Fractional inverse and inverse fractional domination

P. D. Johnson Jr.
Department of Mathematics and Statistics
Auburn University, AL 36849
johnspd@auburn.edu

M. Walsh
Department of Mathematical Sciences
Indiana University-Purdue University
Fort Wayne, IN 46805
walshm@ipfw.edu

Abstract

We examine the inverse domination number of a graph, as well as two reasonable candidates for the fractional analogue of this parameter. We also examine the relations among these and other graph parameters. In particular, we show that both proposed fractional analogues of the inverse domination number are no greater than the fractional independence number. These results establish the fractional analogue of a well-known conjecture about the inverse domination and vertex independence numbers of a graph.

1 Domination and its variants

G will be a finite, simple graph throughout, with edge set E and vertex set V, with $n = |V|$. As usual, if $v \in V$, $N(v) = \{u \in V; uv \in E\}$ and $N[v] = \{v\} \cup N(v)$. If $S \subseteq V$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = S \cup N(S)$.

A set $S \subseteq V$ is *dominating* in G if $V = N[S]$. The domination number of G is defined as:

$$\gamma(G) = \min[|S|; S \subseteq V \text{ is dominating}]$$

We call a dominating set S *efficient* if, for any $u, v \in S$, the intersection $N[u] \cap N[v] = \emptyset$; in other words, every vertex in G is dominated by exactly one vertex in S. Not every graph admits an efficient dominating set; for those that do, however, any efficient dominating set is minimum.

We can also characterize domination algebraically. Let A be an adjacency matrix of G with respect to some fixed ordering of the vertices, I be the $n \times n$ identity matrix, and for each $S \subseteq V$ let f_S be the characteristic vector of S with respect to our chosen vertex ordering. Then S is dominating if and only if $(A+I)f_S \geq \bar{1}$, and efficient dominating if and only if equality holds.

A *fractional dominating function on* G is a function $f : V \longrightarrow [0,1]$ such that for every $v \in V$, $\sum_{u \in N[v]} f(u) \geq 1$. That is, representing f as a vector, $(A+I)f \geq \bar{1}$.

The *fractional domination number* of G is

$$\gamma_f(G) = \min[\sum_{v \in V} f(v); f \text{ is a fractional dominating function on } G.]$$

This minimum is always achieved, by the compactness of the unit interval. It is immediately perceived that finding the fractional domination number of a graph is the linear relaxation of the problem of finding the domination number.

Likewise, we can say that a function $f : V \longrightarrow [0,1]$ is *fractional efficient dominating* if $\sum_{u \in N[v]} f(u) = 1$ for every $v \in V$; algebraically, this means that $(A+I)f = \bar{1}$. By a conventional argument from linear programming, it is easy to see that if f is any non-negative function whose vector form satisfies this matrix equation, and if G has no isolated vertices, then f is a minimum fractional dominating function.

In [2], Kulli and Sigarkanti introduced the *inverse domination number*, $\gamma'(G) = \min\ [|U|; U \subseteq V \setminus S$ dominating, S minimum dominating], and noted that $\gamma'(G)$ is well-defined if G has no isolated vertices.

Proposition 1.1 *If f is a minimal fractional dominating function of G, and G has no isolated vertices, then $1-f$ is a fractional dominating function of G.*

Proof. Let $v \in V$. If there exists a vertex $u \in N[v]$ such that $f(u) = 0$, then $1 - f(u) = 1$ and hence v is dominated by $1 - f$. We assume therefore that $f(u) > 0$ for all $u \in N[v]$; in particular, $f(v) > 0$.

Let $w \in N(v)$ be a vertex such that $f(v) + f(w) \leq 1$; if no such vertex exists, then we can reduce the value of $f(v)$ and maintain domination, contradicting minimality. So $[1 - f(v)] + [1 - f(w)] = 2 - [f(v) + f(w)] \geq 1$, and hence v is dominated by $1 - f$. □

A dominating function g of $G = (V, E)$ that satisfies the inequality $g(v) \leq 1 - f(v)$ for some minimum fractional dominating function f of

G and for all vertices $v \in V$ is called an *inverse fractional dominating function* of G; the previous result guarantees that such a function exists for any graph with no isolated vertices. (As such, for the remainder of this paper we will assume that graphs under discussion have no isolated vertices.) Then the *inverse fractional domination number* of G is defined as

$$(\gamma_f)'(G) = \min_g \left[\sum_{v \in V} g(v)\right]$$

where the minimum is taken over all inverse fractional dominating functions. Again, the minimum is achieved, by a standard compactness argument and by Proposition 1.1.

Since every inverse fractional dominating function of G is a fractional dominating function of G, $\gamma_f(G) \leq (\gamma_f)'(G)$; one of our ambitions is to determine when equality holds. We are far from settling this question, but the following easy observations suggests that $\gamma_f = (\gamma_f)'$ "usually".

Proposition 1.2 *If G has a minimum fractional dominating function f satisfying $f(v) \leq \frac{1}{2}$ for all $v \in V$ then $\gamma_f(G) = (\gamma_f)'(G)$.*

Proof. $f \leq 1 - f$, so f is also a minimum inverse fractional dominating function. □

Corollary 1.2.1 *If G is regular of positive degree, then $\gamma_f(G) = (\gamma_f)'(G)$.*

Proof. If G has degree k, then the constant function $\frac{1}{k+1}$ is a fractional efficient dominating function, and hence a minimum fractional dominating function; and clearly $\frac{1}{k+1} \leq \frac{1}{2}$. □

Corollary 1.2.2 *If G has two or more vertices of degree $n-1$ then $\gamma_f(G) = (\gamma_f)'(G)$.*

Proof. If $v, w \in V, v \neq w$ each have degree $n-1$, then $f(v) = f(w) = \frac{1}{2}$ and $f = 0$ elsewhere is a fractional efficient dominating function. □

We doubt that the converse of Proposition 1.2 holds, although we have no counterexample. However, the following weak converse of Proposition 1.2 has some uses.

Proposition 1.3 *Suppose that, for some $v \in V$, $f(v) > \frac{1}{2}$ for every minimum fractional dominating function f. Then $\gamma_f(G) < (\gamma_f)'(G)$.*

Proof. No minimum inverse fractional dominating function g can be a minimum fractional dominating function, since for any such g, $g(v) \leq 1 - f(v) < \frac{1}{2}$. □

Corollary 1.3.1 *If G has exactly one vertex of degree $n-1$, then $\gamma_f(G) < (\gamma_f)'(G)$.*

Proposition 1.4 *If G is a complete r-partite graph, $r \geq 2$, then $\gamma_f(G) = (\gamma_f)'(G)$ if and only if there is not a unique part of cardinality 1.*

Proof. Note first that, if G does possess a single part of cardinality 1, then by Corollary 1.3.1 we have that $\gamma_f(G) < (\gamma_f)'(G)$.

Otherwise, suppose that the parts of G are of cardinalities $n_1 \leq \ldots \leq n_r$. If $n_1 = n_2 = 1$, the conclusion follows from Corollary 1.2.2. If $n_1 \geq 2$, observe that the function f defined by:

$$f(v) = \left[\left(n_j - 1 \right) \left(\sum_{i=1}^{r} \frac{n_i}{n_i - 1} - 1 \right) \right]^{-1}$$

for all vertices in the j^{th} part, $j = 1, \ldots, r$ is a fractional efficient dominating function, and is therefore a minimum fractional dominating function. Further, the values of f are all at most $\frac{1}{2}$. (Verification of these facts is left to the reader.) The conclusion follows from Proposition 1.2. □

There are other situations in which Proposition 1.3 supplies guidance. For example, in the case of $G = P_{3k}$, a path on $3k$ vertices, $\gamma = \gamma_f = k$. If the vertices of that path are labelled $v_1, \ldots, v_{3k}$, from one end of the path to the other, the characteristic function of the unique minimum dominating set (the set $\{v_{3i-1}; i = 1, \ldots, k\}$) is the unique minimum fractional dominating function. Thus we know that $\gamma_f(G) < (\gamma_f)'(G)$. In fact, in this case $(\gamma_f)' = k + 1$, a value which is achieved with a weight of 1 on each end vertex and $\frac{1}{2}$ on every other vertex not in the dominating set.

2 A different fractional form

At first glance, it may seem likely that $(\gamma_f)' \leq \gamma'$, but in fact we have no proof. The pursuit of this question may be facilitated by the following.

Suppose that $D \subseteq V$ is a minimum dominating set of G. We say that g is a *fractional inverse dominating function* of G (with respect to D) if g is a fractional dominating function of G such that $g(v) = 0$ for every $v \in D$. (To borrow language from analysis, the support of g must be disjoint from D.) The *fractional inverse domination number* of G is then:

$$(\gamma')_f = \min_g \left[\sum_{v \in V} g(v) \right]$$

where the minimum is taken over all fractional inverse dominating functions g (with respect to minimum dominating sets D). This parameter is well-defined when G has no isolated vertices, for the same reason that $\gamma'(G)$

is, and by a compactness argument it is easy to see that the minimum is achieved. Clearly, $(\gamma')_f(G) \leq \gamma'(G)$.

Proposition 2.1 *If* $\gamma(G) = \gamma_f(G)$, *then* $(\gamma_f)'(G) \leq (\gamma')_f(G)$.

Proof. If $\gamma(G) = \gamma_f(G)$ then for any minimum dominating set S, the characteristic function f_S is a minimum fractional dominating function and the conclusion is therefore straightforward from the definitions. □

Another easy result which relates these two fractional analogues of inverse domination is the following, which we state without proof.

Proposition 2.2 *If* G *has a unique vertex* v *of degree* $n-1$, *then* $(\gamma')_f(G) = (\gamma_f)'(G) = \gamma_f(G-v)$.

We wonder if anything can be concluded in those cases where $\gamma' = (\gamma')_f$ or $\gamma' = (\gamma_f)'$? We have no good reason for raising the question except that, in all specific examples where we have worked out the values of the parameters under discussion and either of the above-mentioned equalities holds, it has turned out that $\gamma = \gamma_f$ as well.

3 Convexity

Eventually the questions about the relations among these parameters will bring about an inspection of the sets of dominating sets of functions involved in their definitions. Recall that a set S of vectors is *convex* if for any $v_1, \ldots, v_k \in S$ and nonnegative scalars $\lambda_1, \ldots, \lambda_k$, if $\sum_{i=1}^k \lambda_i = 1$ then $\sum_{i=1}^k \lambda_i v_i \in S$. To prove convexity it is sufficient to verify the case $k = 2$. As we can speak interchangeably of functions on the vertex set and vectors, we shall assume it is clear what it means for a set of such functions to be convex. It is obvious that the sets of all fractional and minimum fractional dominating functions of a graph are convex; this is also the case for minimum inverse fractional dominating functions.

Proposition 3.1 *The set of all minimum inverse fractional dominating functions on a graph* G *is convex.*

Proof. Suppose that g_1, g_2 are minimum inverse fractional dominating functions on G, with respect to minimum fractional dominating functions f_1, f_2 respectively. (In other words, $g_i \leq 1 - f_i$ for $i = 1, 2$.) Let $\lambda \in [0,1]$. Then $\lambda f_1 + (1-\lambda) f_2$ is a minimum fractional dominating function, and $\lambda g_1 + (1-\lambda) g_2 \leq \lambda(1-f_1) + (1-\lambda)(1-f_2) = 1 - (\lambda f_1 + (1-\lambda) f_2)$.

Furthermore, the function $\lambda g_1 + (1-\lambda) g_2$ is fractional dominating, and the sum of its values is equal to the sum of the values of g_1 and g_2, namely

$(\gamma_f)'(G)$; it is therefore a minimum inverse fractional dominating function on G. □

To see that the same is not true of the set of minimum fractional inverse dominating functions, consider the graph P_4. The minimum fractional inverse dominating functions in this case are in fact the characteristic functions of the minimum dominating sets.

The next proposition is an analogue of a well-known result of Grinstead and Slater ([1],[4]) for fractional domination; the proof of it is much the same as the proof of the original, and is related to Proposition 3.1. Let $\text{Aut}(G)$ denote the group of graph automorphisms of G, and for each $v \in V$, let the *automorphism class* of v be $\{\sigma(v); \sigma \in \text{Aut}(G)\}$, the orbit of v under the action of the automorphism group. It is elementary that the automorphism classes partition V.

Proposition 3.2 *If G has no isolated vertices, then there is a minimum inverse fractional dominating function on G which is constant on each automorphism class.*

Proof. Let g be a minimum inverse fractional dominating function on G, $g \leq 1 - f$ for some minimum fractional dominating function f. Define $\hat{f} : V \to [0,1]$ by $\hat{f}(v) = \frac{1}{|\text{Aut}(G)|} \sum_{\sigma \in \text{Aut}(G)} f(\sigma(v))$, and $\hat{g}$ similarly. Both of these new functions are constant on each orbit of the automorphism group $\text{Aut}(G)$ of the graph G.

Clearly the composition of a (minimum) fractional dominating function with an automorphism is (minimum) fractional dominating. The convexity of the sets of minimum fractional dominating functions and of fractional dominating functions therefore implies that $\hat{f}$ is minimum fractional dominating and that $\hat{g}$ is dominating. Further, it is straightforward to verify that $\hat{g} \leq 1 - \hat{f}$, and that $\hat{g}$ has the same total weight as g, namely $(\gamma_f)'(G)$. Thus $\hat{g}$ is a minimum inverse fractional dominating function on G, which is constant on each automorphism class of G. □

P_2 and $C_n, n \geq 3$ are graphs with no minimum fractional inverse dominating function constant on each automorphism class (i.e. the whole vertex set, for these graphs). C_5 with a chord is an example of a non-vertex-transitive graph with no minimum fractional inverse dominating function constant on each automorphism class.

4 The independence question

A modestly notorious problem is to determine whether it is always the case that $\gamma'(G) \leq \alpha(G)$, where the latter denotes the vertex independence number of G. This result was stated in [2], but the proof given there was fallacious. While we have as yet no insight to offer the world regarding this conundrum, we can state that both fractional analogues of this conjecture turn out to be true.

Fractional independence is defined in [3] as follows. A function $f : V \to [0,1]$ is *fractional independent* if, for any pair v, w of adjacent vertices, $f(v) + f(w) \leq 1$. The *fractional independence number* is given by

$$\alpha_f(G) = \max_f \sum_{v \in V} f(v)$$

where the minimum is taken over all fractional independent functions on G.

Theorem 4.1 *For any graph G with no isolated vertices, $(\gamma')_f(G) \leq \alpha_f(G)$.*

Proof. For any dominating set D of a graph G, we shall define a *companion function* $c_D : V \to [0,1]$ as follows:

$$c_D(v) = \begin{cases} 0, & v \in D \\ 1, & v \notin D \text{ and } N(v) \subseteq D \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

It is easily shown that the companion function of any dominating set (or, in fact, any set of vertices) is fractional independent; we endeavour to demonstrate that, for some minimum dominating set D, c_D is also a fractional dominating function. This will suffice to show that the fractional inverse domination number is less than or equal to the fractional independence number.

Let D be a minimum dominating set. We see immediately that c_D dominates any vertex which is not in D, since such a vertex either gets a weight of 1, or else gets a weight of $\frac{1}{2}$ and has a neighbour which also receives a weight of $\frac{1}{2}$. Thus, only vertices within our dominating set D could potentially pose problems for us. In the sequel, we shall abuse notation by writing $c_D(N[v])$ for the sum of the values of the companion function on the closed neighbourhood of v.

Suppose that $v \in D$ is a vertex which is not dominated by the companion function: $c_D(N[v]) < 1$. Due to the structure of c_D, the sum of the weights on the neighbourhood of v must therefore be either 0 or $\frac{1}{2}$. The former is impossible, because D is a minimum dominating set; if every neighbour of v was in D, then there is no need for v to also be in D.

So we can assume that $c_D(N[v]) = \frac{1}{2}$. This means that there exists a unique $u \in N(v)$ such that $u \notin D$. Let $D' = (D - v) \cup \{u\}$; it is easily shown that D' is also a dominating set of G, of the same cardinality as D and hence minimum. We see also now that $c_{D'}(N[v]) = 1$ and $c_{D'}(N[u]) \geq 1$.

We can repeat the technique of the previous paragraph to find a sequence of dominating sets $D_1, \ldots, D_k$; the corresponding sequence of companion functions $c_{D_1}, \ldots, c_{D_k}$ will feature a strictly decreasing number of vertices receiving a weight of $\frac{1}{2}$, which shows that the sequence must at some point terminate. This can only happen when we have found a companion function which is fractional dominating. □

Lemma 4.2 *Let G be a connected graph with $|V| \geq 3$. There is a minimum fractional dominating function f with the property that $f(x) = 0$ for any vertex x where $d(x) = 1$.*

Proof. Suppose f is a minimum fractional dominating function and x is a vertex of degree 1 with $f(x) > 0$. Let y be the unique neighbour of x; since G has more than two vertices, $d(y) > 1$. We can assume that $f(N[x]) = f(x) + f(y) = 1$, since if the sum totals to more than 1 nothing stops us from reducing the weight on x, which contradicts minimality. Then define a new function f' which agrees with f everywhere except x and y, setting $f'(x) = 0$ and $f'(y) = 1$. This new function f' has the same total weight as f, and is also dominating. If there are other vertices of degree 1 at which $f' = f$ is positive, treat them as x was treated to obtain finally a minimum fractional dominating function satisfying the requirement. □

Theorem 4.3 *For any graph G with no isolated vertices, $(\gamma_f)'(G) \leq \alpha_f(G)$.*

Proof. Let f be a minimum fractional dominating function in G, using Lemma 4.2 to ensure that $f(x) = 0$ for any vertex x of degree 1, and define $g : V \to [0,1]$ by $g(v) = \min\{\frac{1}{2}, 1 - f(v)\}$. From the definition, g is fractional independent; if it is also fractional dominating, we are done. So suppose the contrary, and let $U = \{u \in V : g(N[u]) < 1\}$. Call a vertex v *deficient* if $f(v) < \frac{1}{2}$ (and hence $g(v) = \frac{1}{2}$). For any $u \in U$, there must be exactly one deficient vertex in $N[u]$. If there were none at all, then locally $g = 1 - f$ which we know to be dominating by Proposition 1.1; if there were at least two (say v_1, v_2) then $g(N[u]) \geq g(v_1) + g(v_2) = 1$, so again u would be dominated.

Let w be the deficient vertex in $N[u]$ for some $u \in U$, and suppose $u \neq w$. We can assume that u has degree at least 2, since otherwise w would be the sole neighbour of u, and our choice of f guarantees that $f(w) = 1$, contradicting the deficiency of w. For every $v \in N[u] - \{w\}$, $f(v) > \frac{1}{2}$ (if $f(v) = \frac{1}{2}$. then $u \notin U$ as $f(w) = \frac{1}{2}$); since this includes u, we conclude that $f(N[v]) > 1$ for all such v. This in turn implies that $f(N[w]) = 1$, since otherwise f would not be minimal.

Let $\epsilon = \min\{f(u)-\frac{1}{2}, \frac{1}{2}-f(w)\}$; we shall define a new function f' which agrees with f everywhere but for u and w, letting $f'(u) = f(u) - \epsilon$ and $f'(w) = f(w) + \epsilon$. We shall also define g' using f' analogously to the way we defined g using f above. The function f' has the same total weight as f. It is clear that f' dominates all vertices except possibly those in $N(u) - w$. But for any such vertex v, $f'(N[v]) \geq f'(v) + f'(u) = f(v) + f'(u) \geq \frac{1}{2} + \frac{1}{2}$. Therefore f' is a minimum fractional dominating function. As for g', we see that $g' \geq g$ and hence g' dominates everywhere that g does. If $\epsilon = f(u) - \frac{1}{2}$ then $f'(u) = g'(u) = \frac{1}{2}$, and if $\epsilon = \frac{1}{2} - f(w)$ then $g' = 1 - f'$ on $N[u]$; in either case, g' dominates u.

Therefore, we may as well assume that every vertex in U is deficient. Since only one vertex in $N[u]$ can be deficient for any $u \in U$, we may conclude that no two vertices in U are adjacent: U is an independent set. Define a new function $h : V \longrightarrow [0,1]$ which is identical to g on $V - U$, with $h(u), u \in U$ chosen so that $h(N[u]) = 1$; so h is a fractional dominating function. To check for fractional independence, take any edge $e = vw$; if neither of v, w are in U, then $h(v) + h(w) = g(v) + g(w) \leq 1$, and if $v \in U$ and $w \in \overline{U}$ then $h(v) + h(w) \leq h(N[v]) = 1$. Since U is independent there are no other kinds of edges; thus h is fractional independent.

To finish the proof, we need check that $h(v) \leq 1 - f(v)$ for all $v \in V$. If $v \in \overline{U}$ then $h(v) = g(v) \geq 1 - f(v)$, so assume that $v \in U$. Since $1 - f$ dominates, we have that $[1-f](N[v]) = [1-f](N(v)) + [1 - f(v)] \geq 1$. We defined h so that $h(N[v]) = h(N(v)) + h(v) = 1$; combining the equation with the inequality and the facts that $h = g \leq 1 - f$ on $V - U$ and that $v \in U$ implies that $N(v) \subseteq V - U$ gives us $1 - f(v) \geq h(v)$, as required. □

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