On toughness and fractional f-factors*

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Abstract

In this paper, we consider the relationship between the toughness and the existence of fractional f-factors. It is proved that a graph G has a fractional f-factor if $t(G) \geq \frac{b^2+b}{a} - \frac{b+1}{b}$. Furthermore, we show that the result is best possible in some sense.

Keywords. toughness; fractional f-factor; fractional matching;
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1 Preliminaries

The graphs considered here will be finite undirected graph which may have multiple edges but no loops. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. Let S and T be two disjoint subsets of $V(G)$. $E_G(S,T)$ denotes the set of edges of G having one vertex in S and the other in T. For a vertex $x \in V(G)$, we write $N_G(x)$ for the set of vertices of $V(G)$ adjacent to x and use $d_G(x)$ and $\delta(G)$ for the degree and minimum degree of G. A subset S of $V(G)$ is called a covering set (an independent set) of G if every edge of G is incident with at least (at most) one vertex of S. We use $G[S]$ and $G-S$ to denote the subgraph of G induced by S and $V(G) \setminus S$.

Let g and f be two integer-valued functions defined on $V(G)$. A spanning subgraph F of G is called a (g,f)-factor if $g(x) \leq d_F(x) \leq f(x)$ holds for all $x \in V(F)$. A (g,f)-factor is called a k-factor if $g(x) = f(x) = k$. Let $h : E(G) \rightarrow [0,1]$ be a function. A function h is called a fractional (g,f)-factor if and only if $g(x) \leq h(E_x) \leq f(x)$ holds for any vertex $x \in V(G)$, where $h(E_x) = \sum_{e \in E_x} h(e)$, $E_x = \{e \in E(G) | e \text{ is incident with } x \text{ in } E(G)\}$. If $g(x) = f(x)$ or $g(x) = f(x) = k$, then a fractional (g,f)-factor is called

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a fractional f-factor or a fractional k-factor. In particular, a fractional [0,1]-factor is also called a fractional matching and a fractional 1-factor is also called a fractional perfect matching[9,10]. Other terminologies and notations not defined here can be found in [2,10].

A graph is t-tough if for any $S \subseteq V(G)$ and $\omega(G-S) > 1$, we have $|S| \geq t\omega(G-S)$ holds where $\omega(G-S)$ denotes the number of components of $G-S$. A complete graph is t-tough for any positive real number t. If G is not complete, there exists a largest t such that G is t-tough. This number is called the toughness of G and denoted by $t(G)$. We define $t(K_n) = \infty$. If G is not complete, $t(G) = \min\{\frac{|S|}{\omega(G-S)} | S \subseteq V(G), \omega(G-S) \geq 2\}$. The toughness of a graph was first introduced by Chvátal in [4]. Since then, much work has been contributed to the relations between toughness and factors of a graph. The following result confirmed a conjecture stated by Chvátal.

Lemma 1.1 *Let G be a graph. If G is k-tough, $|V(G)| \geq k+1$ and $k|V(G)|$ is even, then G has a k-factor.*

The result is sharp since for any positive real number ϵ, there exists a graph G that has no k-factor with $t(G) \geq k - \epsilon$[5]. P. Katerinis studied toughness and the existence of f-factors and $[a,b]$-factors[6]. Recently many authors are studying fractional factors. In [8] G.Liu and L. Zhang discussed the sufficient condition of fractional k-factors with $k \geq 1$ related to toughness.

Lemma 1.2 *Let G be a graph with $|V(G)| \geq 2$. Then G has a fractional 1-factor if $t(G) \geq 1$.*

Lemma 1.3 *Let $k \geq 2$ be an integer. A graph G has a fractional k-factor if $t(G) \geq k - \frac{1}{k}$.*

J.Cai and G.Liu gave a result of fractional f-factors and Stability Number[3]. In this paper, we consider the relationship between the toughness and the existence of fractional f-factors.

Theorem 1.4 *Let G be a graph and f an integer-valued function on $V(G)$ satisfying $a \leq f(x) \leq b$ with $1 \leq a \leq b$ and $b \geq 2$ for all $x \in V(G)$. If $t(G) \geq \frac{b^2+b}{a} - \frac{b+1}{b}$, then G has a fractional f-factor.*

Obviously, we can obtain Lemma 1.3 with $a = f(x) = b$ for all $x \in V(G)$. From the example of [8], our result is also sharp in the sense of $f(x) = k$ for all $x \in V(G)$.

2 Proof of theorem

In [1] R.P. Anstee gave a necessary and sufficient condition for a graph to have a fractional (g, f)-factor which Liu and Zhang gave a new proof[7].

Lemma 2.1 *A graph G has a fractional f-factor if and only if for any subset $S \subseteq V(G)$,*

$$f(S) - f(T) + d_{G-S}(T) \geq 0,$$

where $T = \{x \in V(G) \setminus S, d_{G-S}(x) \leq f(x) - 1\}$.

To prove the result, the following lemmas are also needed.

Lemma 2.2[4] *If a graph G is not complete, then $t(G) \leq \frac{1}{2}\delta(G)$.*

Lemma 2.3[8] *Let G be a graph and let $H = G[T]$ such that $d_G(x) = k - 1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k where $T \subseteq V(G)$ and $k \geq 2$. Then there exists an independent set I and the covering set $C = V(H) \setminus I$ of H satisfying*

$$|V(H)| \leq (k - \frac{1}{k+1})|I|, \quad |C| \leq (k - 1 - \frac{1}{k+1})|I|.$$

Lemma 2.4[8] *Let G be a graph and let $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k - 1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. let $T_1, \cdots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k - 2$ in G, then G has a maximal independent set I and a covering set $C = V(H) \setminus I$ such that*

$$\sum_{j=1}^{k-1}(k - j)c_j \leq \sum_{j=1}^{k-1}(k - 2)(k - j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for every $j = 1, \cdots, k - 1$.

Proof of Theorem 1.4. Suppose, by the contrary, that there exists an integer-valued function f satisfying all the conditions of the theorem, but G has no fractional f-factors. From Lemma 2.1 there exists a subset S of $V(G)$ such that

$$f(T) - d_{G-S}(T) > f(S), \tag{1}$$

where $T = \{x \in V(G)\setminus S | d_{G-S}(x) \leq f(x) - 1\}$. Obviously, $d_{G-S}(x) \leq b-1$ for all $x \in T$. By Lemma 2.2, we have $\delta(G) \geq 2t(G) \geq 2(\frac{b^2+b}{a} - \frac{b+1}{b}) \geq b+1$. Therefore $S \neq \emptyset$ by (1). Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_b and let $T_0 = \{x \in V(H') | d_{G-S}(x) = 0\}$. Let H be the subgraph obtained from $H' - T_0$ by deleting those components isomorphic to K_b.

If $|V(H)| = 0$, then $a|S| \leq f(S) < f(T) - d_{G-S}(T) \leq b|T_0| + bl$ namely $1 \leq |S| < \frac{b}{a}(|T_0| + l)$. Hence $l + |T_0| > \frac{a}{b}$ and $\omega(G - S) \geq l + |T_0|$. Clearly $l + |T_0| \geq 1$. If $l + |T_0| > 1$ or $\omega(G - S) > l + |T_0|$, then $\omega(G - S) > 1$ and $t(G) \leq \frac{|S|}{\omega(G-S)} < \frac{\frac{b}{a}(|T_0|+l)}{l+|T_0|} = \frac{b}{a}$. This contradicts that $t(G) \geq \frac{b^2+b}{a} - \frac{b+1}{b} >$

$\frac{b}{a}$. If $\omega(G-S) = l + |T_0| = 1$, then $d_{G-S}(x) = b-1$ or $d_{G-S}(x) = 0$ for $x \in V(G) \setminus S$. Since $d_{G-S}(x) + |S| \geq d_G(x) \geq \delta(G) \geq 2t(G)$, we have $|S| \geq 2t(G) - (b-1) \geq t(G) > \frac{b}{a} = \frac{b}{a}(|T_0| + l)$, a contradiction.

Now we consider that $|V(H)| > 0$ and $\delta(H) \geq 1$. Let $H = H_1 \cup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(x) = b-1$ for every vertex $x \in V(H_1)$ and $H_2 = H - H_1$. By Lemma 2.3, H_1 has a maximum independent set I_1 and the covering set $C_1 = V(H_1) - I_1$ such that

$$|V(H_1)| \leq (b - \frac{1}{b+1})|I_1| \tag{2}$$

and

$$|C_1| \leq (b - 1 - \frac{1}{b+1})|I_1|. \tag{3}$$

On the other hand, it is obvious that $\delta(H_2) \geq 1$ and $\Delta(H_2) \leq b-1$. Let $T_j = \{x \in V(H_2) | d_{G-S}(x) = j\}$ for $1 \leq j \leq b-1$. By the definition of H and H_2 we can also see that each component of H_2 has a vertex of degree at most $b-2$ in $G-S$. According to Lemma 2.4, H_2 has a maximal independent set I_2 and a covering set $C_2 = V(H_2) - I_2$ such that

$$\sum_{j=1}^{b-1}(b-j)c_j \leq \sum_{j=1}^{b-1}(b-2)(b-j)i_j, \tag{4}$$

where $c_j = |C_2 \cap T_j|$ and $i_j = |I_2 \cap T_j|$ for every $j = 1, \cdots, b-1$. Set $W = V(G) - S - T$ and $U = S \cup C_1 \cup C_2 \cup (N_G(I_2) \cap W)$. Then since $|C_2| + |(N_G(I_2) \cap W)| \leq \sum_{j=1}^{b-1} j i_j$ we obtain

$$|U| \leq |S| + |C_1| + \sum_{j=1}^{b-1} j i_j \tag{5}$$

and

$$\omega(G-U) \geq t_0 + l + |I_1| + \sum_{j=1}^{b-1} i_j, \tag{6}$$

where $t_0 = |T_0|$. Let $t(G) = t$. Then when $\omega(G-U) > 1$, we have

$$|U| \geq t\omega(G-U). \tag{7}$$

In addition, the above inequation also holds when $\omega(G-U) = 1$, since from Lemma 2.2, $|U| \geq d_{G-S}(x) + |S| \geq d_G(x) \geq 2t > t\omega(G-U)$ for any $x \in T$. By (5), (6) and (7) we have

$$|S| + |C_1| \geq \sum_{j=1}^{b-1}(t-j)i_j + t(t_0 + l) + t|I_1|. \tag{8}$$

From (1), $b|T| - d_{G-S}(T) > a|S|$holds. Then

$$\begin{aligned} & bt_0 + bl + |V(H_1)| + \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j + a|C_1| \\ > \ & a(|S| + |C_1|) \\ \geq \ & \sum_{j=1}^{b-1}(at - aj)i_j + at(t_0 + l) + at|I_1|. \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{j=1}^{b-1}(b-j)c_j + |V(H_1)| + a|C_1| \\ > \ & \sum_{j=1}^{b-1}(at - aj - b + j)i_j + (at - b)(t_0 + l) + at|I_1| \\ \geq \ & \sum_{j=1}^{b-1}(at - aj - b + j)i_j + at|I_1|. \end{aligned}$$

By (2) and (3)

$$\begin{aligned} |V(H_1)| + a|C_1| \ & \leq \ (b - \frac{1}{b+1})|I_1| + a(b - 1 - \frac{1}{b+1})|I_1| \\ & = \ (b - \frac{1}{b+1} + ab - a - \frac{a}{b+1})|I_1|. \end{aligned}$$

Combining the above two inequations with (4) we have

$$\begin{aligned} & \sum_{j=1}^{b-1}(b-2)(b-j)i_j + (b - \frac{1}{b+1} + ab - a - \frac{a}{b+1})|I_1| \\ > \ & \sum_{j=1}^{b-1}(at - aj - b + j)i_j + at|I_1|. \end{aligned}$$

Thus at least one of the following two cases must hold.

case 1. There is at least one j such that

$$(b-2)(b-j) > at - aj - b + j.$$

Then $at < (b-2)(b-j) + aj + b - j = b(b-2) + (a-b+1)j + b$.

If $a = b$, then $at < (a-2)a + j + a \leq (a-2)a + a - 1 + a = a^2 - 1$, contradicts to $t \geq a - \frac{1}{a}$. If $a < b$, then $at < (b-2)b + (a-b+1) + b$.

Because of $\frac{(b-2)b+(a-b+1)+b}{a} \leq \frac{b^2+b}{a} - \frac{b+1}{b}$, we have $t < \frac{b^2+b}{a} - \frac{b+1}{b}$. It is also a contradiction.

case 2. $b - \frac{1}{b+1} + ab - a - \frac{a}{b+1} > at$.

In this case we have

$$t < \frac{b}{a} + b - \frac{1+a}{a(b+1)} - 1 < \frac{b}{a} + b - \frac{1}{b} - 1 \leq \frac{b^2+b}{a} - \frac{b+1}{b}.$$

This also contradicts the condition of Theorem. □

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