On Graceful Generalized Spiders and Caterpillars

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Abstract

In this paper, we, by means of Rosa's α-labelling and k-graceful labelling, prove the generalized spiders, generalized caterpillars and generalized path-block chain the gracefulness to be graceful under some conditions. The some of results are stronger than that obtained in [4].

Mathematics subject classification (MSC2000):05C78

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1. Introduction

Let G be a graph with the vertex set V and edge set E. If f is an injection from V into $\{0,1,2,\cdots,|E|\}$ such that an edge uv label is $|f(u)-f(v)|$ and any two edges labels are distinct, then f is said to be a *graceful labelling* of G and, one often say G is *graceful*. Rosa[3] first introduced this graceful labelling as well as a number of other labelling for decomposing a complete graph into isomorphic subgraphs. Next, Rosa, Ringel and Kötzig have proposed a conjecture stated as this: *Every tree is Skolem-graceful* in [3], or *all tree are perfect* in [1]. Many results have been obtained in [3], [4], [5] and [6], but the conjecture of the perfect tree still is on open. The graceful problem seems to be difficult even for some special graphs. In [3], the author mentioned two important labellings: One is the *m-α-labelling* f defined by Rosa which is a graceful labelling with the additional property that there is an integer m such that for each edge $uv\in E(G)$ either $f(u)\le m<f(v)$ or $f(v)\le m<f(u)$. Another one is called the *m-graceful labelling* (or arbitrary graceful) defined by a proper vertex labelling f of a connected graph G from $V(G)$ into $\{0,1,2,\cdots,|E(G)|+m-1\}$ satisfies $\{|f(u)-f(v)|\mid uv\in E(G)\}=\{m,m+1,\cdots,|E(G)|+m-1\}$.

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All graphs mentioned are simple and continuous. The other terminology can be found in [1].

For $1 \leq i \leq m$, let $G_i(u_i, u_{i+1})$ be a connected graph of order more that two, and there are two vertices u_i and u_{i+1} of G_i that are called the initial vertex and terminal vertex of G_i, respectively. A *graph-block chain* H_m is defined by identifying the terminal vertex u_{i+1} of $G_i(u_i, u_{i+1})$ with the initial vertex u_{i+1} of $G_{i+1}(u_{i+1}, u_{i+2})$ into a new vertex, still write by u_{i+1}, after i runs from 1 to m, we get this graph H_m which looks like a chain, denoted by

$$\begin{aligned} H_{s_k} &= \oplus_{i=1}^{s_k} G_i(u_i, u_{i+1}) \\ &= G_1(u_1, u_2) \oplus G_2(u_2, u_3) \oplus G_3(u_3, u_4) \oplus \cdots \oplus G_{s_k}(u_{s_k}, u_{s_k+1}). \end{aligned}$$

We call u_1, u_{s_k+1} and u_i the *initial vertex*, *terminal vertex* and *node vertex* of the graph-block chain $\oplus_{i=1}^{s_k} G_i(u_i, u_{i+1})$, respectively, and each $G_i(u_i, u_{i+1})$ is said a *node block* of this graph-block chain.

We will use so-called path-block graphs to construct several classes of graphs and then study their graceful property in this paper. Let u_i^k and u_{i+1}^k be two isolated vertices. A *path-block graph* $G_{t_i}(u_i^k, u_{i+1}^k)$ is defined by linking two isolated vertices u_i^k and u_{i+1}^k with t_i disjoint paths

$$P_{j,m_{ij}}^k = v_{i,j,1}^k v_{i,j,2}^k \cdots v_{i,j,m_{ij}}^k, \ m_{ij} \geq 1, \ 1 \leq j \leq t_i.$$

Here, we call u_i^k the initial vertex and u_{i+1}^k the terminal vertex of $G_{t_i}(u_i^k, u_{i+1}^k)$, respectively. Therefore,

$$|V(G_{t_i}(u_i^k, u_{i+1}^k))| = 2 + \sum_{j=1}^{t_i} m_{ij} \text{ and } |E(G_{t_i}(u_i^k, u_{i+1}^k))| = t_i + \sum_{j=1}^{t_i} m_{ij}.$$

Let s_k is a positive integer, given s_k path-block graphs $G_{t_i}(u_i^k, u_{i+1}^k)$ for $1 \leq i \leq s_k$, we have a graph-block chain $\oplus_{i=1}^{s_k} G_{t_i}(u_i^k, u_{i+1}^k)$, for speaking conveniently and imaginably, call it the *s_k-path-block chain* and, denote it by H_{s_k}. In this s_k-path-block chain, here, u_1^k is the *initial vertex*, $u_{s_k+1}^k$ is the *terminal vertex* and, $u_{s_l+1}^k$ is the *node vertex* for $2 \leq l \leq s_k$, and each $G_{t_i}(u_i^k, u_{i+1}^k)$ is the node block. The numbers of all vertices and edges of H_{s_k} are:

$$|V(H_{s_k})| = s_k + 1 + \sum_{i=1}^{s_k} \sum_{j=1}^{t_i} m_{ij} \text{ and } |E(H_{s_k})| = \sum_{i=1}^{s_k} \Big(t_i + \sum_{j=1}^{t_i} m_{ij}\Big).$$

Some special s_k-path-block graph have been proved to be graceful in [4]. We observe some graceful graphs of large order that could be assembled by some s_k-path-block chains together.

Definition 1. Given n (> 1) s_k-path-block chains H_{s_k} with $1 \le k \le n$, *a generalized path-block chain* P_n^* is obtained by linking each pair of the initial vertices $u_1^k \in V(H_{s_k})$ and $u_1^{k+1} \in V(H_{s_{k+1}})$ with an edge $u_1^k u_1^{k+1}$ for $1 \le k \le n-1$.

A tree T is called a *spider* if it has a unique vertex u_0 of degree more than two, namely, $\mathrm{d}(u) \le 2$ for all $u \in V(T) \setminus \{u_0\}$ and, we speak of u_0 to be the center of the spider T. A *caterpillar* is a such tree when deleting all vertices of degree one from this tree the remaining graph just is a path, and this path is said the *spine* of the caterpillar. As known, any caterpillar is graceful.

Definition 2. Given m (> 2) s_k-path-block chains H_{s_k} for $1 \le k \le m$, *a generalized spider* S_m^* with the center u_0 is obtained by linking u_0 to the initial vertex u_1^k of H_{s_k} with an edge $u_0 u_1^k$ for $1 \le k \le m$. Each H_{s_k} is called a leg of S_m^*.

Definition 3. Given m (> 1) s_k-path-block chains H_{s_k} for $1 \le k \le m$. Let T be a caterpillar with m vertices of degree one, say $v_1, v_2, \cdots, v_m$. A *generalized caterpillar* T_m^* is obtained by identifying the vertex v_k of T with the initial vertex u_1^k of H_{s_k} into a new vertex for $1 \le k \le m$. Every H_{s_k} is called a leg of T_m^*, the spine of T is still called the spine of T_m^*.

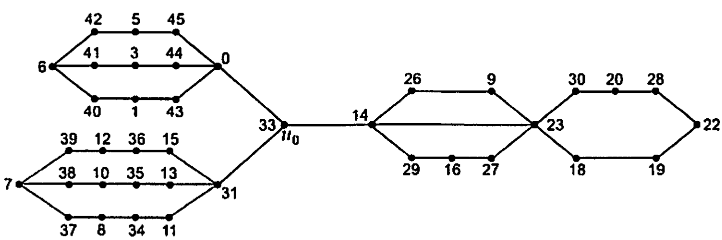

Figure 1. A graceful generalized spider S_3^* with three different legs and the center u_0.

2. Main Results

Lemma 1. *Let f be a graceful labelling of G and let n be a positive integer. Then both $g = |E(G)| - f$ and $h = n + f$ are two graceful labellings of G too.*

For any edge $uv \in E(G)$, it is easy to see that

$$|h(u) - h(v)| = |f(u) - f(v)| = |g(u) - g(v)|.$$

One often say $g = |E(G)| - f$ to be the complementary labelling of f and, $h = n + f$ to be a n-floating labelling of f.

Lemma 2. *If G has a m-α-labelling, then*
(i) G is bipartite;
(ii) G is m-graceful for all positive integers m.

Lemma 3. *Let two graphs G_1 with m_1 edges and G_2 with m_2 edges are connected and disjoint in each other, and let f_i be a graceful labelling of G_i for $i = 1, 2$. If f_1 is a m-α-labelling of G_1 for $m = \frac{m_1}{2}$ when m_1 is even and $m = \frac{m_1-1}{2}$ when m_1 is odd and, there are $f_1(u_0) = m$ ($u_0 \in V(G_1)$) and $f_2(v_0) = 0$ ($v_0 \in V(G_2)$), then the graph $G_1 \oplus G_2$ obtained by identifying u_0 with v_0 is graceful.*

Proof. There is a subset V_1 of $V(G_1)$ so that any $u \in V_1$ satisfies that $f_1(u) \leq m$ since f_1 is an m-α-labelling of G_1. Therefore, we can construct a labelling f of $G_1 \oplus G_2$ as this form:

$$f(v) = \begin{cases} f_1(v), & \text{if } v \in V_1; \\ f_1(v) + m_2, & \text{if } v \in V(G_1) \setminus V_1; \\ f_2(v) + m, & \text{if } v \in V(G_2), \end{cases}$$

note that f is the combination of a m_2-graceful labelling of G_1 and a m-floating labelling of f_2. It is not difficult to verify that the graph $G_1 \oplus G_2$ is graceful under f, as desired. □

Lemma 4. *Let $H_n = \oplus_{i=1}^{n} G_i(u_i, u_{i+1})$ be a graph-block chain where $n > 2$ in which each node block $G_i(u_i, u_{i+1})$ $(1 \leq i \leq n)$ has m_i edges. If for $1 \leq i \leq n-1$, every $G_i(u_i, u_{i+1})$ has an n_i-α-labelling with $f_i(u_i) = 0$ and $f_i(u_{i+1}) = n_i$ where $n_i = \frac{m_i}{2}$ when m_i is even and $n_i = \frac{m_i-1}{2}$ when m_i is odd; and if G_n has a graceful labelling f_n with $f_n(u_n) = 0$. Then H_n is graceful.*

Proof. Using the mathematical induction. When identifying the terminal vertex u_n of the node block $G_{n-1}(u_{n-1}, u_n)$ with the initial vertex u_n of the node block $G_n(u_n, u_{n+1})$, the graph-block chain $G_{n-1} \oplus G_n$ is graceful by Lemma 3, and there is a graceful labelling g such that $g(u_{n-1}) = 0$ for the initial vertex u_{n-1} of the graph-block chain $G_{n-1} \oplus G_n$. Suppose the lemma holds for $G = G_2 \oplus G_3 \oplus \cdots \oplus G_n$, so this graph has a graceful labelling f so that $f(u_2) = 0$. Therefore, the graph $G_1 \oplus G$ obtained by identifying the terminal vertex u_2 of $G_1(u_1, u_2)$ with the initial vertex u_2 of the graph-block chain G is graceful by Lemma 3. The proof is completed. □

We come to consider a special class of s_k-path-block chains, that is, each node block $G_b(u_i^k, u_{i+1}^k)$ is defined by linking two isolated vertices u_i^k and u_{i+1}^k by b disjoint paths with the same length $a-2$, as the form

$$P_{j,a-2}^k = v_{i,j,1}^k v_{i,j,2}^k v_{i,j,3}^k \cdots v_{i,j,a-1}^k, \ a \geq 2, \ 1 \leq j \leq b. \tag{1}$$

To avoid confusion, we use the symbol $G_b^a(u_i^k, u_{i+1}^k)$ instead of $G_b(u_i^k, u_{i+1}^k)$, and refer $G_b^a(u_i^k, u_{i+1}^k)$ as the (a,b)-regular path-block.

Lemma 5. *Let $m = \frac{ab}{2}$ where a is of even and b is of odd, then each (a,b)-regular path-block $G_b^a(u_i^k, u_{i+1}^k)$ has an m-α-labelling f so that $f(u_i^k) = 0$ and $f(u_{i+1}^k) = m$.*

Proof. First case, we deal with the case of $b = 1$. By the notation of the form (1), we give directly a labelling f to $G_1^a(u_i^k, u_{i+1}^k)$ as follows.

(i) Let $f(u_i^k) = 0$ and $f(u_{i+1}^k) = \frac{a}{2}$;

(ii) Let $f(v_{i,j,2t-1}^k) = a + 1 - t$ for $1 \le t \le \frac{a}{2}$ and $f(v_{i,j,2t}^k) = t$ for $1 \le t \le \frac{a}{2} - 1$.

It is not difficult to see f is a $\frac{a}{2}$-α-labelling of $G_1^a(u_i^k, u_{i+1}^k)$.

Second case is on $b \ge 2$, more or less, it is similar with one of $b = 1$. We set a labelling g to $G_b^a(u_i^k, u_{i+1}^k)$ as this:

(iii) Let $g(u_i^k) = 0$, $g(u_{i+1}^k) = \frac{ab}{2}$;

(v) For $1 \le j \le b$, set $g(v_{i,j,2t-1}^k) = (a+1-t)b - (j-1)$ for $1 \le t \le \frac{a}{2}$ and $g(v_{i,j,2t}^k) = 1 + (t+1)b - 2j$ for $1 \le t \le \frac{a}{2} - 1$.

Next, for $1 \le t \le \frac{a}{2} - 1$ we can estimate

$$g(v_{i,j,2t}^k) = 1 + (t+1)b - 2j \le \frac{ab}{2} - 1 < \frac{ab}{2}$$

and for $1 \le t \le \frac{a}{2}$ there is

$$\begin{aligned} g(v_{i,j,2t-1}^k) &= (a+1-t)b - (j-1) \ge (a + 1 - \frac{a}{2})b - (j-1) \\ &= b\Big(\frac{a}{2} + 1\Big) - j + 1 = \frac{ab}{2} + b - j + 1 \ge \frac{ab}{2} + 1 > \frac{ab}{2}, \end{aligned}$$

it shows that g is a $\frac{ab}{2}$-α-labelling of $G_b^a(u_i^k, u_{i+1}^k)$, as desired. □

Lemma 6. *Let $m = \frac{ab}{2}$ where b is of even and $a = 2r$ for an odd r. Then the (a,b)-path-block $G_b^a(u_i^k, u_{i+1}^k)$ has an m-α-labelling f so that $f(u_i^k) = 0$ and $f(u_{i+1}^k) = m$.*

Proof. By the notation of the form (1), we give directly a labelling f of $G_b^a(u_i^k, u_{i+1}^k)$ as this:

1) Let $f(u_i^k) = 0, f(u_{i+1}^k) = \frac{ab}{2}$;

2) For $1 \le j \le b$, let $f(v_{i,j,2t-1}^k) = (a+1-t)b - (j-1)$ with $1 \le t \le \frac{ab}{2}$;

3) When $r \ge 5$ and $1 \le j \le b f(v_{i,j,2}^k) = 1 + 2(b-j)$; and $f(v_{i,j,a-2}^k) = \frac{ab}{2} + 1 - 2j$;

4) For $r \ge 5$ and $2 \le j \le b$, let $f(v_{i,j,6}^k) = 2 + 2(b-j)$ and

$$f(v_{i,j,2t}^k) = \begin{cases} 2 + (t+3)b - 2j, & \text{for } 2 \le t < \frac{a}{2} - 2, t \text{ is even;} \\ 1 + (t-1)b - 2j, & \text{for } 5 \le t < \frac{a}{2} - 1, t \text{ is odd;} \end{cases}$$

5) For $r \geq 5$ and $j = 1$, let $f(v_{i,1,2t}^k) = tb$ with $2 \leq t \leq \frac{a}{2} - 2$ and t is 2 and odd; $f(v_{i,1,2t}^k) = (t+1)b - 1$, with $2 < t \leq \frac{a}{2} - 1$ and t is even.

6) For $r = 3$ and $1 \leq j \leq b-1$, let $f(v_{i,1,2}^k) = 2(b-j)$; for $1 \leq j \leq b-2$, set $f(v_{i,1,4}^k) = 3(b+1) - 2j - 4$, $f(v_{i,b-1,4}^k) = 1$, $f(v_{i,b,2}^k) = 3b - 1$, $f(v_{i,b,4}^k) = 2b$.

When $r = 1, 3$, it is easy to see this labelling is a graceful $\frac{ab}{2}$-α-labelling of $G_b^a(u_i^k, u_{i+1}^k)$.

Similarly, for $r \geq 5$, this labelling f is an injection from $V(G_b^a(u_i^k, u_{i+1}^k))$ into the set $\{0, 1, 2, \cdots, ab\}$ and the labels of all edges of $G_b^a(u_i^k, u_{i+1}^k)$ are: for $1 \leq j \leq b$,

$$|f(v_{i,j,2t-1}^k) - f(v_{i,j,2}^k)| = (a-1-t)b + j, \ t = 1, 2,$$

$$|f(v_{i,j,2t-1}^k) - f(v_{i,j,a-2}^k)| = \left(\frac{a}{2} + 1 - t\right)b + j, \ t = \frac{a}{2} - 1, \frac{a}{2}.$$

And when $2 \leq j \leq b$,

$$|f(v_{i,j,2t-1}^k) - f(v_{i,j,6}^k)| = (a-1-t)b + j - 1, \ t = 3, 4,$$

$$|f(v_{i,j,2t-1}^k) - f(v_{i,j,2t}^k)| = \begin{cases} (a-2-2t)b + j - 1, & \text{for } 2 \leq t < \frac{a}{2} - 2, \ t \text{ is even;} \\ (a+2-2t)b + j, & \text{for } 5 \leq t < \frac{a}{2} - 1, \ t \text{ is odd;} \end{cases}$$

$$|f(v_{i,j,2t+1}^k) - f(v_{i,j,2t}^k)| = \begin{cases} (a-3-2t)b + j - 1, & \text{for } 2 \leq t < \frac{a}{2} - 2, \ t \text{ is even;} \\ (a+1-2t)b + j, & \text{for } 5 \leq t < \frac{a}{2} - 1, \ t \text{ is odd;} \end{cases}$$

When $j = 1$,

$$|f(v_{i,1,2t-1}^k) - f(v_{i,1,2t}^k)| = \begin{cases} (a+1-2t)b, & \text{for } 2 \leq t \leq \frac{a}{2} - 2, \ t \text{ is 2 } \textit{and odd;} \\ (a-2t)b + 1, & \text{for } 2 < t < \frac{a}{2} - 1, \ t \text{ is even;} \end{cases}$$

$$|f(v_{i,1,2t+1}^k) - f(v_{i,1,2t}^k)| = \begin{cases} (a-2t)b, & \text{for } 2 \leq t \leq \frac{a}{2} - 2, \ t \text{ is 2 } \textit{and odd;} \\ (a-1-2t)b + 1, & \text{for } 2 < t \leq \frac{a}{2} - 1, \ t \text{ is even;} \end{cases}$$

Hence the label set of all edges of the $G_b^a(u_i^k, u_{i+1}^k)$ just is equal to $\{1, 2, \cdots, ab\}$, that means the (a, b)-regular path-block $G_b^a(u_i^k, u_{i+1}^k)$ is graceful and, f is a $\frac{ab}{2}$-α-labelling, as required. □

Theorem 7. *Given an n-path-block chain $H_n = \oplus_{i=1}^n G_{b_i}^{a_i}(u_i^k, u_{i+1}^k)$ where every path-block $G_{b_i}^{a_i}(u_i^k, u_{i+1}^k)$ is (a_i, b_i)-regular. If for each pair of a_i and b_i $(1 \leq i \leq n)$ there are that b_i is odd and a_i is even, or b_i is even and $a_i = 2r_i$ when r_i is odd, then H_n is graceful.*

By Lemma 4, 5 and 6, we are able to prove Theorem 7 above. Notice that there may be some two distinct a_s and a_t, so Theorem 7 is very stronger than the result obtained in [4].

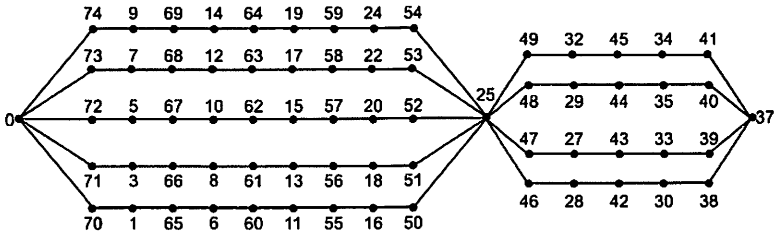

Figure 2. An 2-path-block chain H_2 has a graceful labelling.

Let H_n be a n-path-block chain $\oplus_{i=1}^{n} G_{b_i}^{a_i}(u_i^k, u_{i+1}^k)$ where every $G_{b_i}^{a_i}(u_i^k, u_{i+1}^k)$ is (a_i, b_i)-regular and, for each pair of a_i and b_i $(1 \le i \le n)$ there are that b_i is odd and a_i is even, or b_i is even and $a_i = 2r_i$ when r_i is odd, we then say that H_n satisfies "the OE-condition" in the following results.

Theorem 8. *If H_n satisfies the OE-condition, then a generalized spider S_m^* in which each leg is equal to H_n is graceful.*

Proof. By Theorem 7, $H_n = \oplus_{i=1}^{n} G_{b_i}^{a_i}(u_i^k, u_{i+1}^k)$ is graceful. Therefore,

$$r = |E(H_n)| = \sum_{i=1}^{n} a_i b_i \text{ and } s = |V(H_n)| = \sum_{i=1}^{n} (a_i - 1)b_i + n + 1.$$

H_n has the initial vertex u_1^k, the node vertices $u_2^k, \cdots, u_n^k$ and the terminal vertex u_{n+1}^k. By Lemma 4, there is an $\frac{r}{2} - \alpha$–labelling f of H_n such that $f(u_1^k) = 0$ and $f(u_{n+1}^k) = \frac{r}{2}$. Symmetrically, we can consider u_{n+1}^k to be the initial vertex and u_1^k to be the terminal vertex in H_n, so there is another $\frac{r}{2} - \alpha$–labelling f' of H_n such that $f'(u_{n+1}^k) = 0$ and $f'(u_1^k) = \frac{r}{2}$. Let g and g' be the complementary labellings of f and f' respectively, notice that $g(u_1^k) = g'(u_{n+1}^k) = r$ by the definition of the complementary labelling of a labelling.

A generalized spider S_m^* have $m(1+r)$ edges, let u_0 be its center vertex. For $1 \le k \le m$, let $v_{i,j,l}^k$ $(0 \le l \le a_i)$ be the vertices of the jth $(1 \le j \le b_i)$ path of the ith node block $G_{b_i}^{a_i}(u_i^k, u_{i+1}^k)$ $(1 \le i \le s_k)$ on the kth leg H_{s_k} of S_m^* where $v_{i,j,0}^k = u_i^k$, $v_{i,j,a_i}^k = u_{i+1}^k$. Define a labelling ϕ on the vertices set of S_m^* as follows:

$$\phi(u_{i,j,l}^1) = \begin{cases} f(u_{i,j,l}^1), & \text{if } f(u_{i,j,l}^1) \le \frac{r}{2}; \\ f(u_{i,j,l}^1) + (m-1)r + m, & \text{otherwise.} \end{cases}$$

$$\phi(u_{i,j,l}^2) = \begin{cases} f'(u_{i,j,l}^2) + \frac{1}{2}r + 1, & \text{if } f'(u_{i,j,l}^2) \le \frac{r}{2}; \\ f'(u_{i,j,l}^2) + (m - \frac{3}{2})r + m, & \text{otherwise.} \end{cases}$$

$$\phi(u_{i,j,l}^{2t+1}) = \begin{cases} f(u_{i,j,l}^{2t+1}) + tr + t + 1, & \text{if } f(u_{i,j,l}^{2t+1}) \le \frac{r}{2}; \\ f(u_{i,j,l}^{2t+1}) + (m-t-1)r + m - t, & \text{otherwise.} \end{cases}$$

with t is odd, $1 < 2t+1 \le m$.

$$\phi(u_{i,j,l}^{2t+2}) = \begin{cases} f'(u_{i,j,l}^{2t+2}) + \frac{1}{2}(2t+1)r + t + 2, & \text{if } f'(u_{i,j,l}^{2t+2}) \le \frac{r}{2}; \\ f'(u_{i,j,l}^{2t+2}) + (m-t-\frac{3}{2})r + m - t, & \text{otherwise.} \end{cases}$$

with t is odd, $1 < 2t+2 \le m$.

$$\phi(u_{i,j,l}^{2t+1}) = \begin{cases} g(u_{i,j,l}^{2t+1}) + tr + t + 2, & \text{if } g(u_{i,j,l}^{2t+1}) \le \frac{r}{2}; \\ g(u_{i,j,l}^{2t+1}) + (m-t-1)r + m - t + 1, & \text{otherwise.} \end{cases}$$

with t is even, $1 < 2t+1 \le m$.

$$\phi(u_{i,j,l}^{2t+2}) = \begin{cases} g'(u_{i,j,l}^{2t+2}) + (t+\frac{1}{2})r + t + 2, & \text{if } g'(u_{i,j,l}^{2t+2}) \le \frac{r}{2}; \\ g'(u_{i,j,l}^{2t+2}) + (m-t-\frac{3}{2})r + m - t, & \text{otherwise.} \end{cases}$$

with t is even, $1 < 2t+2 \le m$. Only u_0 is not labelled, so we let $\phi(u_0) = (m-1)r + m$.

It is straight forward to verify that ϕ is a mapping from the vertex set of S_m^* into $\{1, 2, \cdots, ms+1\}$. The labels set of all edges of S_m^* is $\{1, 2, \cdots, m(1+r)\}$. Therefore, S_m^* is a graceful, as desired. □

Corollary 9. *Let S_m^* be a generalized spider with m legs H_{s_k} for $1 \le k \le m$. If each leg H_{s_k} $(1 \le k \le m)$ satisfies the OE-condition and, any two legs of S_m^* have the same number of edges, then S_m^* is graceful.*

Theorem 10. *Let P_t^* be a generalized path-block chain with t s_k-path-block chains H_{s_k} $(1 \le k \le t)$. If each H_{s_k} satisfies the OE-condition and, $|E(H_{s_k})| = |E(H_{s_{k+1}})|$ for any odd k with respect to $1 \le k \le t-1$, then P_t^* is graceful.*

Proof. By theorem 7, every leg $H_{s_k} = \oplus_{i=1}^{s_k} G_{b_i}^{a_i}(u_i^k, u_{i+1}^k)$ of P_t^* is graceful for $1 \le k \le t$, and H_{s_k} has $r_k = \sum_{i=1}^{s_k} a_i b_i$ edges. By Lemma 5 and Lemma 6, for $1 \le k \le t$ and k is odd, there is a $\frac{r_k}{2}$-α-labelling f_k of H_{s_k} such that $f_k(u_1^k) = 0$ and $f_k(u_{s_k+1}^k) = \frac{r_k}{2}$; and for $1 < k \le t$ and k is even, there is a $\frac{r_k}{2}$-α-labelling g_k of H_{s_k} such that $g_k(u_{s_k+1}^k) = r_k$ and $g_k(u_1^k) = \frac{r_k}{2}$.

Let $v^k_{i,j,l}$ $(0 \le l \le a_i)$ be the vertices of the jth $(1 \le j \le b_i)$ path of the ith node block $G^{a_i}_{b_i}(u^k_i, u^k_{i+1})$ $(1 \le i \le s_k)$ on the kth leg H_{s_k} of P^*_t where $v^k_{i,j,0} = u^k_i$ and $v^k_{i,j,a_i} = u^k_{i+1}$. We come to define a labelling ψ of P^*_t as follows:

$$\psi(u^k_{i,j,l}) = \begin{cases} f_k(u^k_{i,j,l}) + \frac{1}{2}\sum\limits_{x=1}^{k-1} r_x + \frac{1}{2}(k-1), & \text{if } f_k(u^k_{i,j,l}) \le \frac{r_k}{2}; \\ f_k(u^k_{i,j,l}) + \sum\limits_{x=k+1}^{t} r_x + \frac{1}{2}\sum\limits_{x=1}^{k-1} r_x + \frac{2t-k-1}{2}, & \text{otherwise.} \end{cases}$$

with $1 \le k \le t$, k is odd.

$$\psi(u^k_{i,j,l}) = \begin{cases} g_k(u^k_{i,j,l}) + \frac{1}{2}\sum\limits_{x=1}^{k-1} r_x + \frac{1}{2}k, & \text{if } g_k(u^k_{i,j,l}) \le \frac{r_k}{2}; \\ g_k(u^k_{i,j,l}) + \sum\limits_{x=k+1}^{t} r_x + \frac{1}{2}\sum\limits_{x=1}^{k-1} r_x + \frac{2t-k}{2}, & \text{otherwise.} \end{cases}$$

with $1 < k \le t$, k is even.

Let $W_1 = \sum\limits_{x=1}^{t} r_x$, $W_k = \sum\limits_{x=k}^{t} r_x$ and $W_{k+1} = \sum\limits_{x=k+1}^{t} r_x$. Under ψ and for $1 \le k \le t$, when k is odd, the edge labels set of H_{s_k} is $\{W_k + t - k, W_k + t - k - 1, \cdots, W_{k+1} + t - k + 1\}$; and when k is oven, the edge labels set of H_{s_k} is $\{W_{k+1} + t - k + 1, W_{k+1} + t - k + 2, \cdots, W_k + t - k\}$. Therefore, the edge labels set of P^*_t is $\{W_1 + t - 1, W_1 + t - 2, \cdots, 2, 1\}$. Hence, P^*_t is graceful. □

Theorem 11. *Let T^*_t be a generalized caterpillar with t legs H_{s_k} that satisfies the OE-condition. If any two legs have the same number of edges and, any vertex w_i on the spine $P = w_1w_2\cdots w_p$ of T^*_t has the same number q of vertices of degree one, then this generalized caterpillar is graceful for $2 \le q \le 4$.*

Proof. In fact, this special T^*_t can be obtained by linking t generalized spider S^*_m. Suppose that each s_k-path-block chain $H_{s_k} = \oplus^{s_k}_{i=1} G^{a_i}_{b_i}(u^k_i, u^k_{i+1})$ has s edges, so S^*_m has $m(s+1)$ edges, and T^*_t has $[tm(s+1)+t-1]$ edges. Let $(v^k_{i,j,l})^h$ be the vertices of the ith node block $G^{a_i}_{b_i}(u^k_i, u^k_{i+1})$ of the kth leg H_{s_k} on the hth generalized spider S^*_m with the center vertex v^h_o. Let ϕ be the graceful labels of the S^*_m that is given by theorem 8, ϕ' is the complementary labels of the ϕ.

When $m = 4$, define the labelling F_1 of the T^*_t:

$$F_1((v^k_{i,j,l})^h) = \begin{cases} \phi(v^k_{i,j,l}) + \frac{h-1}{2}(4s+5), & \text{if } \phi(v^k_{i,j,l}) \le 2s+3; \\ \phi(v^k_{i,j,l}) + \frac{2t-h-1}{2}(4s+5), & \text{otherwise.} \end{cases}$$

with $1 \le h \le t$ and h is odd.

$$F_1((v^k_{i,j,l})^h) = \begin{cases} \phi'(v^k_{i,j,l}) + \frac{h}{2}(4s+5) - 2s - 1, & \text{if } \phi'(v^k_{i,j,l}) \le 2s+3; \\ \phi'(v^k_{i,j,l}) + \frac{2t-h}{2}(4s+5) - 2s - 1, & \text{otherwise.} \end{cases}$$

with $2 \le h \le t$ and h is even.

$$F_1(v^h_0) = \begin{cases} \phi(v^h_0) + (t - \frac{h+1}{2})(4s+5), & \text{if } h \text{ is odd, } 1 \le h \le t; \\ \phi'(v^h_0) + \frac{h}{2}(4s+5) - 2s - 1, & \text{if } h \text{ is even, } 2 \le h \le t. \end{cases}$$

We are easy to show that the F_1 is graceful.

To give a graceful labels of the T^*_t for $m = 2, 3$, we give directly a graceful labelling of S^*_m in the following procedure. Let f_1 be a $\frac{s}{2}$-α-labelling of $H_n = \oplus^n_{i=1} G^{a_i}_{b_i}(u^k_i, u^k_{i+1})$ such that $f_1(v^k_{1,j,0}) = \frac{s}{2}$, $f_1(v^k_{n,j,a_n}) = 0$; and f_2 be another $\frac{s}{2}$-α-labelling of $H_n = \oplus^n_{i=1} G^{a_i}_{b_i}(u^k_i, u^k_{i+1})$ such that $f_2(v^k_{1,j,0}) = 0$, $f_2(v^k_{n,j,a_n}) = \frac{s}{2}$. We have that two g_1 and g_2 are the complementary labellings of two f_1 and f_2. Now, we come to show a graceful labels ψ of the S^*_m for $m = 2, 3$ as follows:

$$\psi(v^k_{i,j,l}) = \begin{cases} f_1(v^k_{i,j,l}) + \frac{k-1}{2}(s+2), & \text{if } f_1(v^k_{i,j,l}) \le \frac{1}{2}s; \\ f_1(v^k_{i,j,l}) + (m - \frac{k+1}{2})s + m - \frac{k-1}{2}, & \text{otherwise.} \end{cases}$$

for $1 \le k \le m$, k is odd.

$$\psi(v^k_{i,j,l}) = \begin{cases} f_2(v^k_{i,j,l}) + \frac{s}{2} + 1, & \text{if } f_2(v^k_{i,j,l}) \le \frac{1}{2}s; \\ f_2(v^k_{i,j,l}) + |m - \frac{3}{2}|s + m - 1, & \text{otherwise.} \end{cases}$$

for $2 \le k \le m$, k is even.

then $\psi(v^h_0) = (m - \frac{1}{2})s + m$.

Let ψ' be the complementary labelling of ψ. We can determine a graceful labelling F_2 of T^*_t when $m = 2$ as follows.

$$F_2((v^k_{i,j,l})^h) = \begin{cases} \psi(v^k_{i,j,l}) + \frac{h-1}{2}(2s+3), & \text{if } \psi(v^k_{i,j,l}) \le s+1; \\ \psi(v^k_{i,j,l}) + (t - \frac{h+1}{2})(2s+3), & \text{otherwise.} \end{cases}$$

with $1 \le h \le t$ and h is odd.

$$F_2((v^k_{i,j,l})^h) = \begin{cases} \psi'(v^k_{i,j,l}) + (h-1)(s+1) + \frac{h}{2}, & \text{if } \psi'(v^k_{i,j,l}) \le s+1; \\ \psi'(v^k_{i,j,l}) + (2t-h-1)(s+1) + \frac{2t-h}{2}, & \text{otherwise.} \end{cases}$$

with $2 \leq h \leq t$ and h is even.

$$F_2(v_0^h) = \begin{cases} \psi(v_0^h) + (t - \frac{h+1}{2})(2s+3), & \text{if } h \text{ is odd, } 1 \leq h \leq t; \\ \psi'(v_0^h) + \frac{h}{2}(2s+3) - s - 1, & \text{if } h \text{ is even, } 2 \leq h \leq t. \end{cases}$$

For $m = 3$, it is very similar, there is a graceful labelling F_3 of T_t^* as this form:

$$F_3((v_{i,j,l}^k)^h) = \begin{cases} \psi(v_{i,j,l}^k) + \frac{h-1}{2}(3s+4), & \text{if } \psi(v_{i,j,l}^k) \leq \frac{3s+4}{2}; \\ \psi(v_{i,j,l}^k) + (t - \frac{h+1}{2})(3s+4), & \text{otherwise.} \end{cases}$$

with $1 \leq h \leq t$ and h is odd.

$$F_3((v_{i,j,l}^k)^h) = \begin{cases} \phi'(v_{i,j,l}^k) + \frac{h-1}{2}(3s+4) + 1, & \text{if } \phi'(v_{i,j,l}^k) \leq \frac{3s+4}{2}; \\ \phi'(v_{i,j,l}^k) + \frac{2t-h-1}{2}(3s+4) + 1, & \text{otherwise.} \end{cases}$$

with $2 \leq h \leq t$ and h is even.

$$F_3(v_0^h) = \begin{cases} \psi(v_0^h) + (t - \frac{h+1}{2})(3s+4), & \text{if } h \text{ is odd, } 1 \leq h \leq t; \\ \phi'(v_0^h) + \frac{h-1}{2}(3s+4) + 1, & \text{if } h \text{ is even, } 2 \leq h \leq t. \end{cases}$$

We have finished the proof. □

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