Some Maximally Tough Circulants

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Abstract

The maximum possible toughness among graphs with n vertices and m edges is considered for $m \geq \lceil n^2/4 \rceil$. We thus extend results known for $m \geq n\lfloor n/3 \rfloor$. When n is even, all of the values are determined. When n is odd, some values are determined, and the difficulties are discussed, leaving open questions.

1 Terminology

A graph $G = (V, E)$ is an (n, m)-graph if $|V| = n$ and $|E| = m$. The toughness [1] of a non-complete graph $G = (V, E)$ is

$$\tau(G) = \min\{\frac{|S|}{\omega(G \setminus S)} \ : S \subseteq V \text{ and } \omega(G \setminus S) > 1\},$$

where $\omega(G \setminus S)$ is the number of components in the graph resulting from the removal of the vertices in S from G. A separating set S for which the toughness $\tau(G) = |S|/\omega(G \setminus S)$ is said to be a τ-set for G. Similarly, a separating set S for which the connectivity $\kappa(G) = |S|$ is called a κ-set. Among all (n, m)-graphs, the maximum toughness [1, 2, 4, 5] is denoted by $T_n(m)$. An (n, m)-graph G is said to be maximally tough if $\tau(G) = T_n(m)$.

For a fixed value of n, a study of the values $T_n(m)$ naturally splits into r-indexed intervals $\lceil nr/2 \rceil \leq m < \lceil n(r+1)/2 \rceil$, in which r is the maximum possible minimum degree. Of course, $r = \lfloor 2m/n \rfloor$. For m in the r^{th} interval, $T_n(m) \leq r/2$. Thus, if we show that $T_n(\lceil nr/2 \rceil) = r/2$, then the computation of T_n is completed in that interval.

A standard approach to constructing maximally tough graphs is to avoid $K_{1,3}$-centers. These are vertices with three neighbors that are not adjacent to each other. Matthews and Sumner [7] show that, if a graph G is $K_{1,3}$-free, then $\tau(G) = \kappa(G)/2$. All standard notation and terminology not listed here can be found in [8].

2 Introduction

In this paper, we improve upon part (ii) of the following result, which gives the maximum toughness when edges are relatively abundant.

Theorem 2.1 **([1, 4]).** *Let $n \geq 3$. If*

(i) r is even, or

(ii) $r \geq 2\lfloor \frac{n}{3} \rfloor$,

then $T_n(\lceil \frac{nr}{2} \rceil) = \frac{r}{2}$.

Theorem 2.1 is a consequence of the fact that the relevant Harary graphs [6] are maximally tough. Part (ii) handles the cases in which $m \geq n\lfloor n/3 \rfloor$. The following result, which we prove at the end of Section 3, improves upon Theorem 2.1(ii) and handles the cases in which n is even and $m \geq n^2/4$.

Theorem 2.2. *Let n be even and $r \geq \frac{n}{2}$. Then $T_n(\frac{nr}{2}) = \frac{r}{2}$.*

When n is odd, a strong result like Theorem 2.2 is not possible. The first failing occurs when $n = 13$ and $r = 7$. In [3], it is shown that

$$T_{13}(46) = \frac{10}{3} < \frac{7}{2} = T_{13}(47). \tag{2.1}$$

We can extend this result somewhat. Of course, in light of Theorem 2.1(i), it suffices to focus on the cases in which r is odd. The following three results are proven in Section 4.

Theorem 2.3. *Let n be odd and $r = \frac{n+(n \bmod 4)}{2}$.*

(a) If $n \equiv 1 \pmod 4$, then $T_n(\lceil \frac{nr}{2} \rceil + 1) = \frac{r}{2}$.

(b) If $n \equiv 3 \pmod 4$ and $n \geq 19$, then $T_n(\lceil \frac{nr}{2} \rceil + 1) \geq \frac{r}{2} - \frac{1}{6}$.

Theorem 2.4. *If $n \equiv 1 \pmod 4$ and $r = \frac{n+1}{2}$, then $T_n(\lceil \frac{nr}{2} \rceil) \geq \frac{r}{2} - \frac{1}{6}$.*

Theorem 2.3(b) leaves room for improvement when $n \equiv 3 \pmod 4$ and $n \geq 19$. However, the following result, which we have been unable to generalize, leaves no room for improvement when $n = 15$ and $r = 9$.

Proposition 2.5. $T_{15}(68) = \frac{9}{2}$.

We see in (2.1) that equality holds in Theorem 2.4, when $n = 13$. A sense of the difficulties involved in establishing equality in general can be gained by reading, in Section 5, the proof of the following result.

Proposition 2.6. $T_{17}(77) = \frac{13}{3}$.

3 Key Construction: n even

Given an integer $n \geq 3$ and a set R of integers modulo n with $R = -R$, the circulant $C(n, R)$ is the graph with vertex set $V = \{0, 1, \ldots, n-1\}$ and edge set $E = \{\{i, i+j\} : j \in R\}$, where addition is taken modulo n. Note that $|V| = n$ and $C(n, R)$ is $|R|$-regular. When r is even, the r-regular Harary graph [6] on n vertices $H(n, r)$ is defined to be the circulant $C(n, \{i : -r/2 \leq i \leq r/2, i \neq 0\})$. When r is odd and n is even, $H(n, r)$ is $C(n, \{i : -(r-1)/2 \leq i \leq (r-1)/2, i \neq 0\} \cup \{\frac{n}{2}\})$. When r and n are both odd, the Harary graph $H(n, r)$ has degree sequence $r+1, r, \ldots, r$ and is not a circulant; it is obtained from $H(n, r-1)$ by adding $(n+1)/2$ "diameter" edges. Although, in general, the circulant $C(n, R)$ is not a Harary graph, it has some properties similar to those of Harary graphs that we shall exploit.

Our proof of Theorem 2.2 comes out of the following construction. Although we shall see that it is a particular circulant graph, we give an alternate description of its construction that lends itself to the proof of Theorem 2.2 and a later construction.

Definition 3.1 (Construction of $G(n, r)$). Given n even with $r \geq n/2$, let $a = r + 1 - n/2$. Form $G(n, r)$ from two $n/2$-cliques with vertex sets $V_1 = \{u_1, \ldots, u_{n/2}\}$ and $V_2 = \{w_1, \ldots, w_{n/2}\}$ by joining each u_i to the a vertices $w_i, \ldots, w_{i+a-1}$, where subscripts are taken modulo $n/2$. See Figure 1. Note that $G(n, r)$ has n vertices and is r-regular. Moreover, when r is odd, it is easy to confirm that $G(n, r) \cong C(n, R)$ for

$$R = \{2i : 1 \leq i \leq \tfrac{n}{2} - 1\} \cup \{2i + 1 : \tfrac{n-r-1}{2} \leq i \leq \tfrac{r+1}{2} - 1\}.$$

Lemma 3.2. *Let n be even and $r \geq \frac{n}{2}$. Then, $\kappa(G(n, r)) = r$.*

Proof. Let S be a κ-set for $G(n, r)$. For $j = 1, 2$, let $S_j = S \cap V_j$ and $W_j = V_j \setminus S_j$. Note that $G \setminus S$ must have two components, one induced by W_1 and the other induced by W_2. Moreover, $G \setminus S$ cannot contain edges joining V_1 to V_2. Since $|W_1| \geq 1$, there is some $u_i \in W_1$ such that $u_{i+1} \in S_1 \subseteq S$. Since $u_i \notin S$, it follows that $w_i, \ldots, w_{i+a-1} \in S_2 \subseteq S$.

We claim that $u_{i+1}, \ldots, u_{i+a-1} \in S_1 \subseteq S$. If not, there is some $2 \leq b \leq a-1$ such that $u_{i+b} \in W_1$. Since $u_{i+b} \notin S$, it follows that

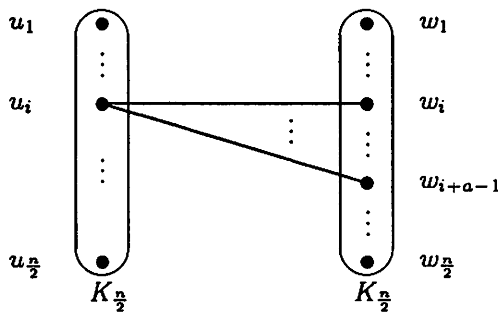

Figure 1: $G(n,r)$

$w_{i+b}, \ldots, w_{i+b+a-1} \in S_2 \subseteq S$. In particular, we have $w_{i+1}, \ldots, w_{i+a} \in S$, and we see that it is not necessary to have $u_{i+1} \in S$ to separate $G(n,r)$. This contradicts the fact that S is a κ-set and establishes our claim.

For each $1 \leq k \leq n/2$, since u_k is joined to w_k, we must have at least one of u_k or w_k in S. Our claim has shown that, for $a-1$ values of k, say $c+1, \ldots, c+a-1$, we must have *both* u_k and w_k in S. Consequently, $|S| \geq n/2 + (a-1) = r$. That is, $\kappa(G(n,r)) \geq r$. Since a separating set of size r is given by $\{w_1, \ldots, w_a, u_2, \ldots, u_{n/2}\}$, $\kappa(G(n,r)) = r$. □

Proof of Theorem 2.2. Since $G(n,r)$ has two disjoint cliques of size $n/2$, it is $K_{1,3}$-free. It now follows from Lemma 3.2 that $\tau(G(n,r)) = r/2$. □

4 Key Construction: n odd

Definition 4.1 (Construction of $G'(n,r)$). Given n, r both odd with $r \geq (n+1)/2$, let $a = r+1-(n+1)/2$. Form $G'(n,r)$ from $G(n-1, r-1)$ by adding a new vertex z that is joined to

$$u_1, \ldots, u_{(r+1)/2}, w_1, \ldots, w_{(r+1)/2}, w_{(n-1)/2},$$

and by joining u_i to w_{i-1}, for each $(r+3)/2 \leq i \leq (n-1)/2$. See Figure 2. Note that $G'(n,r)$ has n vertices, $\deg(z) = r+2$, $\deg(w_{(r+1)/2}) = r+1$, and all other vertices have degree r.

Lemma 4.2. *Let n, r be odd with $r \geq \frac{n+1}{2}$. Then, $\kappa(G'(n,r)) = r$.*

Proof. Let S be a κ-set for $G'(n,r)$. If $z \in S$, then since the $(r-1)$-connected graph $G(n-1, r-1)$ is a subgraph of $G'(n,r) \setminus \{z\}$, we have $|S| \geq 1 + (r-1) = r$. If $z \notin S$, then for each vertex $v \neq z$ in $G'(n,r)$, there are r internally disjoint paths in $G'(n,r)$ between z and v. Hence, $|S| \geq r$. □

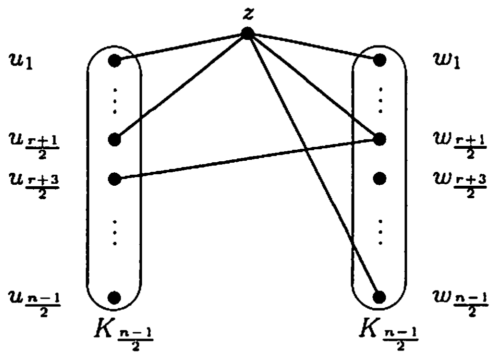

Figure 2: $G'(n, r)$

Proof of Theorem 2.3. Note that r is odd. Let S be a τ-set for $G'(n,r)$. Since $G'(n,r)$ consists of two cliques and the vertex z, we must have $\omega(G'(n,r) \setminus S) \leq 3$. If $\omega(G'(n,r) \setminus S) = 2$, then

$$\frac{|S|}{\omega(G'(n,r) \setminus S)} = \frac{|S|}{2} \geq \frac{\kappa(G'(n,r))}{2} = \frac{r}{2}.$$

So assume that $\omega(G'(n,r) \setminus S) = 3$. Hence, the $r+2$ neighbors of z are in S, and z is a component by itself. For $j = 1, 2$, let W_j be the vertices in V_j that are not adjacent to z. None of the edges joining W_1 to W_2 can remain in $G'(n,r) \setminus S$. However, since at least one vertex from each of W_1 and W_2 must remain, $n \geq 17$.

We claim that at least $(n-r)/2 - 1$ of the vertices in $W_1 \cup W_2$ are in S. Let $w_i \in W_2 \setminus S$. Its neighbors u_i and u_{i+1} in W_1 must be in S. Since the remaining $(n-r)/2 - 3$ vertices in W_1 have a matching with the remaining $(n-r)/2 - 3$ vertices in W_2, there are at least $(n-r)/2 - 3$ more vertices in S from $W_1 \cup W_2$. In total, there are at least $(n-r)/2 - 1$, as claimed.

All together, since $2r = n + (n \bmod 4)$, we see that

$$|S| \geq (r+2) + (\tfrac{n-r}{2} - 1) = \frac{3r + 2 - (n \bmod 4)}{2},$$

and hence

$$\tau(G'(n,r)) = \frac{|S|}{3} \geq \frac{r}{2} + \frac{2 - (n \bmod 4)}{6}.$$

If $n \equiv 1 \pmod 4$, then S cannot be a τ-set. If $n \equiv 3 \pmod 4$, then $\tau(G'(n,r)) \geq r/2 - 1/6$. □

The maximally tough (13, 47)-graph presented in [3] is isomorphic to $G'(13,7)$. Moreover, removing the edge joining the two highest degree vertices yields the maximally tough (13, 46)-graph given there. Theorem 2.4 can be proven by generalizing this approach.

Proof of Theorem 2.4. The result follows easily either by considering the graph obtained from $G'(n,r)$ by removing the edge joining the two highest degree vertices, or by using a construction similar to that of $G'(n,r)$ in which vertex z gets degree r, vertex $w_{(r-1)/2}$ gets degree $r+1$, and all the rest get degree r. □

Proof of Proposition 2.5. The (15, 68)-graph obtained from the graph $C(15, \{\pm 2, \pm 4, \pm 5, \pm 6\})$ by adding the edges $\{0,7\}, \{1,8\}, \ldots, \{7,14\}, \{0,8\}$ has been verified by computer to have toughness 9/2. □

5 Proof of Proposition 2.6: $T_{17}(77) = 13/3$

We shall employ some further notation. For $A \subseteq V(G)$, let $N(A)$ denote the set of neighbors of A, let $N[A] = N(A) \cup A$, and let $\langle A \rangle$ denote the subgraph induced by A. We use $\sim$ for the adjacency relation. For a vertex v in a subgraph H of G, $\deg_H(v)$ denotes the degree in H of v. For any disjoint subsets A and B of $V(G)$, n_{AB} denotes the number of edges with one endpoint in A and the other in B.

Proof. Suppose instead that we have a (17, 77)-graph G with $\tau(G) > 13/3$. Hence, G has toughness 9/2, connectivity 9, 16 vertices of degree 9, and one vertex of degree 10.

Claim 5.1. *Let v_1 and v_2 be distinct nonadjacent vertices. Then, $|V(G) \setminus (N[v_1] \cup N[v_2])| \leq 1$. Moreover, if $\deg_G(v_1) = \deg_G(v_2) = 9$, then $|N(v_1) \cap N(v_2)| \leq 4$.*

Pf. Let $S = N(v_1) \cup N(v_2)$. If $|V(G) \setminus (N[v_1] \cup N[v_2])| > 1$, then $|S| \leq 13$, $\omega(G \setminus S) \geq 3$, and $\tau(G) \leq 13/3$. The rest follows by inclusion-exclusion. Q.E.D.

We consider two main cases.

Case 1: There are distinct nonadjacent degree 9 vertices x and y such that $|V(G) \setminus (N[x] \cup N[y])| = 1$, and hence $|N(x) \cap N(y)| = 4$.

We have a vertex $z \notin N[x] \cup N[y]$. Since $z \nsim x$, $|N(x) \cup N(z)| \geq 14$, by Claim 5.1. Thus z is adjacent to all 5 vertices of $N(y) \setminus N(x)$. Similarly, z is adjacent to all 5 vertices of $N(x) \setminus N(y)$. Thus $\deg_G(z) = 10$, and $N(z) = (N(x) \cup N(y)) \setminus (N(x) \cap N(y))$. Define $X = N(x) \setminus N(y) = \{x_1, \ldots, x_5\}$, $Y = N(y) \setminus N(x) = \{y_1, \ldots, y_5\}$, and $W = N(x) \cap N(y) = \{w_1, \ldots, w_4\}$.

So $V(G) = \{x, y, z\} \cup X \cup Y \cup W$, $N(x) = X \cup W$, $N(y) = Y \cup W$, and $N(z) = X \cup Y$.

Claim 5.2. *(a)* $\langle X \rangle \cong \langle Y \rangle \cong K_5$.

(b) $\forall x_i \in X$, $|N(x_i) \cap (Y \cup W)| = 3$.

(c) $\forall y_i \in Y$, $|N(y_i) \cap (X \cup W)| = 3$.

(d) $\langle W \rangle$ *is one of* K_4, K_4 *minus one edge, or* C_4.

(e) $\forall w \in W$, *if* $\deg_W(w) = 3$, *then* $N(w) \cap X \neq \emptyset$ *and* $N(w) \cap Y \neq \emptyset$.

Pf. First we show that each $x_i \in X$ is adjacent to at least 3 other vertices in X. Since $x_i \nsim y$, by Claim 5.1, $|V(G) \setminus (N[x_i] \cup N[y])| \leq 1$. Moreover, $N[y] \cap X = \emptyset$, so x_i must be adjacent to at least 3 vertices of X. To see why $\langle X \rangle \cong K_5$, consider distinct $x_i, x_j \in X$. We already have $|N(x_i) \cap N(x_j)| = |\{x, z\} \cup (X \setminus \{x_i, x_j\})| = 5$. Thus, by Claim 5.1, $x_i \sim x_j$, and hence $\langle X \rangle \cong K_5$. A similar argument shows $\langle Y \rangle$ is K_5. Part (b) follows since $\deg_G(x_i) = 9$ and x_i is adjacent to x, z, and 4 other elements of X. Part (c) follows similarly. For (d), note that, for each $w \in W$, $w \nsim z$. So, by Claim 5.1, $|V(G) \setminus (N[w] \cup N[z])| \leq 1$. Furthermore, $N[z] \cap W = \emptyset$. So w must be adjacent to at least 2 of the other 3 vertices of W. Hence $\deg_{\langle W \rangle}(w) \geq 2$, and the result follows. For (e), if $N(w) \cap X = \emptyset$, then $|N(w) \cap Y| = 9 - (3 + 2) = 4$. So $\exists y_i \in Y$ such that $y_i \nsim w$, and, by Claim 5.1, $|V(G) \setminus (N[w] \cup N[y_i])| \leq 1$. Hence, $|N(y_i) \cap X| \geq 4$, which contradicts part (c). Q.E.D.

Claim 5.3. *(a)* $\forall x_i \neq x_j \in X$, $N(x_i, x_j) \cap Y \neq \emptyset$.

(b) $\forall y_i \neq y_j \in Y$, $N(y_i, y_j) \cap X \neq \emptyset$.

Pf. Let $x_i \neq x_j \in X$, and suppose that $N(x_i, x_j) \cap Y = \emptyset$. Define $S = \{z\} \cup W \cup X \setminus \{x_i, x_j\}$. Since $|S| = 8$ and $G \setminus S$ is disconnected with $\langle \{x, x_1, x_2\} \rangle$ forming one component, we contradict $\kappa(G) = 9$. Part (b) follows similarly. Q.E.D.

Claim 5.4. *(a) If* $\langle W \rangle \cong K_4$, *then* $n_{WX} = n_{WY} = 8$ *and* $n_{XY} = 7$.

(b) If $\langle W \rangle \cong K_4$ *minus one edge, then* $n_{WX} = n_{WY} = 9$ *and* $n_{XY} = 6$.

(c) If $\langle W \rangle \cong C_4$, *then* $n_{WX} = n_{WY} = 10$ *and* $n_{XY} = 5$.

Pf. By Claim 5.2 (b) and (c), each $x_i \in X$ is adjacent to 3 vertices of $W \cup Y$, and each $y_i \in Y$ is adjacent to 3 vertices of $W \cup X$. So

$$n_{XW} + n_{XY} = 3 \cdot 5 = 15 \text{ and } n_{YW} + n_{YX} = 15. \qquad (5.1)$$

If $\langle W\rangle = K_4$, then each $w_i \in W$ is adjacent to x, y, and 3 vertices of W. So each such w_i is adjacent to 4 vertices of $X \cup Y$. This gives $n_{WX} + n_{WY} = 4 \cdot 4 = 16$. In general, similar counts show that

$$n_{WX} + n_{WY} = \begin{cases} 16 & \text{if } \langle W\rangle \cong K_4, \\ 18 & \text{if } \langle W\rangle \cong K_4 \text{ minus one edge}, \\ 20 & \text{if } \langle W\rangle \cong C_4, \end{cases} \tag{5.2}$$

The results follow by solving the system of equations (5.1) and (5.2). Q.E.D.

Claim 5.5. $\langle W\rangle \cong C_4$.

Pf. Suppose instead that $\langle W\rangle$ is K_4 or K_4 minus one edge. Then there exists $w_1 \in W$ such that $\deg_W(w_1) = 3$, and hence $|N(w_1) \cap (X \cup Y)| = 4$. Let $|N(w_1) \cap Y| = k$. So $|N(w_1) \cap X| = 4 - k$. By symmetry of $\langle X\rangle$ and $\langle Y\rangle$, we may assume that $0 \le k \le 2$. By Claim 5.2(e), $k \ne 0$. Since $|N(w_1) \cap X| = 4 - k$, $|X \setminus N(w_1)| = 5 - (4 - k) = k + 1$. Suppose $x_1, \ldots, x_{k+1} \nsim w_1$. For $1 \le i \le k+1$, $|V(G) \setminus (N[w_1] \cup N[x_i])| \le 1$, by Claim 5.1. Now w_1 is adjacent to k elements of Y. So, for each $1 \le i \le k+1$, x_i must be adjacent to at least $4 - k$ elements of Y. For both $k = 1$ and $k = 2$, this accounts for a total of at least 6 X-Y edges. If $\langle W\rangle \cong K_4$ minus one edge, then, by Claim 5.4, there are no more X-Y edges. So we have $N(x_4, x_5) \cap Y = \emptyset$, contradicting Claim 5.3. If $\langle W\rangle \cong K_4$, then there is exactly one more X-Y edge. So we have, without loss of generality, $x_5 \in X$ such that $x_5 \sim w_1$ and $N(x_5) \cap Y = \emptyset$. This fact and Claim 5.2(b) give $|N(x_5) \cap W| = 3$. So $\exists w_j \in W$ such that $w_j \nsim x_5$. By Claim 5.1, $N(w_j) \cup N(x_5)$ must include all but at most one element of Y. Since x_5 is adjacent to no elements of Y, w_j is adjacent to at least 4 elements of Y. Since $\langle W\rangle \cong K_4$, $|N(w_j) \cap W| = 3$. Since w_j is adjacent to x and y, we have accounted for all vertices adjacent to w_j. That w_j has no neighbors in X contradicts Claim 5.2(e). Q.E.D.

Claim 5.6. $\langle W\rangle \ncong C_4$.

Pf. Suppose $\langle W\rangle \cong C_4$. Thus, $\forall w_i \in W$, $|N(w_i) \cap (X \cup Y)| = 5$.

First, consider the possibility that $\exists w_1 \in W$ such that $N(w_1) \cap X \ne X$ and $N(w_1) \cap Y \ne Y$. Let $|N(w_1) \cap Y| = k$. So $|N(w_1) \cap X| = 5 - k$. By the symmetry between X and Y, we need only consider $k = 1$ or 2. Since $|N(w_1) \cap X| = 5 - k$, $|X \setminus N(w_1)| = k$. So, take $w_1 \nsim x_1, \ldots, x_k$. Since $|N(w_1) \cap Y| = k$, Claim 5.1 implies that, for $1 \le i \le k$, $|N(x_i) \cap Y| = 4 - k$. If $k = 1$, then $w_1 \nsim x_1$ and x_1 is incident to at least 3 X-Y edges. By Claim 5.4(c), this leaves at most 2 edges from $X \setminus \{x_1\}$ to Y. Thus, at least two of the elements of $X \setminus \{x_1\}$ are incident with no edges to Y, contradicting Claim 5.3(a). If $k = 2$, then $w_1 \nsim x_1, x_2$ and each of x_1, x_2 is incident to at least 2 X-Y edges, accounting for at least 4 of the 5 X-Y

edges. Thus at least two of the elements of $X \setminus \{x_1, x_2\}$ are incident with no edges to Y, again contradicting Claim 5.3(a).

Second, consider the possibility that $\forall w_i \in W$, either $N(w_i) \cap X = X$ or $N(w_i) \cap Y = Y$. Assume that $w_1 \nsim w_3$ in $\langle W \rangle \cong C_4$, so $N(w_1) \cap X = X$ and $N(w_3) \cap Y = Y$. By Claim 5.4(c), $n_{WX} = 10$. Thus, one of w_2, w_4 is adjacent to elements of X. By our assumptions, this element, say w_2, is adjacent to all 5 elements of X. Similarly w_4 is adjacent to all 5 elements of Y. This accounts for all W-X edges and all W-Y edges. Now define $S = \{w_1, w_2, x\} \cup Y$. In $G \setminus S$, $\langle X \cup \{z\} \rangle$ is a component. Thus $G \setminus S$ is disconnected with $|S| = 8$, contradicting $\kappa(G) = 9$. Q.E.D.

Obviously, Claims 5.5 and 5.6 contradict each other.

Case 2: For each pair of distinct nonadjacent degree 9 vertices v_1 and v_2, $N[v_1] \cup N[v_2] = V(G)$, and hence $|N(v_1) \cup N(v_2)| = 15$.

Let z be the vertex of G with degree 10. Not all vertices in $N(z)$ can be adjacent, since all vertices in $N(z)$ have degree 9. Therefore, let x and y be non-adjacent degree 9 vertices in $N(z)$. Hence, $|N(x) \cup N(y)| = 15$. Define $Z = N(x) \cap N(y) = \{z, z_1, z_2\}$, $X = N(x) \setminus N(y) = \{x_1, \ldots, x_6\}$, and $Y = N(y) \setminus N(x) = \{y_1, \ldots, y_6\}$. So $V(G) = \{x, y\} \cup X \cup Y \cup Z$, $N(x) = X \cup Z$, and $N(y) = Y \cup Z$.

Claim 5.7. *(a)* $\langle X \rangle \cong \langle Y \rangle \cong K_6$.

(b) $\forall x_i \in X$, $|N(x_i) \cap (Y \cup Z)| = 3$.

(c) $\forall y_i \in Y$, $|N(y_i) \cap (X \cup Z)| = 3$.

(d) $\forall x_i \in X$, $N(x_i) \cap Y \neq \emptyset$.

(e) $\forall y_i \in Y$, $N(y_i) \cap X \neq \emptyset$.

Pf. If $x_i \nsim x_j \in \langle X \rangle$, then $y \notin N[x_i] \cup N[x_j]$, contradicting the case hypothesis. Thus $\langle X \rangle \cong K_6$. Similarly, $\langle Y \rangle \cong K_6$. Parts (b) and (c) now follow since each vertex in $X \cup Y$ has degree 9. For (d), suppose $\exists x_1 \in X$ with $N(x_1) \cap Y = \emptyset$. Define $S = Z \cup X \setminus \{x_1\}$. Thus, $|S| = 8$ and $\langle \{x, x_1\} \rangle$ is a component of $G \setminus S$, contradicting $\kappa(G) = 9$. Part (e) is similar. Q.E.D.

If $\langle Z \rangle$ has no edges, then $z_1 \nsim z_2$ and $z \notin N[z_1] \cup N[z_2]$, which contradicts the case hypothesis.

If at least one of z_1, z_2 has degree 2 in $\langle Z \rangle$, then take z_1 to be such a vertex. Now z_1 is adjacent to x, y, and 2 elements of Z. So z_1 is adjacent to exactly 5 elements of $X \cup Y$. Thus there exists $\hat{y} \in Y$ such that $z_1 \nsim \hat{y}$. By the case hypothesis, $N[z_1] \cup N[\hat{y}] = V(G)$. Hence, $\hat{y}$ is adjacent to all elements of $X \setminus N(z_1)$. If z_1 is adjacent to 0, 1, or 2 elements of X, then

$|N(\hat{y}) \cap X| \geq 4$, contradicting Claim 5.7(c). If z_1 is adjacent to 3, 4, or 5 elements of X, then z_1 is adjacent to 2, 1, or 0 elements of Y, respectively, and we repeat the previous argument exchanging the roles of X and Y.

We conclude that the degree in $\langle Z \rangle$ of both z_1 and z_2 must be at most 1 (but not both 0). Take z_1 to have degree 1 in $\langle Z \rangle$. Hence, $\langle Z \rangle$ has exactly three possible structures, which we denote by H_1, H_2, and H_3, where

- H_1 has exactly two edges, $\{z, z_1\}$ and $\{z, z_2\}$,
- H_2 has exactly one edge $\{z_1, z\}$, and
- H_3 has exactly one edge $\{z_1, z_2\}$.

Claim 5.8. *(a)* $|N(z_1) \cap (X \cup Y)| = 6$.

(b) If $\langle Z \rangle$ is H_1, then $n_{ZX} = n_{ZY} = 9$ and $n_{XY} = 9$.

(c) If $\langle Z \rangle$ is H_2 or H_3, then $n_{ZX} = n_{ZY} = 10$ and $n_{XY} = 8$.

Pf. Part (a) follows, since z_1 is adjacent to x, y, and one element of Z. For parts (b) and (c), the number of edges connecting Z to $X \cup Y$ is

$$n_{ZX} + n_{ZY} = \begin{cases} 18 & \text{if } \langle Z \rangle \text{ is } H_1, \\ 20 & \text{if } \langle Z \rangle \text{ is } H_2 \text{ or } H_3. \end{cases} \tag{5.3}$$

By parts (b) and (c) of Claim 5.7, the number of edges connecting X to $Y \cup Z$ is $n_{XZ} + n_{XY} = 6 \cdot 3 = 18$. Similarly, $n_{YZ} + n_{YX} = 18$. Solving this system of equations gives the desired result. Q.E.D.

Let $|N(z_1) \cap X| = k$. By Claim 5.8(a), $|N(z_1) \cap Y| = 6 - k$.

Subcase (a): $k = 1$, 2, 4, or 5. By the symmetry between X and Y, we may assume that $k = 1$ or 2. So $\exists y_1 \in Y$ such that $y_1 \nsim z_1$. By the case hypothesis, $N[y_1] \cup N[z_1] = V(G)$. Hence y_1 is adjacent to the $6 - k \geq 4$ elements of X that are not adjacent to z_1. However, this contradicts Claim 5.7(c).

Subcase (b): $k = 3$. Take $z_1 \nsim x_4, x_5, x_6$, and $z_1 \nsim y_4, y_5, y_6$. By the case hypothesis, for $i = 4, 5, 6$, $N[z_1] \cup N[x_i] = V(G)$. So each of x_4, x_5, x_6 is adjacent to each of y_4, y_5, y_6. This accounts for 9 X-Y edges. By Claim 5.8, there are no more X-Y edges. Hence $N(x_1) \cap Y = \emptyset$, contradicting Claim 5.7(d).

Subcase (c): $k = 6$ or 0. First, consider $\langle Z \rangle = H_1$ or H_3. Note that, in H_1 and H_3, z_1 and z_2 are interchangeable. So, if $1 \leq |N(z_2) \cap X| \leq 5$, we can apply subcase (a) or (b) arguments to z_2 to get a contradiction. Hence, we may assume that $|N(z_1) \cap X| = 6$ or 0, and $|N(z_2) \cap X| = 6$ or 0. By Claim 5.8, $n_{ZX} \leq 10$. So, one of $N(z_1) \cap X$ or $N(z_2) \cap X$ must be empty.

Take $|N(z_1) \cap X| = 6$ and $|N(z_2) \cap X| = 0$. This gives $|N(z_1) \cap Y| = 0$ and $|N(z_2) \cap Y| = 6$. If $\langle Z \rangle = H_3$, then, by Claim 5.8(c), $n_{ZY} = 10$, and hence $|N(z) \cap Y| = 4$. Thus, there are 2 elements of Y that are not in $N[z] \cup N[z_1]$, contradicting Claim 5.1. If $\langle Z \rangle = H_1$, then, by Claim 5.8(b), $n_{ZY} = n_{ZX} = 9$, and hence $|N(z) \cap X| = 3$ and $|N(z) \cap Y| = 3$. So take $z \nsim x_i$, for $i = 4, 5, 6$, and $z \nsim y_j$, for $j = 4, 5, 6$. For $j = 4, 5, 6$, since $z \notin N[x_4] \cup N[y_j]$, we have $x_4 \sim y_j$, by the case hypothesis. Since $|N(z_1) \cap X| = 6$, x_4 is adjacent to z_1 and 3 elements of Y. This contradicts Claim 5.7(b).

To complete subcase (c), consider $\langle Z \rangle = H_2$. By the symmetry between X and Y, we may assume that $k = |N(z_1) \cap X| = 6$ and $|N(z_1) \cap Y| = 0$. Since $z_1 \nsim z_2$ and z_1 is not adjacent to elements of Y, the case hypothesis gives $|N(z_2) \cap Y| = 6$. Thus, z_2 is adjacent to exactly one element of X. By Claim 5.8(c), $n_{ZX} = 10 = n_{ZY}$, and so $|N(z) \cap X| = 3$ and $|N(z) \cap Y| = 4$. Take $z \nsim x_4, x_5, x_6, y_5, y_6$. Now $y_5 \sim z_2$, and, by Claim 5.7(c), y_5 is adjacent to exactly 3 elements of $X \cup Z$. So y_5 is adjacent to no more than 2 of x_4, x_5, x_6. Say $y_5 \nsim x_4$. However, $z \notin N[y_5] \cup N[x_4]$, contradicting the case hypothesis. □

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