On the (s, t)-Fibonacci and Fibonacci Matrix Sequences

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Abstract

It is always fascinating to see what results when seemingly different areas mathematics overlap. This article reveals one such result; number theory and linear algebra are intertwined to yield complex factorizations of the classic Fibonacci, Pell, Jacobsthal, and Mersenne numbers. Also, in this paper we define a new matrix generalization of the Fibonacci numbers, and using essentially a matrix approach we show some properties of this matrix sequence.

Keywords: Fibonacci numers; Pell numbers; Jacobsthal numbers; Mersenne numbers;

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1 Introduction

There is a huge interest of modern science in the application of the Golden Section and Fibonacci numbers [8, 13 − 16]. The Fibonacci numbers f_n are the terms of the sequence $\{0, 1, 1, 2, 3, 5, ...\}$ wherein each term is the sum of the two previous terms, beginning with the values $f_0 = 0$, and $f_1 = 1$. Pell numbers are defined by a recurrence relation similar to that for the Fibonacci numbers, and grow exponentially, proportionally to powers of the silver ratio. Pell numbers arise in the approximation of the square root of 2, in the definition of square triangular numbers, in the construction of nearly-isosceles integer right triangles, and in certain combinatorial enumeration problems [1]. Pell numbers are the sequence of numbers $\{p_n\}_{n\in\mathbb{N}}$ defined by the linear recurrence equation $p_{n+1} = 2p_n + p_{n-1}$ with $p_0 = 0$ and $p_1 = 1$. Falcón and Plaza [3] introduced a general Fibonacci sequence that generalizes, between others, both the classic Fibonacci sequence and the Pell sequence. In [3], Falcón and Plaza showed the relation between the 4-triangle longest-edge (4TLE) partition and the Fibonacci numbers, as another example of the relation between geometry and numbers, and many properties of these numbers are deduced directly from elementary matrix algebra. In [4], many properties of these numbers are deduced and related with the so-called Pascal 2-triangle. In [5], the 3-dimensional k-Fibonacci spirals are studied from a geometric point of view. These curves appear naturally

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from studying the k-th Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ and the related hyperbolic k-Fibonacci functions.

The Jacobsthal numbers are the sequence of numbers $\{j_n\}_{n\in\mathbb{N}}$ defined by the linear recurrence equation $j_{n+1} = j_n + 2j_{n-1}$ with $j_0 = 0$ and $j_1 = 1$ (see [9], [10]). Microcontrollers (and other computers) use conditional instructions to change the flow of execution of a program. In addition to branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction. This winds up being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 on 5 bits, 21 on 6 bits, 43 on 7 bits, 85 on 8 bits, ..., which are exactly the Jacobsthal numbers (see [7]).

A Mersenne number is an integer in the form:

$$M_n = 2^n - 1, \quad n \geq 1.$$

Mersenne numbers are very good test cases for the special number field sieve algorithm, so often the largest number factorised has been a Mersenne number. As of March 2007, $2^{21039} - 1$ is the record-holder, after a calculation taking about a year on a couple of hundred computers, mostly at NTT in Japan and at EPFL in Switzerland [11].

In tis paper, we present the (s,t)th Fibonacci and Fibonacci matrix sequences in an explicit way and, by easy arguments, many properties are proven. In particular the (s,t)th Fibonacci matrices are related with the so-called (s,t)th Fibonacci numbers.

2 (s,t)-Fibonacci numbers and complex factorization

Definition 1 *(Falcon and Plaza [3]) For any integer number $k \geq 1$, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1$$

with initial conditions

$$F_{k,0} = 0; \quad F_{k,1} = 1.$$

Now, we introduce a new generalization of the kth Fibonacci numbers. It should be noted that the recurrence formula of these numbers depends on two real parameters.

Definition 2 *For any real numbers s,t; the (s,t)th Fibonacci sequence, say $\{F_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$F_{n+1}(s,t) = sF_n(s,t) + tF_{n-1}(s,t) \quad \text{for } n \geq 1, \tag{1}$$

with $F_0(s,t) = 0$, $F_1(s,t) = 1$.

We assume $t \neq 0$, as well as $s^2 + 4t \neq 0$.

Particular cases of the previous definition are:

- If $s = t = 1$, the classic Fibonacci sequence is obtained:

$$F_{n+1}(1,1) = F_n(1,1) + F_{n-1}(1,1) \quad \text{for } n \geq 1,$$

with $F_0(1,1)=0,\ F_1(1,1)=1$:

$$\{F_n(1,1)\}_{n\in\mathbb{N}}=\{0,1,1,2,3,5,8,13,...\}.$$

- If $s=2$ and $t=1$, the classic Pell sequence appears:

$$F_{n+1}(2,1)=2F_n(2,1)+F_{n-1}(2,1)\quad\text{for } n\geq 1,$$

with $F_0(2,1)=0,\ F_1(2,1)=1$:

$$\{F_n(2,1)\}_{n\in\mathbb{N}}=\{0,1,2,5,12,29,...\}.$$

- If $s=1$ and $t=2$, the classic Jacobsthal sequence is obtained:

$$F_{n+1}(1,2)=F_n(1,2)+2F_{n-1}(1,2)\quad\text{for } n\geq 1,$$

with $F_0(1,2)=0,\ F_1(1,2)=1$:

$$\{F_n(1,2)\}_{n\in\mathbb{N}}=\{0,1,1,3,5,11,21,...\}.$$

- If $s=3$ and $t=-2$, the Mersenne sequence appears:

$$F_{n+1}(3,-2)=3F_n(3,-2)-2F_{n-1}(1,2)\quad\text{for } n\geq 1,$$

with $F_0(3,-2)=0,\ F_1(3,-2)=1$:

$$\{F_n(3,-2)\}_{n\in\mathbb{N}}=\{0,1,3,7,15,...\}.$$

In the sequel we will write simply F_n, f_n, p_n, j_n, and m_n instead of $F_n(s,t)$, $F_n(1,1)$, $F_n(2,1)$, $F_n(1,2)$, and $F_n(3,-2)$, respectively.

Binet's formulas are well known in the Fibonacci numbers theory [7]. In our case, Binet's formula allows us to express the (s,t)th Fibonacci number in function of the roots α and β of the following characteristic equation, associated to the recurrence relation (1) :

$$r^2=sr+t. \tag{2}$$

Theorem 3 *The nth (s,t)-Fibonacci number is given by*

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},$$

where α and β are the roots of the characteristic equation (2), *and $\alpha>\beta$.*

Proof. The roots of characteristic equation (2) are $\alpha=\frac{s+\sqrt{s^2+4t}}{2}$, and $\beta=\frac{s-\sqrt{s^2+4t}}{2}$. Note that

$$\alpha+\beta=s\ \text{ and }\ \alpha\beta=-t,$$
$$\alpha-\beta=\sqrt{s^2+4t}.$$

Therefore, the general term of the (s,t)-Fibonacci sequence may be expressed in the form:

$$F_n=C_1\alpha^n+C_2\beta^n,$$

for some coefficients C_1 and C_2. The constants C_1 and C_2 are determined by the initial conditions

$$\begin{aligned} 0 &= C_1 + C_2, \\ 1 &= C_1\alpha + C_2\beta. \end{aligned}$$

Now, after some algebra we get $C_1 = \frac{1}{\alpha-\beta} = -C_2$ from where the result is obtained. ■

Next Theoren gives us the complex factorization of the (s,t)th Fibonacci sequence. In [2], it is showed how the classic Fibonacci and Lucas numbers arise as determinants of some tridiagonal matrices, and given the complex factorizations of the classic Fibonacci and Lucas sequences.

Theorem 4 *The (s,t)th Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ satisfy, for $n \geq 1$,*

$$F_{n+1} = \prod_{k=1}^{n} \left(s - 2i\sqrt{t}\cos\frac{\pi k}{n+1} \right), \tag{3}$$

and

$$F_{n+1} = i^n t^{n/2} \frac{\sin\left[(n+1)\cos^{-1}\left(-\frac{is\sqrt{t}}{2t}\right)\right]}{\sin\left(\cos^{-1}\left(-\frac{is\sqrt{t}}{2t}\right)\right)}.$$

Proof.

In order to derive (3), we introduce the sequence of matrices $\{M_n(s,t), n = 1,2,...\}$, where $M_n(s,t)$ is the $n \times n$ tridiagonal matrix with entries $m_{k,k} = s$, $1 \leq k \leq n$, and $m_{k-1,k} = m_{k,k-1} = i\sqrt{t}$, $2 \leq k \leq n$, with $i = \sqrt{-1}$. That is,

$$M_n(s,t) = \begin{pmatrix} s & i\sqrt{t} & & & \\ i\sqrt{t} & s & i\sqrt{t} & & \\ & i\sqrt{t} & s & \ddots & \\ & & \ddots & \ddots & i\sqrt{t} \\ & & & i\sqrt{t} & s \end{pmatrix}.$$

Then the successive determinants of $M_n(s,t)$ are given by the recursive formula:

$$\begin{aligned} |M_1(s,t)| &= s, \\ |M_2(s,t)| &= s^2 + t, \\ |M_n(s,t)| &= s\,|M_{n-1}(s,t)| + t\,|M_{n-2}(s,t)|, \; n \geq 3. \end{aligned}$$

Therefore, we write

$$F_{n+1} = |M_n(s,t)|, n \geq 1. \tag{4}$$

We now introduce another sequence of matrices $\{G_n, n = 1,2,...\}$, where G_n is the $n \times n$ tridiagonal matrix with entries $g_{j,j} = 0$, $1 \leq j \leq n$, and $g_{j-1,j} = g_{j,j-1} = 1$, $2 \leq j \leq n$. That is,

$$G_n = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}.$$

Then, note that $M_n(s,t) = sI + i\sqrt{t}G_n$. Let λ_j, $j = 1, 2, .., n$, be the eigenvalues of the matrix G_n (with associated eigenvectors x_j). Then, for each j,

$$\begin{aligned} M_n(s,t)\, x_j &= \left[sI + i\sqrt{t}G_n\right] x_j \\ &= \left[s + i\sqrt{t}\lambda_j\right] x_j. \end{aligned}$$

Therefore, $\mu_k = s + i\sqrt{t}\lambda_k$, $k = 1, 2, .., n$, are the eigenvalues of $M_n(s,t)$. It was determined the λ_k's as

$$\lambda_k = -2\cos\frac{\pi k}{n+1}, \quad k = 1, 2, ..., n,$$

in [2]. Thus, if we consider that the determinant of $M_n(s,t)$ is the product of the its eigenvalues and the equality (4), then we have

$$F_{n+1} = \prod_{k=1}^{n}\left(s - 2i\sqrt{t}\cos\frac{\pi k}{n+1}\right).$$

Now we will think of Chebyshev polynomials of the second kind as being generated by determinants of successive matrices of the form

$$A(x,n) = \begin{pmatrix} 2x & 1 & & & \\ 1 & 2x & 1 & & \\ & 1 & 2x & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2x \end{pmatrix},$$

where $A(x,n)$ is $n \times n$. Hence, we obtain

$$|M_n(s,t)| = i^n t^{n/2}\left|A\left(-\frac{is\sqrt{t}}{2t}, n\right)\right|. \tag{5}$$

Then the successive determinants of $A\left(-\frac{is\sqrt{t}}{2t}, n\right)$ are given by the recursive formula:

$$\begin{aligned} \left|A\left(-\frac{is\sqrt{t}}{2t}, 1\right)\right| &= -\frac{is\sqrt{t}}{t}, \\ \left|A\left(-\frac{is\sqrt{t}}{2t}, 2\right)\right| &= -\frac{s^2}{t} - 1, \\ \left|A\left(-\frac{is\sqrt{t}}{2t}, n\right)\right| &= -\frac{is\sqrt{t}}{t}\left|A\left(-\frac{is\sqrt{t}}{2t}, n-1\right)\right| - \left|A\left(-\frac{is\sqrt{t}}{2t}, n-2\right)\right|. \end{aligned}$$

Note that this family of determinants can be expressed as the Chebyshev polynomials of the second kind for $y = -\frac{is\sqrt{t}}{2t}$. Thus, for $\varphi = \cos^{-1}\left(-\frac{is\sqrt{t}}{2t}\right)$ we get

$$\left|A\left(-\frac{is\sqrt{t}}{2t}, n\right)\right| = \frac{\sin\left[(n+1)\varphi\right]}{\sin\varphi}. \tag{6}$$

It now folows from (4), (5) and (6) that

$$F_{n+1} = i^n \frac{\sin\left[(n+1)\cos^{-1}\left(-\frac{is\sqrt{t}}{2t}\right)\right]}{\sin\left(\cos^{-1}\left(-\frac{is\sqrt{t}}{2t}\right)\right)}.$$

Consequently, the proof is completed. ■

3 (s,t)-Fibonacci matrix sequence

In this section, a new matrix generalization of the Fibonacci numbers is introduced. It should be noted that the recurrence formula of these matrices depends on two integral parameters.

Definition 5 *For any integer numbers s,t; the (s,t)th Fibonacci matrix sequence, say $\{\mathcal{F}_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$\mathcal{F}_{n+1}(s,t) = s\mathcal{F}_n(s,t) + t\mathcal{F}_{n-1}(s,t) \quad \textit{for } n \geq 1,$$

with $\mathcal{F}_0(s,t) = I$, $\mathcal{F}_1(s,t) = \begin{pmatrix} s & 1 \\ t & 0 \end{pmatrix}$, *where I is the 2×2 unit matrix.*

We assume $t \neq 0$, as well as $s^2 + 4t \neq 0$. In the sequel we will write simply $\mathcal{F}_n$ instead of $\mathcal{F}_n(s,t)$.

Now, we have the following lemma.

Lemma 6 *For any integer $n \geq 1$ holds:*

$$\mathcal{F}_n = \begin{pmatrix} F_{n+1} & F_n \\ tF_n & tF_{n-1} \end{pmatrix}.$$

Proof. *(By induction)* For $n = 1$:

$$\mathcal{F}_1 = \begin{pmatrix} s & 1 \\ t & 0 \end{pmatrix} = \begin{pmatrix} F_2 & F_1 \\ tF_1 & tF_0 \end{pmatrix}$$

since $F_0 = 0$, $F_1 = 1$, and $F_2 = s$. Let us suppose that the formula is true for $n-1$:

$$\mathcal{F}_{n-1} = \begin{pmatrix} F_n & F_{n-1} \\ tF_{n-1} & tF_{n-2} \end{pmatrix}.$$

Then,

$$\begin{aligned} \mathcal{F}_n &= s\mathcal{F}_{n-1} + t\mathcal{F}_{n-2} \\ &= s\begin{pmatrix} F_n & F_{n-1} \\ tF_{n-1} & tF_{n-2} \end{pmatrix} + t\begin{pmatrix} F_{n-1} & F_{n-2} \\ tF_{n-2} & tF_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} F_{n+1} & F_n \\ tF_n & tF_{n-1} \end{pmatrix}. \end{aligned}$$

■

Particular cases are:

- If $s = t = 1$, for the classic Fibonacci sequence, we obtain:

$$\mathcal{F}_n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}.$$

- If $s = 2$ and $t = 1$, for the classic Pell sequence we have:

$$\mathcal{F}_n = \begin{pmatrix} p_{n+1} & p_n \\ p_n & p_{n-1} \end{pmatrix}.$$

- If $s = 1$ and $t = 2$, for the classic Jacobsthal sequence we get:

$$\mathcal{F}_n = \begin{pmatrix} j_{n+1} & j_n \\ 2j_n & 2j_{n-1} \end{pmatrix}.$$

- If $s = 3$ and $t = -2$, for the Mersenne sequence we obtain:

$$\mathcal{F}_n = \begin{pmatrix} m_{n+1} & m_n \\ -2m_n & -2m_{n-1} \end{pmatrix}.$$

Lemma 7 *$\mathcal{F}_{m+n} = \mathcal{F}_m\mathcal{F}_n$ for any integers $m, n \geq 0$.*

Proof. We use the second principle of finite induction on n to prove this lemma. Let $n = 0$. Then, the lemma yields $\mathcal{F}_m = \mathcal{F}_m\mathcal{F}_0$ since $\mathcal{F}_0 = I$. Now assume that

$$\mathcal{F}_{m+n} = \mathcal{F}_m\mathcal{F}_n \quad \text{for } n \leq N.$$

Then,

$$\begin{aligned} \mathcal{F}_{m+N+1} &= s\mathcal{F}_{m+N} + t\mathcal{F}_{m+N-1} \\ &= s\mathcal{F}_m\mathcal{F}_N + t\mathcal{F}_m\mathcal{F}_{N-1} \\ &= \mathcal{F}_m\left[s\mathcal{F}_N + t\mathcal{F}_{N-1}\right] \\ &= \mathcal{F}_m\mathcal{F}_{N+1}. \end{aligned}$$

■

3.1 Sum formulas for (s,t)th Fibonacci matrix sequence

In this section, we now shall show some properties for the sum of terms of the (s,t)-Fibonacci matrix sequence.

Theorem 8 *For $s + t \neq 1$*

$$\sum_{k=1}^{n} \mathcal{F}_k = \frac{1}{s+t-1}\begin{pmatrix} F_{n+2} + tF_{n+1} - s - t & F_{n+1} + tF_n - 1 \\ tF_{n+1} + t^2F_n - t & tF_n + t^2F_{n-1} - t \end{pmatrix}. \quad (7)$$

Proof. In this proof, we assume that $s + t \neq 1$. Let S_n be the sum of the first n terms $\mathcal{F}_k$. That is,

$$S_n = \mathcal{F}_1 + \mathcal{F}_2 + ... + \mathcal{F}_n.$$

The argument here is the same that used in the proof of the sum of the n first terms of a geometric numerical progression: since, from Lemma 7,

$$S_n\mathcal{F}_1 = \mathcal{F}_2 + \mathcal{F}_3 + ... + \mathcal{F}_{n+1},$$

then

$$S_n(\mathcal{F}_1-\mathcal{F}_0)=\mathcal{F}_{n+1}-\mathcal{F}_1.$$

Since $det(\mathcal{F}_1-\mathcal{F}_0)=1-(s+t)\neq 0$, $\mathcal{F}_1-\mathcal{F}_0$ is invertible. And, therefore,

$$S_n=(\mathcal{F}_{n+1}-\mathcal{F}_1)(\mathcal{F}_1-\mathcal{F}_0)^{-1}.$$

Note, now, that

$$\mathcal{F}_{n+1}-\mathcal{F}_1=\begin{pmatrix} F_{n+2}-s & F_{n+1}-1 \\ tF_{n+1}-t & tF_n \end{pmatrix}.$$

On the other hand,

$$\mathcal{F}_1-\mathcal{F}_0=\begin{pmatrix} s-1 & 1 \\ t & -1 \end{pmatrix}\Rightarrow(\mathcal{F}_1-\mathcal{F}_0)^{-1}=\frac{1}{s+t-1}\begin{pmatrix} 1 & 1 \\ t & 1-s \end{pmatrix}.$$

Therefore we get

$$S_n=\frac{1}{s+t-1}\begin{pmatrix} F_{n+2}+tF_{n+1}-s-t & F_{n+1}+tF_n-1 \\ tF_{n+1}+t^2F_n-t & tF_n+t^2F_{n-1}-t \end{pmatrix}$$

which completes the proof. ■

Corollary 9 *If $s=t=1$, for the classic Fibonacci sequence:*

$$\begin{aligned}\sum_{k=1}^{n} f_{k+1} &= f_{n+2}+f_{n+1}-2=f_{n+3}-2,\\ \sum_{k=1}^{n} f_k &= f_{n+1}+f_n-1=f_{n+2}-1,\\ \sum_{k=1}^{n} f_{k-1} &= f_n+f_{n-1}-1=f_{n+1}-1,\end{aligned}$$

and, if $s=2$ and $t=1$, for the classic Pell sequence:

$$\begin{aligned}\sum_{k=1}^{n} p_{k+1} &= \frac{1}{2}(p_{n+2}+p_{n+1}-3),\\ \sum_{k=1}^{n} p_k &= \frac{1}{2}(p_{n+1}+p_n-1),\\ \sum_{k=1}^{n} p_{k-1} &= \frac{1}{2}(p_n+p_{n-1}-1),\end{aligned}$$

and, if $s=1$ and $t=2$, for the classic Jacobsthal sequence:

$$\begin{aligned}\sum_{k=1}^{n} j_{k+1} &= \frac{1}{2}(j_{n+2}+2j_{n+1}-3),\\ \sum_{k=1}^{n} j_k &= \frac{1}{2}(j_{n+1}+2j_n-1),\\ \sum_{k=1}^{n} j_{k-1} &= \frac{1}{2}(j_n+2j_{n-1}-1).\end{aligned}$$

Proof. The terms a_{11}, a_{12} and a_{22} of both sides of the equality (7) are equal, the formulas is obtained. ■

By summing up the first n even terms of the (s,t)th Fibonacci matrix sequence we obtain the following theorem.

Theorem 10 *For $t-s \neq 1$ and $t+s \neq 1$,*

$$\sum_{k=1}^{n} \mathcal{F}_{2k} = \frac{1}{(s-t+1)(t+s-1)} \begin{pmatrix} F_{2n+3} - t^2 F_{2n+1} - s^2 + t^2 - t & (1-t) F_{2n+2} + stF_{2n+1} - s \\ tF_{2n+2} - t^3 F_{2n} - st & (t-t^2) F_{2n+1} + st^2 F_{2n} + t^2 - t \end{pmatrix}.$$

Proof. In this proof, we assume that $t - s \neq 1$ and $t + s \neq 1$. The proof is similar to the proof of Theorem 8, and we only show an outline of it. The sum is,

$$S_{2n} = \mathcal{F}_2 + \mathcal{F}_4 + ... + \mathcal{F}_{2n}.$$

Since $\det(\mathcal{F}_2 - \mathcal{F}_0) \neq 0$, $\mathcal{F}_2 - \mathcal{F}_0$ is invertible. By multiplying by $\mathcal{F}_2$ and, after some algebra, we get:

$$\begin{aligned} S_{2n} &= (\mathcal{F}_{2n+2} - \mathcal{F}_2)(\mathcal{F}_2 - \mathcal{F}_0)^{-1} \\ &= \frac{1}{(s-t+1)(t+s-1)} \begin{pmatrix} F_{2n+3} - s^2 - t & F_{2n+2} - s \\ tF_{2n+2} - st & tF_{2n+1} - t \end{pmatrix} \begin{pmatrix} 1-t & s \\ st & 1-t-s^2 \end{pmatrix} \\ &= \frac{1}{(s-t+1)(t+s-1)} \begin{pmatrix} F_{2n+3} - t^2 F_{2n+1} - s^2 + t^2 - t & (1-t) F_{2n+2} + stF_{2n+1} - s \\ tF_{2n+2} - t^3 F_{2n} - st & (t-t^2) F_{2n+1} + st^2 F_{2n} + t^2 - t \end{pmatrix} \end{aligned}$$

from where the result is obtained. ■

Corollary 11 *If $s = t = 1$, for the classic Fibonacci sequence:*

$$\begin{aligned} \sum_{k=1}^{n} f_{2k+1} &= f_{2n+3} - f_{2n+1} - 1 \\ &= f_{2n+2} - 1, \\ \sum_{k=1}^{n} f_{2k} &= f_{2n+1} - 1, \\ \sum_{k=1}^{n} f_{2k-1} &= f_{2n}, \end{aligned}$$

and, if $s = 2$ and $t = 1$, for the classic Pell sequence:

$$\begin{aligned} \sum_{k=1}^{n} p_{2k+1} &= \frac{1}{4}(p_{2n+3} - p_{2n+1} - 4) \\ &= \frac{1}{2}(p_{2n+2} - 2), \\ \sum_{k=1}^{n} p_{2k} &= \frac{1}{2}(p_{2n+1} - 1), \\ \sum_{k=1}^{n} p_{2k-1} &= \frac{1}{2} p_{2n}. \end{aligned}$$

It is seen that in a similar way, many formulas for partial sums of the (s,t)th Fibonacci matrix sequence may be obtained and particularized for different values of s and t.

Let x be a non-null real number. Next Theorem gives us the value for the sum of the first (s,t)th Fibonacci matrices with weights x^{-k}.

Theorem 12 *For each non-vanishing real number x :*

$$\sum_{k=1}^{n} \frac{\mathcal{F}_k}{x^k} = \frac{1}{x^2 - sx - t} \begin{pmatrix} -\frac{F_{n+2}}{x^{n-1}} - t\frac{F_{n+1}}{x^n} + sx + t & -x\frac{F_{n+1}}{x^n} - t\frac{F_n}{x^n} + x \\ -t\frac{F_{n+1}}{x^{n-1}} + xt - t^2\frac{F_n}{x^n} & -t\frac{F_n}{x^{n-1}} - t^2\frac{F_{n-1}}{x^n} + t \end{pmatrix}.$$

Proof. The proof is similar to the proof of Theorem 8. And we only consider the sum $S_n = \sum_{k=1}^{n} \frac{\mathcal{F}_k}{x^k}$, and by using elementary matrix algebra we have desired result. ■

Now, we will obtain a closed expression for $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{F_k}{x^k}$.

Corollary 13 *For each real number, such that $x > \frac{s+\sqrt{s^2+4t}}{2}$:*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{F_k}{x^k} = \frac{x}{x^2 - sx - t}. \tag{8}$$

Proof. For $t = 1$, eq. (8) is known as Livio's formula [12]. It should be noted that the denominator of the right-hand side of Livio's formula is precisely the characteristic sth Fibonacci polynomial. Now, and also by using elementary algebra, we will obtain a closed expression for $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{F_k}{x^k}$. The proof of (8) is based in the so-called Binet's formula for the (s,t)th Fibonacci sequence. From Theorem 3 and 12, we have

$$\lim_{n\to\infty} \frac{F_n}{x^n} = \frac{\left(\frac{\alpha}{x}\right)^n - \left(\frac{\beta}{x}\right)^n}{\alpha - \beta} = 0,$$

since $x > \alpha$. And, therefore

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{F_k}{x^k} = \frac{x}{x^2 - sx - t}.$$

■

Particular cases are:

- If $s = t = 1$, for the classic Fibonacci sequence is obtained:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{f_k}{x^k} = \frac{x}{x^2 - x - 1},$$

which for $x = 10$ gives: $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{f_k}{10^k} = \frac{10}{89}$ [4].

- If $s = 2$ and $t = 1$, for the classic Pell sequence appears:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{p_k}{x^k} = \frac{x}{x^2 - 2x - 1},$$

which for $x = 10$ gives: $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{p_k}{10^k} = \frac{10}{79}$ [4].

- If $s = 1$ and $t = 2$, for the classic Jacobsthal sequence results:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{j_k}{x^k} = \frac{x}{x^2 - x - 2},$$

which for $x = 10$ gives: $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{j_k}{10^k} = \frac{10}{88}$.

• If $s = 3$ and $t = -2$, for the classic Mersenne sequence is obtained:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{m_k}{x^k} = \frac{x}{x^2 - 3x + 2},$$

which for $x = 10$ gives: $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{j_k}{10^k} = \frac{10}{72}$.

3.2 Generating function for the (s,t)th Fibonacci matrix sequence

In this section, the generating functions for the Fibonacci, Pell, Jacobsthal, and Mersenne sequences are given. As a result, these sequences are seen as the coefficients of the power series of the corresponding generating function.

Let us suppose that (s,t)th Fibonacci matrices are the coefficients of a potential series centered at the origin, and let us consider the corresponding analytic function $A(x)$. The function defined in such a way is called the generating function of the (s,t)th Fibonacci matrix sequences. So,

$$A(x) = \mathcal{F}_0 + \mathcal{F}_1 x + \mathcal{F}_2 x^2 + \ldots + \mathcal{F}_n x^n + \ldots.$$

And then,

$$sA(x)\,x = s\mathcal{F}_0 x + s\mathcal{F}_1 x^2 + s\mathcal{F}_2 x^3 + \ldots + s\mathcal{F}_n x^{n+1} + \ldots,$$

$$tA(x)\,x^2 = t\mathcal{F}_0 x^2 + t\mathcal{F}_1 x^3 + t\mathcal{F}_2 x^4 + \ldots + t\mathcal{F}_n x^{n+2} + \ldots.$$

From where, since $\mathcal{F}_{n+1} = s\mathcal{F}_n + t\mathcal{F}_{n-1}$, $\mathcal{F}_0 = I$, and $\mathcal{F}_1 = \begin{pmatrix} s & 1 \\ t & 0 \end{pmatrix}$, it is obtained

$$A(x)\left[1 - sx - tx^2\right] = \mathcal{F}_0 + \mathcal{F}_1 x - s\mathcal{F}_0 x,$$

and, therefore, since the term a_{12} of both sides are equal, we have the following particular cases:

• If $s = t = 1$, the generating function for the Fibonacci sequence $\{f_n\}_{n\in\mathbb{N}}$ is

$$A(x) = \frac{x}{1 - x - x^2}.$$

• If $s = 2$ and $t = 1$, the generating function for the Pell sequence $\{p_n\}_{n\in\mathbb{N}}$ is

$$A(x) = \frac{x}{1 - 2x - x^2}.$$

• If $s = 1$ and $t = 2$, the generating function for the Jacobsthal sequence $\{j_n\}_{n\in\mathbb{N}}$ is

$$A(x) = \frac{x}{1 - x - 2x^2}.$$

• If $s = 3$ and $t = -2$, the generating function for the Mersenne Sequence is

$$A(x) = \frac{x}{1 - 3x + 2x^2}$$

In [17], the dynamic behaviour of the one-dimensional family of maps $A_{a,b}(x) = 1/(1 - ax - bx^2)$ is examined, for specific values of the control parameters a and b. This is the generating function of the generalized Fibonacci sequence described in the (1). Also, that paper it is observed that, as the parameters change, the behaviour of the maps progresses from periodicity through bifurcations to a state of chaos. Period doubling bifurcations and periodic windows are visualised, in a manner similar to the logistic map.

4 Conclusions

New (s,t)th Fibonacci and Fibonacci matrix sequences have been introduced and studied. Many of the properties of these sequences are proved.

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