On Incidence Graphs [1]

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Abstract

An incidence graph of a given graph G, denoted by $I(G)$, has its own vertex set $V(I(G)) = \{(ve) \mid v \in V(G),\ e \in E(G)$ and v is incident to e in $G\}$ such that the pair $(ue)(vf)$ of vertices $(ue), (vf) \in V(I(G))$ is an edge of $I(G)$ if and only if there exists at least one case of $u = v$, $e = f$, $uv = e$ or $uv = f$. In this paper we carry out a constructive definition on incidence graphs, and investigate some properties of incidence graphs and some edge-colorings on several classes of them.

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1 Introduction and concepts

Since there are many researching works on incidence graphs generated from some given graphs, so we shall study incidence graphs in this paper. All graphs mentioned are simple, finite, undirected graph. The other terminology not defined here can be found in [2], [5] and [8]. Given a graph G, the **incidence graph** of G, denoted by $I(G)$, has the vertex set

$$V(I(G)) = \{(ve) \mid v \in V(G), e \in E(G) \text{ and } v \text{ is incident to } e \text{ in } G\},$$

where the symbol (ve) stands for a vertex of $I(G)$, and first letter v denotes a vertex of G, second letter e indicates an edge of G who has the vertex

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v as one of its two end-points. We use the notation $(ue)(vf)$ to denotes an edge with end-vertices (ue) and (vf) in $I(G)$. So the pair $(ue)(vf)$ of vertices $(ue), (vf) \in V(I(G))$ is an edge of $I(G)$ if and only if there exists at least one case of $u = v$, $e = f$, $uv = e$ or $uv = f$.

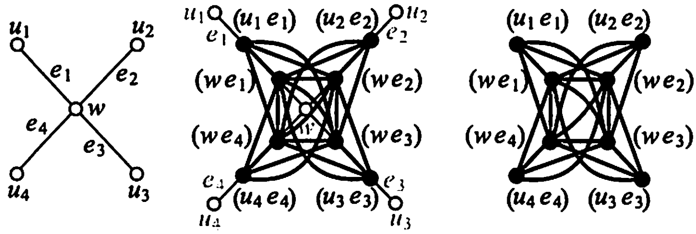
For the purpose of convenience, we use the notation $[k]$ to denite an integer set $\{1, 2, \dots, k\}$. A proper coloring $f : V(I(G)) \to [k]$ is an **incidence coloring** of a graph G, we then say that G is **k-incidence colorable**. The minimum number, for which there exists an incidence coloring of G, is called the **incidence chromatic number** of G, and is denoted by $\chi_i(G)$. This concept was first developed by Brualdi and Massey in 1993 (cf. [7]), and determined the incidence coloring numbers on trees T, complete graphs K_n and complete bipartite graphs $K_{m,n}$ are determined by $\chi_i(K_n) = n$ $(n \geq 2)$, $\chi_i(K_{m,n}) = m + 2$ $(m \geq n \geq 2)$, and $\chi_i(T) = \Delta(T) + 1$ $(|V(T)| \geq 2)$ respectively. And they posed the Incidence Coloring Conjecture (ICC): For every graph G, $\chi_i(G) \leq \Delta(G) + 2$. Chen et al.[4] determined the incidence coloring number of paths, cycles, fans, wheels, adding-edge wheels and complete 3-partite graphs. The **arboricity** $a(G)$ is the minimum k such that the graph G has an edge partition $\{E_1, E_2, \dots, E_k\}$ in which each induced subgraph G_i by E_i is a **forest**. In 1997, Guiduli [6] showed that incidence coloring is a special case of directed star arboricity that is introduced by Algor and Alon (cf. [1]). They pointed out that the ICC was solved in the negative following an example in [1]. It is shown that $\chi_i(G) \leq \Delta(G) + O(\log(\Delta(G)))$ in the article [1].

All above motivate us to research incidence graphs. In the second section we carry out a constructive definition on incidence graphs, and investigate some properties of incidence graphs and some edge-colorings on several classes of them.

2 Main results on incidence graphs

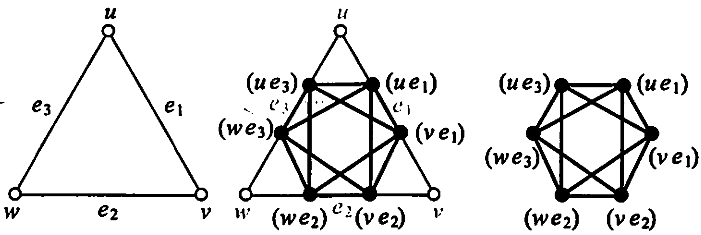
2.1 The I-construction

We can construct the incidence graph $I(G)$ of G by using the following method. We make a subdivision of G, denote by H, by replacing each edge $e = uv$ of G by a path $P = u(ue)(ve)v$ with 4 vertices, where u, (ue), (ve) and v denote the vertices of P respectively. Next we color only vertices (ue) and edges $(ue)(ve)$ in H with red color, color the rest vertices and edges of H with blue color. Now, we add a new red edge $(ue)(vf)$ to H if the distance between two red vertices (ue) and (vf) of H is either 2 or 3. The resulting graph is called the **near-incidence graph** of G, written by $I^*(G)$. It is easy to obtain the incidence graph $I(G)$ of G by deleting all blue vertices and edges from the near-incidence graph $I^*(G)$. Two examples of incidence graphs are shown in Figure 1 and Figure 2, respectively. We call this method the I-**construction**. Clearly, the I-construction is equivalent to the definition of incidence graphs.



(a) (b) (c)

Figure 1. (a) A star $K_{1,4}$; (b) the near-incidence graph $I^*(K_{1,4})$; (c) the incidence graph $I(K_{1,4})$.



(a) (b) (c)

Figure 2. (a) a complete graph K_3; (b) the near-incidence graph $I^*(K_3)$; (c) the incidence graph $I(K_3)$.

Let $\mathrm{d}_H(x)$ be the degree number of a vertex x and let $N(x)$ be the **neighborhood** of x in which each vertex is adjacent to x in a graph H. Obviously, the degree number of the vertex x is equal to the cardinality of $N(x)$, i.e., $\mathrm{d}_H(x) = |N(x)|$. To illustrate some properties of incidence graphs we present that the **join graph** $G + H$ of two graphs G and H is defined by adjoining each vertex of G to every vertex of H such that $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. A bipartite graph $K_{m,n}$ is defined by $K_{m,n} = \overline{K}_m + \overline{K}_n$, where $\overline{K}_s$ indicates the complementary graph of a complete graph K_s. Especially, a **star** is $K_{1,n} = K_1 + \overline{K}_n$. The join of $P_n + K_1$ is called a **fan** with $n+1$ vertices, denoted by F_{n+1}, and u is called the center of the fan.

Let P_n be a path of G, thus, $I(P_n) = P^2_{2(n-1)} \subseteq G$. Furthermore, we are easy to obtain the independent numbers $\alpha(G) \leq \alpha(I(G))$, and the diameters $D(G) \leq D(I(G))$ because $P_n \subset P^2_{2(n-1)} = I(P_n)$. Let C_n be a cycle of G, clearly, $I(C_n) = C^2_{2n}$. Since there is a star $K_{1,\Delta}$ in G where

$\Delta = \Delta(G)$, we have $I(K_{1,\Delta}) = K_\Delta + \overline{K}_\Delta \subseteq I(G)$. Immediately, the independent numbers $\alpha(I(G)) \geq \alpha(K_{1,\Delta}) = \Delta(G)$. For each vertex u of G, u corresponds a subgraph $K_s + \overline{K}_s$ of $I(G)$, where $s = \mathrm{d}_G(u)$.

From the facts above, every result in the following Theorem 1 can be deduced directly from the definition of incidence graphs.

Theorem 1. *Let $I(G)$ be the incidence graph of G.*

(i) $K_{\Delta(G)+1} \subset K_{\Delta(G)} + \overline{K}_{\Delta(G)} \subseteq I(G)$.

(ii) The independent numbers of two graphs G and $I(G)$ hold $\alpha(I(G)) \geq \alpha(G)$, and $\alpha(I(G)) \geq \Delta(G)$.

(iii) The diameters of two graphs G and $I(G)$ hold $D(I(G)) \geq D(G)$.

(iv) Then $I(G)$ is $2k$-connected if G is k-edge connected.

(v) The girth $g(I(G)) = 3$ if G contains at least two incident edges.

(v) There is at least a perfect matching in $I(G)$.

(vi) For any subgraph $H \subseteq G$, then $I(H) \subseteq I(G)$.

In Figure 1.(c), it is not difficult to see that $K_4 + \overline{K}_4 \subseteq I(K_{1,4})$ and $\alpha(I(K_{1,4})) = \alpha(K_{1,4})$. But, the figures (a) and (c) described in Figure 2 show us that $\alpha(I(K_3)) \geq \alpha(K_3)$.

2.2 Some properties of incidence graphs

Theorem 2. *Let $I(G)$ be the incidence graph of G, then $|V(I(G))| = 2|E(G)|$ and*

$$2|E(I(G))| = \sum_{k=2}^{3} \left[(-1)^{k-1} k \sum_{v \in V(G)} \mathrm{d}_G^{k-1}(v) \right]. \tag{1}$$

Proof. According to the definition of incidence graphs we know that every edge $e = uv \in E(G)$ corresponds two vertices (ue) and (ve) of $I(G)$, it means $|V(I(G))| = 2|E(G)|$. Let $P = xuvy$ be a path of G. We write three edges of this path P as $e_1 = xu$, $e = uv$ and $e_2 = vy$. Clearly, the vertex (ue_1) is not adjacent to the vertex (ve_2) in the incidence graph $I(G)$. We compute the degree number of a vertex (ue) of $I(G)$ for an fixed edge $e = uv$ and the end-vertex u.

For every $x \in N(u) \setminus \{v\}$, we have an edge $e_1 = xu$, the vertex (ue_1) in $I(G)$ is adjacent to the vertex (ue) since these two vertices are constructed by a vertex u of G. And more, the vertex (xe_1) in $I(G)$ is adjacent to the vertex (ue) since $e_1 = xu$ in G. Furthermore, for each vertex $t \in N(v)\setminus\{u\}$, so $e_2 = vt$, the vertex (ve_2) in $I(G)$ is adjacent to the vertex (ue) since $e = uv$ in G. Therefore, the degree of a vertex (ue) of $I(G)$ is equal to

$$\mathrm{d}_{I(G)}((ue)) = 2\mathrm{d}_G(u) - 2 + \mathrm{d}_G(v). \tag{2}$$

Analogously, there exists $\mathrm{d}_{I(G)}((ve)) = 2\mathrm{d}_G(v) - 2 + \mathrm{d}_G(u)$. Hence, each edge $e = uv \in E(G)$ contributes

$$\mathrm{d}_{I(G)}((ue)) + \mathrm{d}_{I(G)}((ve)) = 3[\mathrm{d}_G(u) + \mathrm{d}_G(v)] - 4.$$

Let $N(u) = \{v_i \mid 1 \le i \le \mathrm{d}_G(u)\}$ and $N_e(u) = \{e_i = uv_i \mid v_i \in N(u)\}$. We have

$$\mathrm{d}_{I(G)}((ue_i)) + \mathrm{d}_{I(G)}((v_ie_i)) = 3[\mathrm{d}_G(u) + \mathrm{d}_G(v_i)] - 4 \text{ for } i = 1, 2, \ldots, \mathrm{d}_G(u).$$

Adding all of the above equations together yields the following

$$\sum_{v_i \in N(u)} [\mathrm{d}_{I(G)}((ue_i)) + \mathrm{d}_{I(G)}((v_ie_i))] = 3\Big[\mathrm{d}^2_G(u) + \sum_{v_i \in N(u)} \mathrm{d}^2_G(v_i)\Big] - 4\mathrm{d}_G(u), \tag{3}$$

Notice that $u \in N(v_i)$ for $1 \le i \le \mathrm{d}_G(u)$, then the form (3) can be thought of as the sum on the $\mathrm{d}_G(u)$ edges e_i for $1 \le i \le \mathrm{d}_G(u)$. Therefore,

$$2|E(I(G))| = \sum_{x \in V(I(G))} \mathrm{d}_{I(G)}(x) = -4|E(G)| + 3 \sum_{u \in V(G)} \mathrm{d}^2_G(u), \tag{4}$$

Since $2|E(G)| = \sum_{u \in V(G)} \mathrm{d}_G(u)$, it is easy to see the form (4) is just the equality (1). □

The following is a collection of several observations on incidence graphs.

Corollary 3. *Let $I(G)$ be the incidence graph of G. There are the following assertions.*

(i) Then $3\delta(G) - 2 \le \delta(I(G)) \le \Delta(I(G)) \le 3\Delta(G) - 2$.
(ii) If G is k-regular, then $I(G)$ is $(3k-2)$-regular, and vice versa.
(iii) Let H be a subgraph of G, then $\Delta(I(H)) \le \Delta(I(G))$.
(iv) If G is connected, then $I(G) \ne K_n$ for all $n \ge 3$.
(v) If $\Delta(G) \ge 3$, then $I(G)$ is not planar.
(vi) G is eulerian if and only if $I(G)$ is eulerian.

Proof. (i), (ii) and (iii) are deduced directly by the equality (2).

(iv) By the contradiction. We may suppose $I(G) \cong K_n$ for $n \ge 3$, that is, $I(G)$ has n vertices and is $(n-1)$-regular. Since $I(K_n)$ is $(3(n-1)-2)$-regular by the assertion (ii) above, we have $n - 1 = 3(n-1) - 2 = 3n - 5$, or $2n = 4$, a contradiction.

(v) Notice that $K_{3,3} \subset K_\Delta + H_\Delta \subseteq I(G)$ by the assertion (1) of Theorem 1, we are done.

(vi) Assume that G is eulerian, so each degree number $\mathrm{d}_{I(G)}((ue))$ is even from the equality (2), it means that $I(G)$ is eulerian.

Conversely, every $d_{I(G)}((ue))$ is even by the assumption of $I(G)$ being eulerian. Since the equality (2) shows us that every $d_G(v)$ is even, thus, G is eulerian. □

As it is well known that every $2k$-regular graph is 2-factorable first proved by J. Petersen (cf. [2]). Since $I(C_n) = C_{2n}^2$ is 4-regular so that every $I(C_n)$ contains two edge-disjoint Hamilton cycles for $n \geq 3$.

Lemma 4. *Let T be a tree of order at least 3, then $I(T)$ has a Hamiltonian cycle.*

Proof. If T is a star on n vertices and $n \geq 3$, let w be its center vertex and v_i be its leaves. We can write all edges of the star T by such forms $e_i = wv_i \in E(T), 1 \leq i \leq n-1$. Immediately, $C = (we_1)(v_1e_1)(we_2)(v_2e_2)\cdots(we_{n-1})(v_{n-1}\,e_{n-1})(we_1)$ is just a Hamilton cycle of $I(T)$, see for an example shown in Figure 1.(a) and (c).

Suppose that T is not a star. We shall apply the induction on orders of trees. For $n = 4$, thus $T = P_4$, a path of order 4, it is not hard to find a Hamilton cycle in $I(P_4)$. Assume that this lemma is true for all orders of trees that are smaller. Let S be the set of **end-node vertices** of T, i.e., for each $u \in S$ the neighborhood $N(u)$ contains at least $d_T(u) - 1$ leaves of T.

Case 1. If $d_T(u) = 2$ for an end-node vertex $u \in S$. Since T is not a star, so there is a path $P = veue'we_iw_i$ on four vertices, where three edges $e = vu$, $e' = uw$ and $e_i = ww_i$. Clearly, v is a leaf of T. By the induction hypothesis, the incidence graph $I(T - v)$ contains a Hamilton cycle C. Notice that the degree number of the vertex (ue') in $I(T - v)$ is just two, that is, (ue') is adjacent to (we') and (we_i) in C. Hence, $C-(ue')(we')+\{(we')(ue), (ue)(ve), (ve)(ue')\}$ is a Hamilton cycle of $I(T)$.

Case 2. $d_T(u) = k \geq 3$ for an end-node vertex $u \in S$.

Let the neighborhood $N(u) = \{v, v_1, \ldots, v_{k-2}, w\}$. Here, each vertex of $N(u) \setminus \{w\}$ is a leaf of T in T. The edges incident to u are denoted by $e = vu$, $e' = uw$ and $e_i = uv_i$ for $2 \leq i \leq k - 2$. We have a Hamilton cycle C^* of the incidence graph $I(T - v)$ by the induction hypothesis. Notice that each vertex (v_ie_i) in $I(T - v)$ is adjacent to (ue') and (ue_j) for $1 \leq j \leq k - 2$. And, in $I(T)$, the vertex (ve) is adjacent to each one of $\{(ue), (ue'), (ue_i) \mid 1 \leq i \leq k - 2\}$, as well as the vertex (ue) is adjacent to every one of $\{(ve), (ue'), (ue_i), (v_ie_i) \mid 1 \leq i \leq k - 2\}$. There must exist an edge $(v_ie_i)(ue_i) \in E(C^*)$. Immediately, thus, $C^* - (v_ie_i)(ue_i) + \{(ue_i)(ve), (ve)(ue), (ue)(v_ie_i)\}$ is a Hamilton cycle of $I(T)$. □

Theorem 5. *For any connected graph G on n (≥ 3) vertices, the incidence graph $I(G)$ is hamiltonian.*

Proof. Let T be a spanning tree of G. Since $I(T) \subseteq I(G)$ by the assertion (vi) of Theorem 1, so we conclude that $I(G)$ is hamiltonian since $I(T)$ contains a Hamilton cycle from Lemma 4. □

2.3 Edge-colorings of some incidence graphs

Analogously to the incidence chromatic number of a graph, we define the incidence chromatic index on graphs as following. A proper edge coloring $f : E(I(G)) \to [k]$ is called an **incidence edge k-coloring** of a graph G, so G is said to be **k-incidence edge colorable.** The minimum number of k over all incidence edge k-colorings of G, $\chi'_i(G)$, is called the **incidence chromatic index** of G. By means of the assertion (vi) in Corollary 3, we have the following Lemma 6.

Lemma 6. *C_n is a cycle of order n, then* $\chi'_i(C_n) = 4$.

Lemma 7. *For a star $K_{1,n}$ ($n \geq 1$),* $\chi'_i(K_{1,n}) = \Delta(I(K_{1,n})) = 2n - 1$.

Proof. Write $V(K_{1,n}) = \{u_i \mid i \in [n]\} \cup \{w\}$ and $E(K_{1,n}) = \{e_i = wu_i \mid i \in [n]\}$, where w is the center of the star $K_{1,n}$. Suppose that n is even. Clearly, there are n perfect matchings in $I(K_{1,n})$ as the forms $M_j = \{(we_{j+i-1})(u_ie_i) \mid i \in [n]\}$ for each integer $j \in [n]$ and the indices $j + i - 1$ are under model n. Since the graph $I(K_{1,n}) - V^*$ obtained by deleting the vertex set $V^* = \{(u_ie_i) \mid i \in [n]\}$ is congruent to K_n (see Figure 1.(c)) and $\chi'(K_n) = n - 1$ for even n, we assign a color to each perfect matching M_j ($1 \leq j \leq n$), we are done.

Now, let n be odd. Coloring $n - 1$ perfect matchings M_j of $I(K_{1,n})$ with $n - 1$ colors such that each perfect matching receives only a color, where $2 \leq j \leq n$. We use the first matching $M_1 = \{(we_i)(u_ie_i) \mid i \in [n]\}$ to form n edge-disjoint independent sets of $I(K_{1,n})$ in the following:

$S_1 = \{(we_n)(u_ne_n), (we_1)(we_{n-1}), (we_2)(we_{n-2}), \ldots, (we_{(n-1)/2})(we_{(n+1)/2})\}$,

$S_2 = \{(we_1)(u_1e_1), (we_2)(we_n), (we_3)(we_{n-1}), \ldots, (we_{(n+1)/2})(we_{(n+3)/2})\}$,

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$S_n = \{(we_{n-1})(u_{n-1}e_{n-1}), (we_{n-2})(we_n), (we_1)(we_{n-2}), \ldots, (we_{(n-3)/2})(we_{(n-1)/2})\}$.

Obviously, we can assign a new color to each set S_i for every $i \in [n]$, and then we have $\chi'_i(K_{1,n}) = 2n - 1$. Therefore, we obtain the ordinary chromatic number $\chi'(K_n + \overline{K}_n) = 2n - 1$ since $I(K_{1,n}) = K_n + \overline{K}_n$. □

A double star, denoted by $S_{m,n}$, is a tree with two center vertices u, v and $m + n - 2$ leaves. We have $V(S_{m,n}) \setminus \{u, v\} = L(S_{m,n})$, $V(S_{m,n}) = N(u) \cup N(v)$ and the diameter $D(S_{m,n}) = 3$.

Lemma 8. *Let $S_{m,n}$ be a double star with $m \leq n$, then $\chi_i'(S_{m,n}) = 2n + m - 2$.*

Proof. Let $m = d_{S_{m,n}}(u)$ and $n = d_{S_{m,n}}(v)$. We have two neighborhoods $N(u) = \{u_1, u_2, \ldots, u_{m-1}, v\}$ and $N(v) = \{v_1, v_2, \ldots, v_{n-1}, u\}$. Let e be the edge between u and v in $S_{m,n}$.

We consider the structure of the incidence graph $I(S_{m,n})$ by our I-construction in the following. Since the vertex u of $S_{m,n}$ corresponds a subgraph $K_m + \overline{K}_m$ of $I(S_{m,n})$, so the vertex (ve) is in $\overline{K}_m$ and (ue) is in K_m. Similarly, it follows the fact that the vertex v corresponds a subgraph $K_n + \overline{K}_n$ of $I(S_{m,n})$, we are easy to see that the vertex (ue) is in $\overline{K}_n$ and (ve) is in K_n. Furthermore, in $I(S_{m,n})$, the vertex (ve) is adjacent to each vertex of $V(K_m)$ and the vertex (ue) is adjacent to each vertex of $V(K_n)$. Clearly, $V(I(S_{m,n})) = V(K_m + \overline{K}_m) \cup V(K_n + \overline{K}_n)$. Therefore, no vertex of $V(K_m + \overline{K}_m) \setminus \{(ue)\}$ is adjacent to any one of $V(K_n + \overline{K}_n) \setminus \{(ve)\}$.

By Lemma 7, we can use $2n-1$ colors to form a proper edge-coloring of the subgraph $K_n + \overline{K}_n$ of $I(S_{m,n})$. Let S be the set of the $2n-1$ colors used above, and let S' the set of those colors that occur at the vertex (ue) of $V(K_n + \overline{K}_n)$. Next, we can apply another $m-1$ new colors and the colors of $S \setminus S'$ to product a proper edge-coloring of the subgraph $K_m + \overline{K}_m$ of $I(S_{m,n})$. Hence, two edge-colorings above imply a proper $(2n+m-2)$-edge coloring of $I(S_{m,n})$. Note the maximum degree $\Delta(I(S_{m,n})) = 2n + m - 2$, we are done. □

Theorem 9. *For any tree T, $\chi_i'(T) = \Delta(I(T))$.*

Proof. Applying the induction on orders of trees in the following. From Lemma 7 and Lemma 8, this theorem is true if T is a star or a double star, so that we may assume the diameter $D(T) \geq 4$ and this theorem holds for smaller orders of trees.

Let S be the set of end-node vertices of T, i.e., for each $w \in S$ the neighborhood $N(w)$ contains at least $d_T(w) - 1$ leaves of T. We take a vertex $u \in S$ such that $d_T(u) \leq d_T(w)$ for all $w \in S$. Let $N(u) = \{u_1, u_2, \ldots, u_{m-1}, v\}$, where the vertex v does not belong to S since $D(T) \geq 4$. Let $H = T - \{u_1, u_2, \ldots, u_{m-1}\}$. Clearly, $\Delta(I(H)) = \Delta(I(T))$ since T neither is a star nor a double star. By the induction hypothesis, the incident graph $I(H)$ has a proper $\Delta(I(H))$-edge coloring f.

By Lemma 8 and the fact that $d_T(u) \leq \Delta(T)$ (note that u corresponds a subgraph $K_m + \overline{K}_m$ of $I(T)$ where $m = d_T(u)$), we can make a proper $\Delta(I(H))$-edge coloring of $I(T)$ by using the edge coloring f and $\Delta(I(H))$ colors. □

Theorem 10. *For $n \geq m \geq 1$, then $\chi_i'(K_{m,n}) = \Delta(I(K_{m,n})) = 2n+m-2$.*

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $W = \{w_1, w_2, \ldots, w_m\}$ be the two partite sets of $K_{m,n}$, i.e., $V(K_{m,n}) = U \bigcup W$. Note that each vertex u_j corresponds a complete subgraph $K_m^j \subset K_m^j + \overline{K}_m^j$ for $1 \leq j \leq n$, and each vertex v_i corresponds a subgraph $K_n^i \subset K_n^i + \overline{K}_n^i$ for $1 \leq i \leq m$. Clearly,

(F): K_m^j, K_m^s, K_n^i and K_n^t are pairwise disjoint for $1 \leq j, s \leq n$ and $1 \leq i, t \leq m$.

We have two cases in the following.

Case 1. If n is even. Therefore, the ordinary chromatic index $\chi'(K_n^i) = n - 1$ (cf. [2]). By the fact **(F)** above, we can use $n - 1$ colors to form a proper edge coloring for each of K_m^j, K_m^s, K_n^i and K_n^t for $1 \leq j, s \leq n$ and $1 \leq i, t \leq m$. Let $S = (\bigcup_{i=1}^m E(K_n^i)) \bigcup (\bigcup_{j=1}^n E(K_m^j))$, thus, the graph $H = T(K_{m,n}) - S$ is just a bipartite graph, and $\Delta(H) = (2n + m - 2) - (n - 1) = n + m - 1$. It is easy to see that the ordinary chromatic index $\chi'(H) = \Delta(H)$ (cf. [2]). Combining two colorings above products a proper $\Delta(I(K_{m,n}))$-edge coloring of $I(K_{m,n})$.

Case 2. If n is odd. Therefore, the ordinary chromatic index $\chi'(K_n^i) = n$ (cf. [2]). By the fact **(F)** above, there is a proper n-edge coloring for each of K_m^j, K_m^s, K_n^i and K_n^t for $1 \leq j, s \leq n$ and $1 \leq i, t \leq m$.

Let e_{ij} denote an edge between u_i and v_j in $K_{m,n}$ for $1 \leq i \leq n$, $1 \leq j \leq m$. All edges as the form $(u_j e_{ij})(v_i e_{ij})$ form a perfect matching M of $I(K_{m,n})$. Note that there are $n - 1$ colors that occur at each vertex $(v_i e_{ij})$ of $V(K_n^i)$, so we can use the rest one to color the edge $(u_j e_{ij})(v_i e_{ij})$. Let $F = T(K_{m,n}) - S - M$, where S is described in Case 1. We use $n + m - 2$ new colors to color F such that $\chi'(F) = \Delta(F) = n + m - 2$ since F is bipartite.

Two colorings above can be used to yield a proper $\Delta(I(K_{m,n}))$-edge coloring of $I(K_{m,n})$. □

We give a box $L(e)$ to each edge $e \in E(G)$ where $L(e)$ is a positive integer set, so we can define a **list-edge-coloring** h of G if h is a proper edge coloring of G and $h(e) \in L(e)$ for all $e \in E(G)$. Let $f(x)$ be a positive integer-valued function defined on $E(G)$, we say that G is **f-edge-choosable** if G has a list-edge-coloring for **any list box** $L(e)$ with $|L(e)| \geq f(e)$ for each edge $e \in E(G)$. If $f(e)$ is a constant k, we also say G to be **k-list-edge-choosable**. The least integer k such that G is k-edge-choosable is called the **edge-choosability index** (list chromatic index) of G, write it by $\chi'_l(G)$. We are ready to show the following:

Theorem 11. *Let F_{n+1} be a fan with $n + 1$ vertices for $n \geq 13$, then $\chi'_i(F_{n+1}) = \Delta(I(F_{n+1})) = 2n + 1$.*

Proof. Let $P_n = v_1 v_2 \ldots v_n$ be a path of order n for $n \geq 2$ and K_1 be a complete graph with one vertex u. A fan $F_{n+1} = P_n + K_1$ has the edge set $E(F_{n+1}) = \{e_{i,i+1} = v_iv_{i+1} \mid 1 \le i \le n-1\} \cup \{f_i = uv_i \mid 1 \le i \le n\}$. Let $H = I(F_{n+1})$. We are easy to see that the degree $d_H((xe))$ of a vertex (xe) in H is one of $2n+1, n+4, n+2, 7, 6, 5$. Note that the vertex u of F_{n+1} corresponds a subgraph $K_n + \overline{K}_n$, and furthermore the graph $H' = H - E(K_n + \overline{K}_n)$ satisfies $4 \le d_{H'}((xe)) \le 7$ for any $(xe) \in V(H')$. By the proof of Lemma 8, we can give a proper edge coloring of the subgraph $K_n + \overline{K}_n$ by using $2n+1$ colors, let C denote the set of these $2n+1$ colors. An edge s of H' is at most incident to n edges of the subgraph $K_n + \overline{K}_n$, and let all colors these n edges are used is denoted the set $C(s)$. Therefore, we can set a list box $\pi(s) = C \setminus C(s)$ for every edge $s \in E(H')$, clearly, $n \le |\pi(s)| \le n+1$. For $n \ge 13 = 2\Delta(H') - 1$, we can obtain a proper edge coloring of H' since in [3] the author pointed out that $\chi'_l(G) \le 2\Delta(G) - 1$ by coloring greedily in an arbitrary order. Immediately, $I(F_{n+1})$ admits a proper $(2n+1)$-edge coloring. □

In fact, $\chi'_i(F_{n+1}) = \Delta(I(F_{n+1})) = 2n+1$ holds for $2 \le n \le 12$.

Lemma 12. *Let w be a vertex out of a graph G and u a vertex in G. Let H be a graph obtained by adjoin w with u. Let $n = d_G(u)$ and $M = \max\{d_G(x) \mid x \in N(u)\}$. If $2n + 1 + 3n(n + 2M - 1)/2 \le \Delta(I(G))$, then $\chi'_i(H) \le \chi'_i(G)$.*

Proof. Let $e = wu$ in H. Note that the vertex u corresponds a subgraph $K_n + \overline{K}_n$ in $I(G)$. Each vertex in $K_n \subseteq K_n + \overline{K}_n$ is at most adjacent to $2n + M - 1$ vertices, and each each vertex in $\overline{K}_n \subseteq K_n + \overline{K}_n$ is at most adjacent to $n + 2M - 1$ vertices. All edges that are incident to vertices of $K_n + \overline{K}_n$ are at most $n^2 + n(n+1)/2 + nM + 2n(M-1) = 3n(n+2M-1)/2$. Suppose that $I(G)$ is colored well by $\chi'_i(G)$ colors. Since $G \subset H$, add $2n+1$ new colors to the edges which are incident to two vertices (we) and (ue) of $I(H)$, we obtain a proper edge coloring of $I(H)$ that implies that $\chi'_i(H) \le \chi'_i(G)$. □

Note that there only two vertices of degree 2 in a fan, and they are not adjacent. Therefore, we are not difficult to have the following by Lemma 12:

Corollary 13. *For a wheel $W_{n+1} = C_n + K_1$ for $n \ge 13$, where C_n is a cycle on n vertices, then $\chi'_i(W_{n+1}) = \Delta(I(W_{n+1})) = 2n+1$.*

Corollary 14. *Let H be a graph with at least two leaves w_1 and w_2. Let u_1 and u_2 be two non-adjacent vertices of a graph G. We have a graph I obtained by identifying u_i with w_i for $i = 1, 2$. Let $n_i = d_G(u_i)$ and $M_i = \max\{d_G(x) \mid x \in N(u_i)\}$ for $i = 1, 2$. If $2n_i + 1 + 3n_i(n_i + 2M_i - 1)/2 \le \Delta(I(G))$ for $i = 1, 2$, then $\chi'_i(G + u_1u_2) \le \chi'_i(G)$.*

3 Some problems

For a simple graph G, we wish to know whether there are some relations among G, its near-incidence graph $I^*(G)$ and $I(G)$. Our work on the incidence chromatic index of graphs implies the following conjecture:

Conjecture 1. *Let G be a simple graph, then $\chi_i'(G) = \Delta(I(G))$.*

Furthermore, we have the following conjectures:

Conjecture 2. *Let $G = (V, E)$ be a k-regular connected graph with order not less than 3.*

1. If $k \equiv 0 \pmod 2$, the incidence graph $I(G)$ contains at least $\frac{3k-2}{2}$ edge-disjoint Hamilton cycles.

2. If $k \equiv 1 \pmod 2$, $I(G)$ contains at least $\frac{3(k-1)}{2}$ edge-disjoint Hamilton cycles and one 1-factor.

Conjecture 3. *Suppose that G has at least two vertices of maximum degree and the distance between two vertices of maximum degree is at least 3, then $\chi_i(G) = \Delta(G) + 1$.*

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