ON MAXIMAL FRACTIONAL INDEPENDENT SETS IN GRAPHS

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ABSTRACT. We study convexity with respect to a definition of fractional independence in a graph G that is quantified over neighbourhoods rather than edges. The graphs that admit a so-called universal maximal fractional independent set are characterized, as are all such sets. A characterization is given of the maximal fractional independent sets which can not be obtained as a proper convex combination of two other such sets.

1. INTRODUCTION

Fractional dominating sets of graphs, also called dominating functions, were first studied by Hedetniemi, Hedetniemi and Wimer [9]. Since then, a considerable amount of work has been done on the "size" of fractional dominating sets, and the convexity of such sets (e.g. [3, 4, 5. 6, 10, 12]; also see [7, 8]).

A problem stated by Haynes, Hedetniemi and Slater [8] (Chapter 3, page 85) asks if it is possible to define a fractional independent set in a graph as a function $f : V \to [0, 1]$ such that:

(1) the characteristic function of an independent set is a fractional independent set, and
(2) there is a concept of maximality so that:
   (a) the characteristic function of a maximal independent set is a maximal fractional independent set (MFIS), and
   (b) every MFIS is a minimal fractional dominating set.

Such a definition was presented and studied in the Ph.D. Thesis of K. Reji Kumar [10].

We consider convexity questions for Kumar's definition of maximal fractional independent sets. In particular, we give a complete characterization of the graphs that admit a universal maximal fractional independent set. That is, we characterize the graphs with a MFIS f such that any convex combination of f and another MFIS is a MFIS. For these graphs we are also able to describe all possible universal MFISs, as well as all possible "sizes" of such sets.

2. DEFINITIONS AND PRELIMINARIES

We consider only finite simple graphs. Terminology not explicity defined here follows West [13], or Haynes, Hedeiniemi and Slater [7].

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Let G be a graph, and $f : V \to [0,1]$ a function. For a subset $X \subseteq V$, denote by $f(X)$ the quantity $\sum_{x \in X} f(x)$. The *boundary of* f is the set $B_f = \{x : f(N[x]) = 1\}$, and the *positive set of* f is the set $P_f = \{x : f(x) > 0\}$. Here, and elsewhere, $N[x] = N(x) \cup \{x\}$ denotes the *closed neighbourhood* of the vertex x.

For subsets X and Y of the vertex set of a graph G, we say that X *dominates* Y if every vertex in $Y - X$ is adjacent to a vertex in X. If $Y = \{v\}$ we may omit the brackets and say that X dominates v. In the terminology introduced in this paragraph, a subset $D \subseteq V$ is a dominating set if it dominates V.

A function $f : V \to [0,1]$ is a called a *fractional dominating set*, or *dominating function* of the graph G, if $f(N[x]) \geq 1$ for every $x \in V$. A fractional dominating set is called *minimal* if there is no fractional dominating set $g \neq f$ such that $g(w) \leq f(w)$ for all $w \in V$. The 0-1 valued (minimal) fractional dominating sets are precisely the characteristic functions of (minimal) dominating sets.

It is clear that if f and g are fractional dominating sets of the graph G then, for any $\lambda \in [0,1]$, so is the convex combination $h_\lambda = \lambda f + (1-\lambda)g$. The same statement is not true for minimal fractional dominating sets, as is evident from the following results.

Theorem 2.1. [2] *A fractional dominating set f is minimal if and only if B_f dominates P_f.*

Corollary 2.2. [2] *Let f and g be minimal fractional dominating sets of the graph G. Then, for any $\lambda \in (0,1)$, the convex combination $h_\lambda = \lambda f + (1-\lambda)g$ is a minimal fractional dominating set if and only if $B_f \cap B_g$ dominates $P_f \cup P_g$.*

A *universal* minimal fractional dominating set f is one for which any proper convex combination of f and a minimal fractional dominating set g is again a minimal fractional dominating set. The universal minimal fractional dominating sets have been characterized [3]. This leads to a characterization of the graphs for which the set of minimal fractional independent sets is convex, that is, every minimal fractional dominating set is universal [11]. No characterization is known of the graphs that admit a universal minimal fractional dominating set.

A *basic minimal fractional dominating set* is one which can not be expressed as a proper convex combination of two other minimal fractional dominating sets. Kumar and Arumugam [12] characterized the basic minimal fractional dominating sets.

Theorem 2.3. [12] *A minimal fractional dominating set f is basic if and only if there is no other minimal fractional dominating set g with $B_f = B_g$ and $P_f = P_g$.*

The following definition, from the Ph.D. Thesis of Kumar [10], answers the question of Haynes *et al.* mentioned in the Introduction. Let G be a graph. A function $f : V \to [0,1]$ is an *independent function*, or *fractional independent set*, if $f(N[x]) = 1$ for every vertex x with $f(x) > 0$. A fractional independent set is *maximal* if it is also a fractional dominating set.

This notion of maximality resembles the folklore theorem that an independent set in a graph is maximal if and only if it is a dominating set. On the other hand, the notion of maximality does not allow one to increase some function values of a fractional independent set and arrive at a MFIS. For example, let g be the fractional independent set of the path on four vertices obtained by assigning zero to the two end vertices and $\frac{1}{2}$ to the two central vertices. There is no MFIS f such that $g(x) \leq f(x)$ for each vertex x.

Clearly, the charactistic function of an independent set is a fractional independent set, and that the charactistic function of a maximal independent set is a maximal fractional independent set. It follows immediately from the definition that if f is a MFIS of G, then P_f dominates V.

In contrast to fractional dominating sets, the collection of fractional independent sets of a graph is not necessarily convex. For example, consider a path on three vertices x, y, z. If f is the characteristic function of $\{y\}$, and g is the characteristic function of $\{x, z\}$, then $\frac{1}{2}f + \frac{1}{2}g$ is not a fractional independent set because it is positive at y but $(\frac{1}{2}f + \frac{1}{2}g)(N[y]) > 1$.

A complete characterization of the graphs for which the collection of (maximal) fractional independent sets is convex is included among the result in the next section.

3. A CHARACTERISATION OF THE GRAPHS WITH A UNIVERSAL MFIS

A motivation for studying convex combinations of (maximal) fractional independent sets arises from considering the "size" of such sets. The *aggregate* of a fractional independent set f of a graph G is $agg(f) = \sum_{x \in V} f(x)$. It is natural to wonder which values can occur as the aggregate of a (maximal) fractional independent set of a graph G. In particular, f and g are MFISs of G with $agg(f) < agg(g)$, is there a (maximal) fractional independent set of G with aggregate α for every $\alpha \in [agg(f), agg(g)]$? When f and g are fractional dominating sets, this has been investigated by Cockayne *et. al.* [2]. As in that paper, the question leads to the study of convex combinations of (maximal) fractional independent sets, since for $\lambda \in [0, 1]$, the aggregate of $h_\lambda = \lambda f + (1 - \lambda)g$ is $\lambda agg(f) + (1 - \lambda)agg(g)$.

The first proposition follows immediately from the definition of a fractional independent set.

Proposition 3.1. *Let $G = (V, E)$ be a graph. A function $f : V \to [0, 1]$ is a fractional independent set if and only if $P_f \subseteq B_f$.*

Proposition 3.2. *If f is MFIS in G, then P_f dominates V.*

Proof. By definition of maximality, f is a fractional dominating set. Hence $f(N[v]) \geq 1$ for each vertex v. Therefore P_f dominates v. □

Corollary 3.3. *If f is MFIS in G, then B_f dominates V.*

We now show that either all nontrivial convex combinations of two MFISs of a graph G are MFISs, or none are.

Lemma 3.4. *Let f and g be MFISs of G, and $\lambda \in (0,1)$. Then, $h_\lambda = \lambda f + (1-\lambda)g$ is a MFIS if and only if $P_f \cup P_g \subseteq B_f \cap B_g$.*

Proof. The boundary of h_λ is $B_{h_\lambda} = B_f \cap B_g$, and the positive set is $P_{h_\lambda} = P_f \cup P_g$.
Suppose first that h_λ is a MFIS. Then, by Proposition 3.1, $P_f \cup P_g \subseteq B_f \cap B_g$.
Conversely, suppose that $P_f \cup P_g \subseteq B_f \cap B_g$. Then, by Proposition 3.1, the function h_λ is a fractional independent set. Since f and g are MFISs, for any vertex v, $h_\lambda(N[v]) = \lambda f(N[v]) + (1-\lambda)g(N[v]) \geq \lambda + (1-\lambda) = 1$. Hence, h_λ is a MFIS. □

A fractional independent set f is called *universal* if any convex combination of f and a MFIS g is a maximal fractional independent set.
It is easy to see that the collection of maximal fractional independent sets of a complete graph is convex. The following proposition implies that this is essentially the only graph for which the set of maximal fractional independent sets is convex.

Proposition 3.5. *Suppose that the graph G is connected but not complete. Then, there exist maximal fractional independent sets f and g such that no proper convex combination of f and g is a fractional independent set.*

Proof. Since G is connected and not complete, there exist vertices u, v and w such that $uv, vw \in E$, but $uw \notin E$. Let f be the characteristic function of any maximal independent set that contains v, and let g be the characteristic function of any maximal independent set that contains u and w. Then, for any $\lambda \in (0,1)$ we have $(\lambda f + (1-\lambda)g)(v) > 0$ and $(\lambda f + (1-\lambda)g)(N[v]) \geq 1 + \lambda > 1$. □

Lemma 3.6. *If f is a universal MFIS of G, then $B_f = V$.*

Proof. Let $v \in V$, and let S be a maximal independent set of G such that $v \in S$. The characteristic function χ_S of S is a MFIS, and $v \in P_{\chi_S}$. Since f is universal, it follows from Proposition 3.1 that $P_f \cup P_{\chi_S} \subseteq B_f \cap B_{\chi_S}$. Hence $v \in B_f$. □

Corollary 3.7. *If f is a universal MFIS of G, then*

$$P_f \subseteq \bigcap_g B_g,$$

where the intersection is over all MFISs g of G.

Lemma 3.8. *If the subgraph of G induced by $N[v]$ is not complete, then there exists a MFIS g such that $v \notin B_g$.*

Proof. The function g can be taken to be the characteristic function of a maximal independent set containing a pair of neighbours of v which are not adjacent. □

Let G be a graph with a universal MFIS f. By the above corollary and lemma, the positive values of f can occur only at vertices whose closed neighbourhood induces a complete graph. Define $C = C(G) = \{x : G[N[x]] \text{ is complete}\}$.

Corollary 3.9. *If f is a universal MFIS of G, then $P_f \subseteq C(G)$.*

Proposition 3.10. *Let G be a graph that has a universal MFIS. If x and y are vertices of G such that both $G[N[x]]$ and $G[N[y]]$ are complete graphs, then either $N[x] = N[y]$ or $N[x] \cap N[y] = \emptyset$.*

Proof. Let f be a universal MFIS of G. Suppose $N[x] \neq N[y]$ but $N[x] \cap N[y] \neq \emptyset$. Since both $G[N[x]]$ and $G[N[y]]$ are complete graphs, some vertex in $N[x]$ is not adjacent to some vertex in $N[y]$. Any vertex $z \in N[x] \cap N[y]$ is adjacent to every vertex in $N[x] \cup N[y]$, so $G[N[z]]$ is not complete. Consequently, $z \notin P_f$. Hence, $f(N[z]) \geq f(N[x]) + f(N[y]) \geq 1 + 1 = 2$. Thus $z \notin B_f = V$, a contradiction. □

Note that the above proposition does not preclude there being vertices $u \in N[x]$ and $v \in N[y]$ such that $uv \in E$.

Let G be a graph with a universal MFIS f. By Proposition 3.10 the relation $\sim$ on $C(G)$ is an equivalence relation.

Corollary 3.11. *Let f be a MFIS of the graph G. Then $f([x]) = 1$ for each equivalence class $[x]$ of $\sim$.*

Proof. Since $B_f = V$, $f(N[v]) = 1$ for any vertex v. Let $x \in C(G)$. By Corollary 3.9, $1 = f(N[x]) = f(N[x] \cap C(G)) = f([x])$. □

Proposition 3.12. *Suppose that G has a universal MFIS. If $y \notin C$, then there is a unique equivalence class $[x]$ of $\sim$ that contains a vertex adjacent to y. Further, y is adjacent to all elements of $[x]$.*

Proof. By Proposition 3.2 and Corollary 3.9, every vertex $y \notin C$ is adjacent to a vertex in C. By definition of $\sim$, if y is adjacent a vertex of an equivalence class $[x]$ then it is adjacent to every vertex of $[x]$. Since f is a universal MFIS, $y \in B_f$. Therefore, by Proposition 3.10, there is a unique equivalence class $[x]$ that contains a vertex adjacent to y. □

Corollary 3.13. *If G has a universal MFIS, then any set obtained by selecting one vertex from each equivalence class of $\sim$ is a minimum independent dominating set of G.*

An independent dominating set X of G is called *perfect* if every vertex of G not in X is adjacent to exactly one vertex in X.

Corollary 3.14. *If G has a universal MFIS, then G has a perfect minimum independent dominating set (formed by choosing one element from each equivalence class of $\sim$).*

The following theorem summarizes our work. It characterizes the graphs that admit a universal MFIS, and all such functions.

Theorem 3.15. *A graph G has a universal MFIS if and only if there exists a unique partition of V into sets that induce maximal cliques. Further, if V has such a partition, then a function f is a universal MFIS if and only if $P_f \subseteq C(G)$ and $f([x]) = 1$ for each equivalence class $[x]$ of $\sim$. The aggregate of any universal MFIS equals the number of equivalence classes, which is the number of cliques in the partition.*

Proof. Suppose G has a universal MFIS. By Proposition 3.12, each equivalence class of $\sim$ gives rise to a unique maximal clique, and the vertex sets of these cliques partition $V(G)$.

Now suppose there exists a unique partition of V into sets that induce maximal cliques. Let f be any function such that $P_f \subseteq C(G)$ and $f([x]) = 1$ for each equivalence class $[x]$ of $\sim$. Then f is a MFIS of G and $B_f = V$.

We claim that f is a universal MFIS. Let g be a MFIS of G. It must be shown that $P_f \cup P_g \subseteq B_g$. The containment $P_g \subseteq B_g$ follows from the fact that g is a fractional independent set. The containment $P_f \subseteq B_g$ follows from Corollary 3.7. This proves the claim.

Suppose now that f is a universal MFIS of G. By Corollories 3.9 and 3.11, $P_f \subseteq C(G)$ and $f([x]) = 1$ for each equivalence class $[x]$ of $\sim$. The aggregate of f equals the number of equivalence classes. □

The graphs that admit a universal MFIS can all be constructed from a disjoint union of t disjoint cliques (possibly of different sizes), say $G_1, G_2, \ldots, G_t$, where $t \in Z$, by selecting for each $i, 1 \leq i \leq t$ a non-empty subset $X_i \subseteq V(G_i)$, and adding any subset of edges joining vertices in

$$\bigcup_{i=1}^{t} \overline{X_i}.$$

Furthermore, any graph constructed in this way admits a universal MFIS.

The following is a consequence of our characterization above.

Proposition 3.16. *If there exists a unique partition of V into sets that induce maximal cliques, then every maximal independent set of G is a maximum independent set. That is, if G admits a universal MFIS, then G is well-covered.*

Proof. Any maximal independent set includes exactly one vertex from each clique in the partition. □

4. Basic maximal fractional independent sets

In this brief final section we note that Theorem 2.3 also holds for maximal fractional independent sets. A MFIS h of a graph G is called a *basic maximal fractional independent set* (BMFIS) of G if there do not exist different MFISs f and g, and $\lambda \in (0, 1)$, such that $h = \lambda f + (1 - \lambda)g$. It is easy to see that for $n \geq 1$ the characteristic function of a single vertex subset of $V(K_n)$ is a BMFIS of K_n.

Lemma 4.1. *Any BMFIS of a graph G is a basic minimal fractional dominating set of G.*

Proof. Suppose the graph G has a BMFIS f which is not a basic minimal fractional dominating set. Then, there exist different fractional dominating sets f_1 and f_2 of G and $\lambda \in (0,1)$ such that $f = f_1 + (1-\lambda)f_2$. As before, $P_f = P_{f_1} \cup P_{f_2}$ and $B_f = B_{f_1} \cap B_{f_2}$. Since f is a fractional independent set, $P_f \subseteq B_f$. Hence, $P_{f_1} \subseteq P_f \subseteq B_f \subseteq B_{f_1}$. Thus $P_{f_1} \subseteq B_{f_1}$ and, by a similar argument, $P_{f_2} \subseteq B_{f_2}$. It follows that f_1 and f_2 are fractional independent sets. Further, since f_1 and f_2 are also fractional dominating sets, each of them is a MFIS. Therefore, f is a proper convex combination of two different maximal fractional independent sets, a contradiction. □

Theorem 4.2. *A maximal fractional independent set f of a graph G is basic if and only if there is no other maximal fractional independent set g such that $B_f = B_g$ and $P_f = P_g$.*

Proof. Let f be a BMFIS of G. Then, by Lemma 4.1, f is a basic minimal fractional dominating set of G. The implication now follows from Theorem 2.3 and Lemma 4.1.

Conversely, suppose there is no other MFIS g of G with $B_f = B_g$ and $P_f = P_g$ but f is not basic. Then, f is a proper convex combination of some two different MFISs f_1 and f_2. But, for any $\lambda \in (0,1)$, if $h = \lambda f_1 + (1-\lambda)f_2$, then $B_f = B_h$ and $P_f = P_h$. Lemma 3.4 now gives a contradiction. □

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