\max\{\theta(i), \theta'(i)\} \leq \min\{i, 2n-i\}$, $i \in [2n-1]$.

(iii) $|\theta(i+1) - \theta(i)| = |\theta'(i+1) - \theta'(i)| = 1$.

Proposition 3.2 *The sequences f and θ (resp. f' and θ') coincide.*

Proof. The number $\theta(i)$ (resp. $\theta'(i)$) of arcs of the meandric curve which could intersect the i-line and lie above (resp. below) the horizontal line is the number of the arcs that "open" before the i-position (i.e. those with $u_\nu = a$ (resp. $u'_\nu = a$) for $1 \leq \nu \leq i$) and do not "close" before the i-position, (i.e. those ν with conjugate indices $\lambda > i$ and $u_\lambda = \bar{a}$ (resp. $u'_\lambda = \bar{a}$)). Therefore, $\theta(i) = |p_i|_a - |p_i|_{\bar{a}}$ and $\theta'(i) = |p'_i|_a - |p'_i|_{\bar{a}}$; hence, from Proposition 2.1 we conclude that $\theta(i) = f(i)$ and $\theta'(i) = |f'(i)|$, for $i \in [2n-1]$. □

Now, we determine two specific Dyck paths, both passing through the point $(i, \theta(i))$. The first, which we denote by $\overline{f}(i)$, uses the maximum permitted values of $f(i)$, whereas the second, which we denote by $\underline{f}(i)$, uses the minimum permitted values of $f(i)$. Notice that the highest vertices of $\overline{f}(i)$ on either side of i are peaks of two pyramids (see Figure 5). The left pyramid starts at $(\theta(i), \theta(i))$ and ends at $(i, \theta(i))$ and therefore its peak is the point $\left(\frac{i+\theta(i)}{2}, \frac{i+\theta(i)}{2}\right)$. The right pyramid starts at $(i, \theta(i))$ and ends at $(2n-\theta(i), \theta(i))$, and therefore its peak is the point $\left(\frac{i+2n-\theta(i)}{2}, \frac{2n+\theta(i)-i}{2}\right)$. We easily deduce the following proposition.

Proposition 3.3 *For every $i \in [2n-1]$ we have that $\underline{f}(i) \leq f(i) \leq \overline{f}(i)$.*

Similarly, the proposition holds for $f'(i)$.

For example, for the meander of Figure 1, we have the following:

	1	2	3	4	5	6	7	8	9	10	11
$f(i)$	1	2	3	2	1	2	1	0	1	2	1
$\overline{f}(i)$	1	2	3	2	1	2	3	4	3	2	1
$\underline{f}(i)$	1	0	1	0	1	0	1	0	1	0	1

The main problem is to find the number of meanders of order n with given numbers of cuttings $\theta(i), \theta'(i)$ at i. For the existence of such meanders, the first two conditions of Proposition 3.1 must hold.

This problem is related to the one of finding the number $\Pi_n(i; (\theta(i), \theta'(i)))$ of polygons of Π_n with the Dyck path P passing through the point $(i, \theta(i))$ and the Dyck path P' passing throught the point $(i, -\theta'(i))$.

We can now deduce the following proposition.

Proposition 3.4 *For the polygons of Π_n we have*

$$\Pi_n(i; (\theta(i), \theta'(i))) = w_n(i; \theta(i)) \cdot w_n(i; \theta'(i)),$$

where

$$w_n(i; r) = \frac{(r+1)^2}{(i+1)(2n-i+1)} \binom{i+1}{\frac{i-r}{2}} \binom{2n-i+1}{\frac{2n-i-r}{2}}.$$

Proof. It is known [3], that the number of Dyck paths from $(0,0)$ to (i,k) is equal to

$$\frac{k+1}{i+1} \binom{i+1}{\frac{1}{2}(i+k)+1} \text{ or equivalently to } \frac{k+1}{i+1} \binom{i+1}{\frac{i-k}{2}}.$$

Hence, the number of Dyck paths from $(0,0)$ to $(i, \theta(i))$ is equal to

$$\frac{\theta(i)+1}{i+1} \binom{i+1}{\frac{i-\theta(i)}{2}},$$

while the number of Dyck paths from $(i, \theta(i))$ to $(2n, 0)$ is equal to

$$\frac{\theta(i)+1}{2n-i+1}\binom{2n-i+1}{\frac{2n-i-\theta(i)}{2}}$$

(considering that this is equivalent to the number of Dyck paths from $(0,0)$ to $(2n-i, \theta(i))$). Thus, the total number of Dyck paths passing through the point $(i, \theta(i))$ is equal to

$$w_n(i;\theta(i)) = \frac{(\theta(i)+1)^2}{(i+1)(2n-i+1)}\binom{i+1}{\frac{i-\theta(i)}{2}}\binom{2n-i+1}{\frac{2n-i-\theta(i)}{2}}.$$

□

Similarly, the above formula holds for the Dyck paths P' passing through the point $(i, -\theta'(i))$.

From the above we conclude the following inequality:

$$M_n(i;(\theta(i),\theta'(i))) < \Pi_n(i;(\theta(i),\theta'(i))).$$

For the meandric polygon of Figure 2, where $n = 6$, $i = 5$, $\theta(i) = 1$ and $\theta'(i) = 3$, there exist $w_6(5;1) = \frac{2^2}{6\cdot 8}\binom{6}{2}\binom{8}{3} = 70$ Dyck paths of length 12, passing through the point $(5,1)$ and $w_6(5;-3) = \frac{4^2}{6\cdot 8}\binom{6}{1}\binom{8}{2} = 56$ Dyck paths of length 12 passing through the point $(5,-3)$.

The above results can be determined by the following procedure (see Figure 5) at the lattice defined by $\underline{f}(i)$, $\overline{f}(i)$, $\underline{f}'(i)$, $\overline{f}'(i)$ of the meandric polygon of Figure 2.

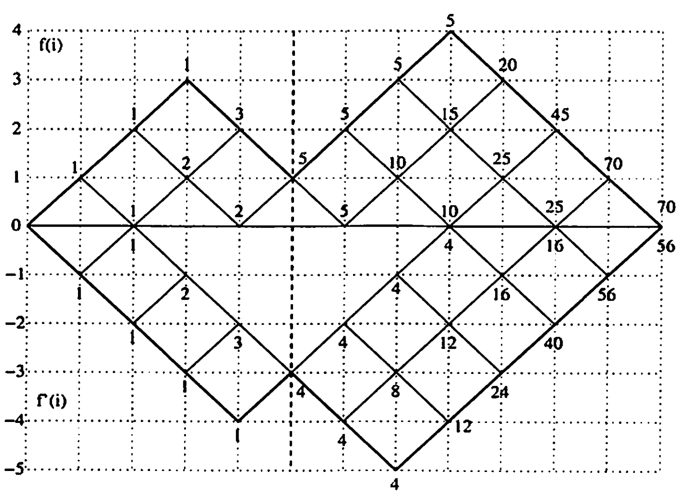

Figure 5: Finding the numbers $w_n(5;1)$ and $w_n(5;-3)$

The binary subtrees of T_{11} with vertices 1 and 3 at their 5^{th} level define $5 \cdot 14 = 70$ and $4 \cdot 14 = 56$ Dyck paths of length 12 passing through the points $(5, 1)$ and $(5, -3)$, respectively (see Figure 6).

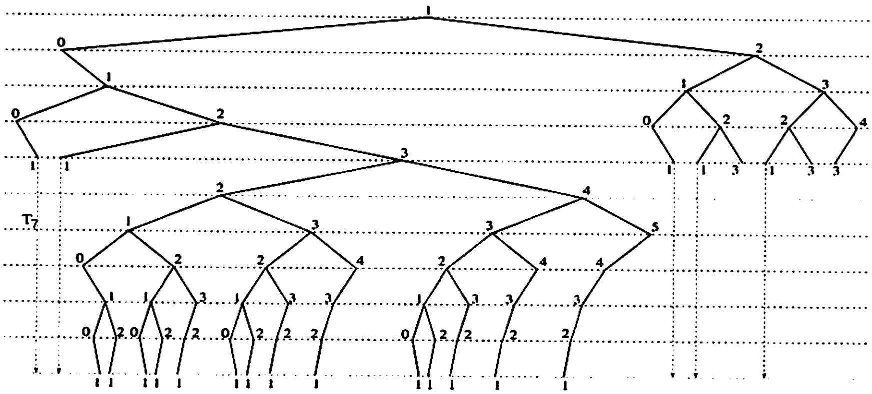

Figure 6: The subtrees of T_{11}

Thus, we have $M_n(5; (1,3)) < 70 \cdot 56 = 3920$ and we restric ourselves to the study of these polygons instead of a total of $C_6^2 = 132^2 = 17424$. Applying the criteria of Proposition 2.6, and finally the condition of Proposition 2.5 for the 3920 polygons, we obtain that $M_6(5; (1,3)) = 468$.

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