ALGORITHMS FOR THE OPTIMAL HAMILTONIAN PATH IN HALIN GRAPHS

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ABSTRACT. This paper deals with the problem of construction hamiltonian paths of optimal weights in Halin graphs. There are three versions of the hamiltonian path: none or one or two of endvertices are specified. We present $O(|V|)$ algorithms to all the versions of the problem.
Keywords: halin graph, edge cutset, hamiltonian path

1. INTRODUCTION AND TERMINOLOGY

Routing design problems are of major importance in combinatorial optimization, and many algorithmic ideas have been applied to them during the past 20 years. We will concern with the problem closely related to the minimum travelling salesman problem (TSP), namely, the problem of finding a hamiltonian path of minimum weight. The approximation algorithms of this problem have been studied by Jerome Monnot [7]. It is well known that the problem of optimal hamiltonian path is NP-hard. In [6], Lou proved there is a hamiltonian path between two distinct vertices in Halin paths. In the light of this, we restrict the problem to Halin graphs and give an $O(|V|)$ algorithm.

All graphs considered in this paper are Halin graphs. For the terminology and notation not defined in this paper, reader can refer to [1].

A Halin graph is constructed as follows: start with a tree T in which each nonleaf has degree at least 3. Embed the tree in a plane and then connect all the leaves of T with a cycle C such that the resulting graph $H = T \cup C$ is planar. Suppose T has at least two nonleaves. Let w be a nonleaf of T which is adjacent to only one other nonleaf of T. Then the set of leaves of T adjacent to w, which we denote by $C(w)$, comprises a consecutive subsequence of the cycle C. We call the subgraph of H induced by $\{w\} \cup C(w)$ a *fan* and call w the *centre* of the fan. In Fig.1. the black vertices are the centres of the fans which are indicated by dotted lines.

Let $H = T \cup C$ be a Halin graph. Given a fan F of H, let $H \times F$ denote the graph obtained from H by shrinking F to form a new "pseudo-vertex", denoted by v_F; that is, $V(H \times F) = \{v_F\} \cup \{V(H) \backslash V(F)\}$ and the edges of $H \times F$ are defined as follows:

(1) An edge with both ends in F is deleted;
(2) An edge with both ends in $H - F$ remains unchanged;
(3) An edge with one end-vertex in $H - F$ and the other in F now joins the incident vertex of $H - F$ and the pseudo-vertex v_F.

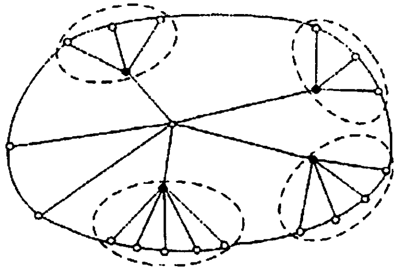

FIGURE 1. A Halin graph

Denote the closure of H under the operation of "Shrinking Fan" by H^*.

A graph G is called a *wheel* if G consists of a cycle every vertex of which is joined to a single common vertex by an edge.

An edge cutset of a connected graph $G = (V, E)$ is a set of edges whose removal leaves a disconnected graph. If it consists of exactly k edges, then we call it a k-edge cutset. Let $H = T \cup C$ be a Halin graph. Given a fan F of H, the three edges connecting $V(F)$ to $V(H-F)$ compose a 3-edge cutset of H. We denote this 3-edge cutset by $EC_3(F)$. Let F be a fan in $H' \in H^*$ and $EC_{ex}(F)$ be the corresponding 3-edge cutset in H with respect to $EC_3(F)$. Let F_{ex} be the subgraph of H which F is fully restored to. Equivalently, F_{ex} is the component of $H - EC_{ex}(F)$ which F stands for.

2. PRELIMINARY RESULTS

Lemma 1. *(Cornuejols[3]) A Halin graph $H = T \cup C$ which is not a wheel has at least two fans.*

Proof. Suppose H is not a wheel and let T' be the subtree of T obtained by deleting all leaves. Then T' has more than one vertex and so has at least two leaves. Each of these is the centre of a fan of H. □

Lemma 2. *(Cornuejols[3]) If F is a fan in a Halin graph H, then $H \times F$ is a Halin graph.*

Proof. Obviously. □

3. MAIN RESULTS

Theorem 3. *Let H be a Halin graph and F be a fan in H. Every hamiltonian path $HP = u \ldots\ldots v$ contains exactly two edges of $EC_3(F)$ if $u, v \notin V(F)$ or $u, v \in V(F)$, while contains exactly one or three edges of $EC_3(F)$ if just one of u and v lies in F.*

Proof. Immediate. □

Corollary 4. *Let H be a Halin graph and F be a fan in $H' \in H^*$. Every hamiltonian path $HP = u \ldots\ldots v$ in H contains exactly two edges of $EC_{ex}(F)$ if $u, v \notin F_{ex}$ or $u, v \in F_{ex}$, while contains exactly one or three edges of $EC_{ex}(F)$ if just one of u and v lies in F_{ex}.*

4. Description of the algorithm

Let H be a Halin graph and $\alpha(u,v)$ be the cost of the edge $e = (u,v)$, $u, v \in V(H)$. For $e \in E(H)$, we abbreviate it by α_e. The hamiltonian paths have three versions: first, the version where no endvertex is specified (denoted by HP); second, the version where just one endvertex is specified (denoted by HP_u); third, the version where two endvertices are specified (denoted by $HP_{u,v}$). We focus on searching an optimal $HP_{u,v}$ in this section.

Cornuejols, Naddef and Pulleyblank[3] gave a linear algorithm to search Min TSP in Halin graphs. When (u,v) is an edge, Min $HP_{u,v}$ can be reduced to Min TSP as following: for each edge (x,y) where ($x = u$ and $y \neq v$) or ($x \neq u$ and $y = v$), change the cost $\alpha(x,y)$ to be $\alpha(x,y) + n * \alpha_{max}$ where $\alpha_{max} = max\{\alpha_e | e \in E(H)\}$ and $n = |V(H)|$. After this amendment, Min TSP must contain the edge (u,v); otherwise, the cost will be larger than $4n * \alpha_{max}$. But if TSP uses the edge (u,v), the cost will be no greater than $3n * \alpha_{max}$. So we can get the Min TSP containing the edge (u,v). Hence Min $HP_{u,v}$ can be obtained by Min TSP deleting the edge (u,v) and the cost of Min $HP_{u,v}$ equals the cost Min TSP minus $\alpha(u,v) + 2n * \alpha_{max}$.

When (u,v) is not an edge in H, we describe the method of shrinking a fan which does not contain any endvertex. It is a restricted usage to the idea of shrinking a fan in [3, 4]. For precision, we give a detailed introduction as follows: let HP be a hamiltonian path in H and F be any fan of H which does not contain the vertex u or v. Let w_c be the centre of F and let $w_1, w_2, \ldots, w_r$ for $r \geqslant 2$ be the vertices of F, which belong to C (in anti-clockwise order). Denote the $EC_3(F) = \{j, k, l\}$. See Fig. 2.

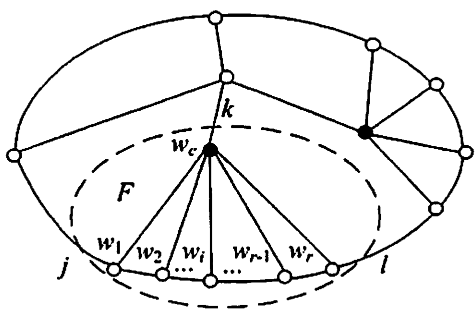

FIGURE 2. A fan (not containing any endvertex)

By Theorem 3, *HP* uses exactly two of $\{j, k, l\}$.

- If it uses j and k, then it must traverse the vertices of F in the order w_1, $w_2, \ldots, w_r, w_c$ or in reverse order;
- If it uses k and l, then it must traverse the vertices of F in the order w_c, $w_1, w_2, \ldots, w_r$ or in reverse order;
- If it uses j and l, then it must traverse the vertices of F in the order w_1, $w_2, \ldots, w_i, w_c, w_{i+1}, \ldots, w_r$ for some $i \in \{1, 2, \ldots, r-1\}$, or in reverse order.

Let K be the sum of the costs of the edges of C in F. Then

- if *HP* uses j and k, the edges in F contribute $C_{jk} \equiv K + \alpha(w_c, w_r)$ to the cost;
- if *HP* uses k and l, the edges in F contribute $C_{kl} \equiv K + \alpha(w_c, w_1)$ to the cost;
- if *HP* uses j and l, the edges in F contribute at least $C_{jl} \equiv K + min\{ \alpha(w_c, w_i) + \alpha(w_c, w_{i+1}) - \alpha(w_i, w_{i+1}), 1 \leqslant i \leqslant r-1\}$ to the cost, and the minimum is realizable.

We shrink a fan F to a pseudo-vertex, denoted by u_F. At first, set the cost function in $H \times F$ to be α' defined by

$$\alpha'_e \equiv \begin{cases} \alpha_e & \text{if } e \in E(H)\backslash\{j,k,l\}, \\ \alpha_j + \frac{(C_{jl}+C_{jk}-C_{kl})}{2} & \text{if } e = j, \\ \alpha_k + \frac{(C_{kl}+C_{jk}-C_{jl})}{2} & \text{if } e = k, \\ \alpha_l + \frac{(C_{jl}+C_{kl}-C_{jk})}{2} & \text{if } e = l. \end{cases} \tag{1}$$

Then, store the structure of F into u_F, including the vertices $w_1, w_2, \ldots, w_r, w_c$ and the edges among them. In addition, the value of i which makes C_{jl} minimum is also stored, but we do not store the structure contained in any pseudo-vertex in F.

Secondly, we introduce the strategy to shrink a fan which contains exactly one endvertex. Let F' be any fan of H which contains just one of u and v. Without loss of generality, we suppose F' contains the vertex u. The situation is the same when F' contains the vertex v except the direction is reverse.

And we suppose the fans, which do not contain any endvertex, have been shrunk. Let x_c be the centre of F' and let $x_1, x_2, \ldots, x_r$ ($r \geq 2$) be the vertices of F' which belong to C (in anti-clockwise order). Denote the $EC_3(F') = \{j, k, l\}$. See Fig. 3. By Corollary 4, *HP* uses exactly one or three of $\{j, k, l\}$.

If the vertex u is not the centre of F', this situation is denoted by Case 1; the other situation is denoted by Case 2.

In Case 1, we assume that $x_i = u$. Consider the situation that *HP* just uses one of $\{j, k, l\}$, firstly.

(1) If it uses j, then the traversal of F' depends on i:
   - If $2 \leqslant i < r$, then the order is $x_i (= u), x_{i+1}, \ldots, x_r, x_c, x_{i-1}, \ldots, x_1$;

- If $i = r$, then the order is $x_i(= u)$, x_{i-1}, ..., x_j, x_c, x_{j-1}, ..., x_1 for some $j \in \{2, \dots, r\}$.

This situation happens when $2 \leqslant i \leqslant r$, that is $u \neq x_1$. Define this type of traversal by Clockwise Traversal (CT);

(2) If it uses k, then the traversal of F' depends on i:
- If $i = 1$, the order is $x_i(= u)$, x_{i+1}, ..., x_r, x_c;
- If $i = r$, the order is $x_i(= u)$, x_{i-1}, ..., x_1, x_c.

This situation happens only when $u = x_1$ or $u = x_r$. Define this type of traversal by Up Traversal (UT);

(3) If it uses l, then the traversal of F' depends on i:
- If $i = 1$, then the order is $x_i(= u)$, x_{i+1}, ..., x_j, x_c, x_{j+1}, ..., x_r for some $j \in \{1, \dots, r-1\}$;
- If $2 \leqslant i < r$, then the order is $x_i(= u)$, x_{i-1}, ..., x_1, x_c, x_{i+1}, ..., x_r.

This situation happens when $1 \leqslant i < r$, that is $u \neq x_r$. Define this type of traversal by Anti-Clockwise Traversal (ACT).

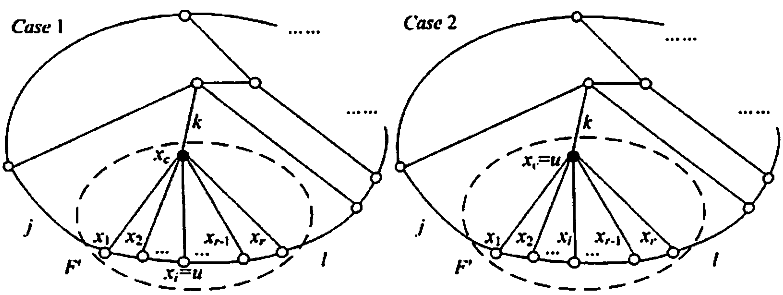

FIGURE 3. A fan (containing precisely one endvertex)

Now, we examine the situation that HP uses all of $\{j, k, l\}$, in Case 1.

Since HP uses three edges of a 3-edge cutset, the traversal of F' in HP is not consecutive. That is, it contains exactly two sequences (denoted by S_1 and S_2) of the vertices of F' such that $V(S_1) \cup V(S_2) = V(F')$ and $V(S_1) \cap V(S_2) = \emptyset$. Without loss of generality, we suppose that the endvertices u and v only belong to S_1. It helps us to know whether one sequence contains any endvertex, which is not only clear but also useful for the construction of HP. Thus, $HP{=}S_1P_1S_2P_2$ such that the two sequences P_1 and P_2 satisfy $V(P_1) \cup V(P_2) = V(H \backslash F')$ and $V(P_1) \cap V(P_2) = \emptyset$.

In Case 1, there are only two possible situations of S_1 and S_2:

(1) $S_1 = x_i(= u)$, x_{i-1}, ..., x_1 and $S_2 = x_c$, x_{i+1}, ..., x_r (or in reverse order) for $1 \leqslant i < r$.

This situation happens when $u \neq x_r$.

Define this type of traversal by discrete Clockwise Traversal (dCT);

(2) $S_1 = x_i(= u), x_{i+1}, \ldots, x_r$ and $S_2 = x_c, x_{i-1}, \ldots, x_1$ (or in reverse order) for $2 \leqslant i \leqslant r$.
This situation happens when $u \neq x_1$.
Define this type of traversal by discrete Anti-Clockwise Traversal ($dACT$).

In Case 2, the vertex u is the centre of F'. Investigate the situation that HP just uses one of $\{j, k, l\}$, firstly.

(1) If it uses j, then it must traverse the vertices of F' in the order $x_c(= u)$, $x_r, x_{r-1}, \ldots, x_1$. We consider it as belonging to the type of Clockwise Traversal (CT);
(2) HP cannot just use the edge k only;
(3) If it uses l, then it must traverse the vertices of F' in the order $x_c(= u), x_1, \ldots, x_{r-1}, x_r$. We consider it as belonging to the type of Anti-Clockwise Traversal (ACT).

In Case 2, if HP uses all of $\{j, k, l\}$, then there is only one possible situation: $S_1 = x_c(= u)$ and $S_2 = x_1, x_2, \ldots, x_{r-1}, x_r$ (or in reverse order). Define this type of traversal by Up-Down Traversal (UDT).

The method of shrinking a fan F not containing any endvertex could be viewed as a reassignment of the costs of $EC_3(F)$ and a storage of the structure. It can preserve the cost of the optimal hamiltonian path. But the traversals of a fan F' containing exactly one endvertex are much more complicated. Similarly, we store the structure of F', too. In addition, we choose to store the costs of these traversals by means of some functions of the shrunk vertices. Since the number of the situations in one certain traversal type can be larger than one, we store the situation which makes the cost of one certain traversal type minimum. If there is more than one getting the minimum, choose one arbitrarily.

Suppose we shrink a fan F' to a pseudo-vertex, denoted by v_F. Define the functions of v_F, as follows:

- $CT(v_F) \equiv$ the minimum cost of all traversals of the type CT;
- $UT(v_F) \equiv$ the minimum cost of all traversals of the type UT;
- $ACT(v_F) \equiv$ the minimum cost of all traversals of the type ACT;
- $dCT(v_F) \equiv$ the minimum cost of all traversals of the type dCT;
- $dACT(v_F) \equiv$ the minimum cost of all traversals of the type $dACT$;
- $UDT(v_F) \equiv$ the minimum cost of all traversals of the type UDT.

If there is no traversal of some certain type, then the value of corresponding function is set to be $+\infty$.

We provide the calculations of these functions in details. Let F' be a fan and it will be shrunk into v_F. Let x_c be the centre of F' and $x_1, x_2, \ldots, x_r$ ($r \geq 2$) be the vertices of F' which belong to C. See Fig. 3. Suppose x_k is the endvertex, or the pseudo-vertex containing it.

The formulas are defined as following. We assume that $\sum_{i=j}^{j-1} \alpha(x_i, x_{i+1}) = 0$ for all values of j and $x_k = x_c$ does not belong to the situation $1 \leqslant k \leqslant r$.

$$CT_1(v_F) = \begin{cases} ACT(x_k) + \sum_{i=k}^{r-1} \alpha(x_i, x_{i+1}) \\ +\alpha(x_r, x_c) + \alpha(x_c, x_{k-1}) + \sum_{i=1}^{k-2} \alpha(x_i, x_{i+1}) & \text{if } 2 \leqslant k < r, \\ \min \left\{ \begin{array}{l} UT(x_k) + \alpha(x_k, x_c) + \alpha(x_c, x_{k-1}) + \sum_{i=1}^{k-2} \alpha(x_i, x_{i+1}), \\ \{CT(x_k) + \sum_{i=j}^{k-1} \alpha(x_i, x_{i+1}) + \alpha(x_j, x_c) + \alpha(x_c, x_{j-1}) \\ + \sum_{i=1}^{j-2} \alpha(x_i, x_{i+1}) \,|\, 2 \leqslant j < r\} \end{array} \right\} & \text{if } k = r, \\ +\infty & \text{otherwise.} \end{cases} \tag{2a}$$

$$CT_2(v_F) = \begin{cases} dACT(x_k) + \sum_{i=k}^{r-1} \alpha(x_i, x_{i+1}) + \alpha(x_r, x_c) \\ +\alpha(x_c, x_k) + \sum_{i=1}^{k-1} \alpha(x_i, x_{i+1}) & \text{if } 1 \leqslant k < r, \\ +\infty & \text{otherwise.} \end{cases} \tag{2b}$$

$$CT_3(v_F) = \begin{cases} UDT(x_k) + \alpha(x_k, x_c) + \alpha(x_c, x_r) + \sum_{i=1}^{r-1} \alpha(x_i, x_{i+1}) & \text{if } 1 \leqslant k < r, \\ +\infty & \text{otherwise.} \end{cases} \tag{2c}$$

$$CT(v_F) = \begin{cases} \alpha(x_c, x_r) + \sum_{i=1}^{r-1} \alpha(x_i, x_{i+1}) & \text{if } x_k = x_c, \\ \min\{CT_1(v_F), CT_2(v_F), CT_3(v_F)\} & \text{otherwise.} \end{cases} \tag{2d}$$

$$UT(v_F) = \begin{cases} ACT(x_k) + \sum_{i=k}^{r-1} \alpha(x_i, x_{i+1}) + \alpha(x_r, x_c) & \text{if } k = 1, \\ CT(x_k) + \sum_{i=1}^{k-1} \alpha(x_i, x_{i+1}) + \alpha(x_1, x_c) & \text{if } k = r, \\ +\infty & \text{otherwise.} \end{cases} \tag{2e}$$

$$ACT_1(v_F) = \begin{cases} \min \left\{ \begin{array}{l} UT(x_k) + \alpha(x_k, x_c) + \alpha(x_c, x_{k+1}) + \sum_{i=k+1}^{r-1} \alpha(x_i, x_{i+1}), \\ \{ACT(x_k) + \sum_{i=k}^{j-1} \alpha(x_i, x_{i+1}) + \alpha(x_j, x_c) + \alpha(x_c, x_{j+1}) \\ + \sum_{i=j+1}^{r-1} \alpha(x_i, x_{i+1}) \,|\, 1 < j \leqslant r-1\} \end{array} \right\} & \text{if } k = 1, \\ CT(x_k) + \sum_{i=1}^{k-1} \alpha(x_i, x_{i+1}) + \alpha(x_1, x_c) \\ +\alpha(x_c, x_{k+1}) + \sum_{i=k+1}^{r-1} \alpha(x_i, x_{i+1}) & \text{if } 2 \leqslant k < r, \\ +\infty & \text{otherwise.} \end{cases} \tag{2f}$$

$$ACT_2(v_F) = \begin{cases} dCT(x_k) + \sum_{i=1}^{k-1} \alpha(x_i, x_{i+1}) + \alpha(x_1, x_c) \\ +\alpha(x_c, x_k) + \sum_{i=k}^{r-1} \alpha(x_i, x_{i+1}) & \text{if } 1 < k \leqslant r, \\ +\infty & \text{otherwise.} \end{cases} \tag{2g}$$

$$ACT_3(v_F) = \begin{cases} UDT(x_k) + \alpha(x_k, x_c) + \alpha(x_c, x_1) + \sum_{i=1}^{r-1} \alpha(x_i, x_{i+1}) & \text{if } 1 < k \leqslant r, \\ +\infty & \text{otherwise.} \end{cases} \tag{2h}$$

$$ACT(v_F) = \begin{cases} \alpha(x_c, x_1) + \sum_{i=1}^{r-1} \alpha(x_i, x_{i+1}) & \text{if } x_k = x_c, \\ \min\{ACT_1(v_F), ACT_2(v_F), ACT_3(v_F)\} & \text{otherwise.} \end{cases} \tag{2i}$$

$$dCT_1(v_F) = \begin{cases} CT(x_k) + \sum_{i=1}^{k-1} \alpha(x_i, x_{i+1}) + \alpha(x_c, x_{k+1}) + \sum_{i=k+1}^{r-1} \alpha(x_i, x_{i+1}) & \text{if } 1 \leqslant k < r, \\ +\infty & \text{otherwise.} \end{cases} \tag{2j}$$

$$dCT_2(v_F) = \begin{cases} dCT(x_k) + \sum_{i=1}^{k-1} \alpha(x_i, x_{i+1}) + \alpha(x_c, x_k) + \sum_{i=k}^{r-1} \alpha(x_i, x_{i+1}) & \text{if } 1 \leqslant k \leqslant r, \\ +\infty & \text{otherwise.} \end{cases} \tag{2k}$$

$$dCT(v_F) = \min\{dCT_1(v_F), dCT_2(v_F)\} \tag{2l}$$

$$dACT_1(v_F) = \begin{cases} ACT(x_k) + \sum_{i=k}^{r-1} \alpha(x_i, x_{i+1}) + \alpha(x_c, x_{k-1}) + \sum_{i=1}^{k-2} \alpha(x_i, x_{i+1}) & \text{if } 2 \leqslant k \leqslant r, \\ +\infty & \text{otherwise.} \end{cases} \tag{2m}$$

$$dACT_2(v_F) = \begin{cases} dACT(x_k) + \sum_{i=k}^{r-1} \alpha(x_i, x_{i+1}) + \alpha(x_c, x_k) + \sum_{i=1}^{k-1} \alpha(x_i, x_{i+1}) & \text{if } 1 \leqslant k \leqslant r, \\ +\infty & \text{otherwise.} \end{cases} \tag{2n}$$

$$dACT(v_F) = \min\{dACT_1(v_F), dACT_2(v_F)\} \tag{2o}$$

$$UDT(v_F) = \begin{cases} UDT(x_k) + \sum_{i=1}^{r-1} \alpha(x_i, x_{i+1}) & \text{if } x_k = x_c, \\ UDT(x_k) + \alpha(x_k, x_c) + \sum_{i=1}^{r-1} \alpha(x_i, x_{i+1}) & \text{if } x_k \neq x_c, \\ +\infty & \text{otherwise.} \end{cases} \tag{2p}$$

In a bid to maintain the meaning that $+\infty$ stands for, if a sub-formula uses a function whose value is $+\infty$, that is, it indicates no such a type of traversal in the pseudo-vertex, then the value of the sub-formula is set to be $+\infty$, immediately. Thus, the value indicates that the corresponding traversal of the sub-formula does not exist. For example, we calculate the function $UDT(v_F)$ when $x_k = x_c$, if $UDT(x_k) = +\infty$, then $UDT(v_F)$ is set to be $+\infty$.

5. Procedure of the Algorithm

Given a Halin graph $H = T \cup C$ and two distinct vertices $u, v \in V(H)$.

The algorithm to search Min $HP_{u,v}$

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CT(u):=ACT(u):=UT(u):=0; dCT(u):=dACT(u):=UDT(u):=+∞;
CT(v):=ACT(v):=UT(v):=0; dCT(v):=dACT(v):=UDT(v):=+∞;
if u and v lie in C then
  begin
  TreeType:=1;
  if u and v lie in the same fan
    then Root:=the centre of the fan;
    else Root:=the centre of the fan containing u;
  end
else begin
    TreeType:=2;
    if u lies in C then Root:=v;
    else if v lies in C then begin swap u and v; Root:=v; end
                      else if neither u nor v lies in C
                        then begin Root:=v; UDT(u):=0;
                          CT(u):=ACT(u):=UT(u):=+∞;
                          end
    end
perform a postorder scan of T, for each fan F has been found do
if the centre of F ≠ Root Then
  begin
  if F does not contain any endvertex then
     begin
     shrink F to v_F
     store the structure of F into v_F
     reassign the costs of edges in EC_3(F)
     end
  else begin
     shrink F to v_F
     store the structure of F into v_F
     Mark v_F containing u or v
     calculate the functions of v_F
     end
  end
let H_w be the wheel we finally get, w_u be u or the pseudo-vertex containing u;
let k be the edge joining w_u and Root, j and l be the other two edges joining w_u
such that the direction of l, w_u, j is clockwise;
Shrink the fan H_w - w_u to the pseudo-vertex w_v;
switch(TreeType)
  begin
  case 1: Cost := min{α(j) + ACT(w_v), α(k) + UT(w_v), α(l) + CT(w_v)};
            let HP' be the path which makes Cost minimum;
         //e.g. HP' = u, ACT of w_v when Cost = α(j)+ACT(w_v).
         //if there is more than one path gets the minimum,
         //then choose one arbitrarily;
            break;
```

$$
41)\quad \text{case 2: } Cost := min \left\{ \begin{array}{l} CT(w_u) + \alpha(j) + ACT(w_v), \\ ACT(w_u) + \alpha(l) + CT(w_v), \\ dCT(w_u) + \alpha(j) + \alpha(k) + \alpha(l) + dCT(w_v), \\ dACT(w_u) + \alpha(j) + \alpha(k) + \alpha(l) + dACT(w_v), \\ UT(w_u) + \alpha(k) + UT(w_v), \\ UDT(w_u) + \alpha(j) + \alpha(k) + \alpha(l) + dCT(w_v), \\ UDT(w_u) + \alpha(j) + \alpha(k) + \alpha(l) + dACT(w_v), \\ UDT(w_v) + \alpha(j) + \alpha(k) + \alpha(l) + dCT(w_u), \\ UDT(w_v) + \alpha(j) + \alpha(k) + \alpha(l) + dACT(w_u) \end{array} \right\}
$$

42) let HP' be the path which makes Cost minimum;
//e.g. $HP' = S_1(dCT \text{ of } w_u), S_2(dCT \text{ of } w_v), S_2(dCT \text{ of } w_u), S_1(dCT \text{ of } w_v)$
//when $Cost = dCT(w_u) + \alpha(j) + \alpha(k) + \alpha(l) + dCT(w_v)$;
//if there is more than one path gets the minimum,
//then choose one arbitrarily;
43) break;
44) end
45) while (there is pseudo-vertex in HP') do
46) begin
47) let w be the first pseudo-vertex in HP';
48) according to the traversal type determined by HP' and the structure information stored, extend w to be the corresponding traversal path in the fan which is shrunk to w;
the direction of the path can be judged by the former vertex in HP' or the starting vertex;
49) end
50) HP' is Min $HP_{u,v}$

6. An example

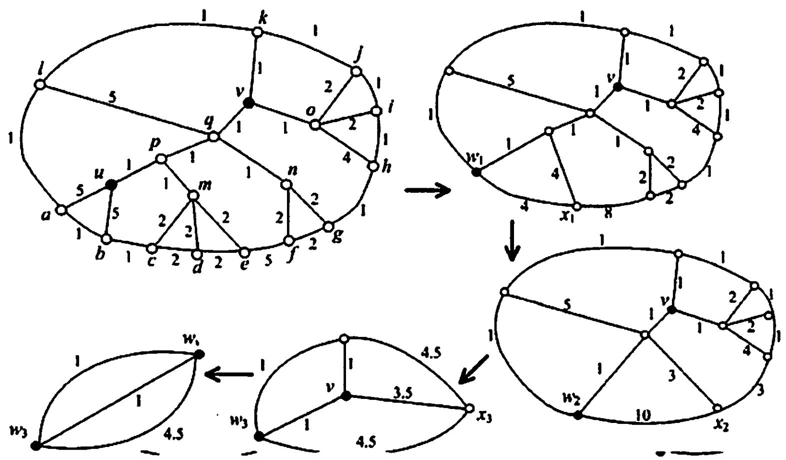

FIGURE 4. An example of the algorithm

The main procedures to search Min $HP_{u,v}$ are shown in Fig. 4. And the values of the functions are illustrated in Table. 1.

TABLE 1. The values of the functions

	CT	UT	ACT	dCT	$dACT$	UDT
w_1	6	$+\infty$	6	$+\infty$	$+\infty$	1
w_2	10	14	$+\infty$	10	$+\infty$	6
w_3	21	$+\infty$	19	14	$+\infty$	18
w_v	5.5	4.5	8	$+\infty$	$+\infty$	4.5

In step 41), we get the $cost = UDT(w_v) + \alpha(j) + \alpha(k) + \alpha(l) + dCT(w_u) = 25$ which is minimum. HP' is extended as following (the pseudo-vertices are underlined):

$HP' = S_1(dCT \text{ of } \underline{w_u}),\ S_2(UDT \text{ of } \underline{w_v}),\ S_2(dCT \text{ of } \underline{w_u}),\ S_1(UDT \text{ of } \underline{w_v})$

$\Rightarrow S_1(dCT \text{ of } \underline{w_3}),\ S_2(UDT \text{ of } \underline{w_v}),\ S_2(dCT \text{ of } \underline{w_3}),\ S_1(UDT \text{ of } \underline{w_v})$

$\Rightarrow CT(\underline{w_2}),\ l,\ S_2(UDT \text{ of } \underline{w_v}),\ S_2(dCT \text{ of } \underline{w_3}),\ S_1(UDT \text{ of } \underline{w_v})$

$\Rightarrow S_1(UDT \text{ of } \underline{w_1}),\ p,\ \underline{x_1},\ S_2(UDT \text{ of } \underline{w_1}),\ l,\ S_2(UDT \text{ of } \underline{w_v}),\ S_2(dCT \text{ of } \underline{w_3}),\ S_1(UDT \text{ of } \underline{w_v})$

$\Rightarrow u,\ p,\ \underline{x_1},\ S_2(UDT \text{ of } \underline{w_1}),\ l,\ S_2(UDT \text{ of } \underline{w_v}),\ S_2(dCT \text{ of } \underline{w_3}),\ S_1(UDT \text{ of } \underline{w_v})$

$\Rightarrow u,\ p,\ m,\ e,\ d,\ c,\ S_2(UDT \text{ of } \underline{w_1}),\ l,\ S_2(UDT \text{ of } \underline{w_v}),\ S_2(dCT \text{ of } \underline{w_3}),\ S_1(UDT \text{ of } \underline{w_v})$

$\Rightarrow u,\ p,\ m,\ e,\ d,\ c,\ b,\ a,\ l,\ S_2(UDT \text{ of } \underline{w_v}),\ S_2(dCT \text{ of } \underline{w_3}),\ S_1(UDT \text{ of } \underline{w_v})$

$\Rightarrow u,\ p,\ m,\ e,\ d,\ c,\ b,\ a,\ l,\ k,\ \underline{x_3},\ S_2(dCT \text{ of } \underline{w_3}),\ S_1(UDT \text{ of } \underline{w_v})$

$\Rightarrow u,\ p,\ m,\ e,\ d,\ c,\ b,\ a,\ l,\ k,\ j,\ o,\ i,\ h,\ S_2(dCT \text{ of } \underline{w_3}),\ S_1(UDT \text{ of } \underline{w_v})$

$\Rightarrow u,\ p,\ m,\ e,\ d,\ c,\ b,\ a,\ l,\ k,\ j,\ o,\ i,\ h,\ \underline{x_2},\ q,\ S_1(UDT \text{ of } \underline{w_v})$

$\Rightarrow u,\ p,\ m,\ e,\ d,\ c,\ b,\ a,\ l,\ k,\ j,\ o,\ i,\ h,\ g,\ f,\ n,\ q,\ S_1(UDT \text{ of } \underline{w_v})$

$\Rightarrow u,\ p,\ m,\ e,\ d,\ c,\ b,\ a,\ l,\ k,\ j,\ o,\ i,\ h,\ g,\ f,\ n,\ q,\ v$

7. THE CORRECTNESS AND THE TIME COMPLEXITY

Theorem 5. *Let H be a Halin graph and F be a fan in H. Every hamiltonian path HP in H induces a hamiltonian path of HP' in $H \times F$, whose cost is no greater than the cost of the HP. Moreover, if HP is the optimal hamiltonian path, then HP' has the same cost as HP.*

Proof. Suppose $H = T \cup C$ and $H \times F = T' \cup C'$. If F does not contain the starting or the ending vertex of HP, then, by the formula (1), HP induces a hamiltonian path of HP' in $H \times F$ whose cost, relative to C', is no greater than the original cost of HP with respect to C. Especially, when HP is optimal, the equality holds.

Otherwise, F contains exactly one endvertex. Since our traversal types enumerate all the situations that HP traverses the vertices of F, such HP' exists. For instance, in the example in Section 6, the hamiltonian path u, p, m, e, d, c, b, a, l, k, j, o, i, h, g, f, n, q, v induces $S_1(UDT$ of $w_1)$, p, m, e, d, c, $S_2(UDT$ of $w_1)$, l, k, j, o, i, h, g, f, n, q, v after shrinking the fan, whose centre is u, into the pseudo-vertex w_1.

The hamiltonian paths, which are the same except the part of traversal in F, induce the same hamiltonian path, if their traversal types in F are the same. By the formulas (2a) $\sim$ (2p), we choose the minimum traversal path of each type. Moreover, the cost of each traversal type is stored in the pseudo-vertex which delivers the information to $H \times F$. Hence, the cost of HP' in $H \times F$, induced by HP in H, is equal to or less than the cost of HP. When HP is optimal, the equality also holds.

The sections 1) $\sim$ 18) and 35) guarantee that the vertices u and v do not lie in the same fan such that the centre of the fan is not the Root. So our traversal types and their functions work. Therefore, the hamiltonian path HP' found in 38) or 41) has the same cost as the optimal hamiltonian path in the original graph H. □

Claim 6. *The hamiltonian path HP' found in 38) or 41) can be extended to a hamiltonian (u,v)-path in the original graph H. And the cost of HP' does not change through the extension.*

Proof. Let v_F be a pseudo-vertex of HP' where F is the fan which is shrunk to v_F. And let H_F be the Halin graph which F belongs to.

If v_F does not contain any endvertex, then the two edges adjacent to v_F, which HP' uses, are determined by HP'. Denote these two edges by e', f'. And their corresponding edges in H_F are denoted by e, f. The traversal path of F could be obtained by the choice of e, f and the structure of F. We substitute the pseudo-vertex v_F by the traversal to achieve the extension. Since the costs of e' and f' were reassigned according to the costs of e and f in H_F and the traversal of F. Therefore, the cost remains the same after the extension.

If v_F contains precisely one endvertex, the traversal type of it is determined by HP'. For example, if $HP' = u$, ACT of v_F, then the traversal type of v_F is Anti-Clockwise Traversal. It means the cost of this part in HP' is calculated by the values of the functions stored in v_F. Therefore, there exists such a traversal of F whose cost equals the value of corresponding function. We use the traversal of F to replace the pseudo-vertex v_F. Hence, the cost does not change. Moreover, the extension is based on the traversal types we define, so it guarantees that the starting and ending vertices of HP' are the vertices u and v, or the pseudo-vertices containing them. □

In the algorithm, sections 1) $\sim$ 18) need $O(1)$ time. The sections 19) $\sim$ 34) are the operations of shrinking fans in H. If a fan F contains r+1 vertices, then it is verified that the time of the shrinking operation is $O(r)$. Moreover, shrinking F reduces the number of vertices of the graph by r. Thus, the total time of the shrinking operations is $O(|V|)$. The time of the postorder scan without shrinking is bounded by $O(|V|)$. The sections 35) $\sim$ 44) need $O(1)$ time. The time of sections 45) $\sim$ 49) is the same as the sections 19) $\sim$ 34). Therefore, the total time for this algorithm is $O(|V|)$.

8. Extensions

8.1. Min HP_u in Halin graphs. Section 4 discusses all types of traversal in a fan containing no endvertex or just one endvertex. We also adopt the "Shrinking Fan" strategy to solve Min HP_u problem. Since just one endvertex is known in HP_u problem, a fan might have none or precisely one endvertex (we can use the trick to avoid two endvertices in a fan as described in Section 5). Hence, the cost of the edges cannot be changed when the fan is shrunk. Similarly with Section 4, we use functions to store all types of traversal in a fan containing no endvertex. Let F be a fan in Halin graph H and $\{k, j, l\}$ be $EC_3(F)$. See Fig. 2. Assume that there is no endvertex of HP_u in F. By Theorem 3, HP_u covers exactly two of $\{k, j, l\}$. The traversal type of HP_u containing $\{k, j\}$ is denoted by C_{kj}. Suppose F is shrunk to v_F. Use function $C_{kj}(v_F)$ to store the minimum cost of all traversals of the type C_{kj}. To an original vertex x in H, $C_{kj}(x)$ is set to be 0. The definitions of C_{kl} and C_{jl} are similar.

Let w_c be the centre of F and let $w_1, w_2, \ldots, w_r$ for $r \geqslant 2$ be the vertices of F, which belong to C. See Fig. 2. For abbreviation, we define

$D_{a,b} \equiv \sum_{i=a}^{b} C_{jl}(w_i) + \sum_{i=a}^{b-1} \alpha(w_i, w_{i+1})$ for $1 \leqslant a < b \leqslant r$.

And assume that $D_{a,b} = 0$ for $b \leqslant a$ and $\alpha(w_0, w_1) = \alpha(w_r, w_{r+1}) = \alpha(w_{r+1}, w_{r+2}) = C_{kl}(w_{r+1}) = 0$. The formulas are defined in the following:

$$
\left.
\begin{aligned}
C_{kj}(v_F) &= \alpha(w_c, w_r) + C_{kj}(w_r) + \alpha(w_{r-1}, w_r) + D_{1,r-1} \\
C_{kl}(v_F) &= \alpha(w_c, w_1) + C_{kl}(w_1) + \alpha(w_1, w_2) + D_{2,r} \\
C_{kj}(v_F) &= \min_{1 \leqslant j \leqslant r} \left\{ \begin{array}{l} D_{1,j-1} + \alpha(w_{j-1}, w_j) + C_{kj}(w_j) + \alpha(w_j, w_c) \\ +\alpha(w_c, w_{j+1}) + C_{kl}(w_{j+1}) + \alpha(w_{j+1}, w_{j+2}) + D_{j+2,r} \end{array} \right\}.
\end{aligned}
\right. \tag{3}
$$

The formulas of CT, ACT, dCT, $dACT$, UT and UDT in Section 4 are supposed to be changed, simultaneously. The definitions of x_k, x_c, x_1, x_2, $\ldots$, x_r are the same as Section 4. The calculations of CT, dCT, UT and UDT are given in the following. The ones of ACT and $dACT$ are similar. Readers can refer to the formulas (2a) $\sim$ (2p).

$$
CT_1(v_F) = \begin{cases}
\begin{array}{l} ACT(x_k) + \alpha(x_k, x_{k+1}) + D_{k+1,r-1} + \alpha(x_{r-1}, x_r) \\ +C_{kj}(x_r) + \alpha(x_r, x_c) + \alpha(x_c, x_{k-1}) + C_{kj}(x_{k-1}) \\ +\alpha(x_{k-2}, x_{k-1}) + D_{1,k-2} \end{array} & \text{if } 2 \leqslant k < r, \\
\min \left\{ \begin{array}{l} UT(x_k) + \alpha(x_k, x_c) + \alpha(x_c, x_{k-1}) + C_{kj}(x_{k-1}) \\ +\alpha(x_{k-2}, x_{k-1}) + D_{1,k-2}, \\ \{CT(x_k) + \alpha(x_{r-1}, x_r) + D_{j+1,r-1} + \alpha(x_j, x_{j+1}) \\ +C_{kl}(x_j) + \alpha(x_j, x_c) + \alpha(x_c, x_{j-1}) + C_{kj}(x_{j-1}) \\ +\alpha(x_{j-2}, x_{j-1}) + D_{1,j-2} \,|\, 2 \leqslant j < r\} \end{array} \right\} & \text{if } k = r, \\
+\infty & \text{otherwise.}
\end{cases}
$$

$$
CT_2(v_F) = \begin{cases}
\begin{array}{l} dACT(x_k) + \alpha(x_k, x_{k+1}) + D_{k+1,r-1} + \alpha(x_{r-1}, x_r) + C_{kj}(x_r) \\ +\alpha(x_r, x_c) + \alpha(x_c, x_k) + \alpha(x_{k-1}, x_k) + D_{1,k-1} \end{array} & \text{if } 1 \leqslant k < r, \\
+\infty & \text{otherwise.}
\end{cases}
$$

$$CT_3(v_F) = \begin{cases} UDT(x_k) + \alpha(x_k, x_c) + \alpha(x_c, x_r) + C_{kj}(x_r) \\ +\alpha(x_{r-1}, x_r) + D_{1,r-1} & \text{if } 1 \leqslant k < r, \\ +\infty & \text{otherwise.} \end{cases}$$

$$CT(v_F) = \begin{cases} \alpha(x_c, x_r) + C_{kj}(x_r) + \alpha(x_{r-1}, x_r) + D_{1,r-1} & x_k = x_c, \\ \min\{CT_1(v_F), CT_2(v_F), CT_3(v_F)\} & \text{otherwise.} \end{cases}$$

$$dCT_1(v_F) = \begin{cases} CT(x_k) + \alpha(x_{k-1}, x_k) + D_{1,k-1} + \alpha(x_c, x_{k+1}) \\ +C_{kl}(x_{k+1}) + \alpha(x_{k+1}, x_{k+2}) + D_{k+2,r} & \text{if } 1 \leqslant k < r, \\ +\infty & \text{otherwise.} \end{cases}$$

$$dCT_2(v_F) = \begin{cases} dCT(x_k) + \alpha(x_{k-1}, x_k) + D_{1,k-1} + \\ \alpha(x_c, x_k) + \alpha(x_k, x_{k+1}) + D_{k+1,r} & \text{if } 1 \leqslant k \leqslant r, \\ +\infty & \text{otherwise.} \end{cases} \quad (4)$$

$$dCT(v_F) = \min\{dCT_1(v_F), dCT_2(v_F)\}$$

$$UT(v_F) = \begin{cases} ACT(x_k) + \alpha(x_k, x_{k+1}) + D_{k+1,r-1} + \alpha(x_{r-1}, x_r) \\ +C_{kj}(x_r) + \alpha(x_r, x_c) & \text{if } k = 1, \\ CT(x_k) + \alpha(x_{r-1}, x_r) + D_{2,r-1} + \alpha(x_1, x_2) \\ +C_{kl}(x_1) + \alpha(x_1, x_c) & \text{if } k = r, \\ +\infty & \text{otherwise.} \end{cases}$$

$$UDT(v_F) = \begin{cases} UDT(x_k) + D_{1,r} & \text{if } x_k = x_c, \\ UDT(x_k) + \alpha(x_k, x_c) + D_{1,r} & \text{if } x_k \neq x_c, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, the other endvertex of HP_u is not specified. That is, each vertex in a fan F can be x_k. Hence, let x_k *enumerate* all vertices in F and store the minimum cost and the corresponding structure while calculating each formula. The values of $D_{1,i}$ and $D_{i,r}$ for $1 \leqslant i \leqslant r$ can be calculated in $2r$ steps. We calculate these two functions at first. Then $D_{i,r-1}$ for $1 \leqslant i < r$ can be obtained by $D_{i,r-1} = D_{i,r} - C_{jl}(x_r) - \alpha(x_{r-1}, x_r)$ in 2 steps. If x_k is x_1 (x_r), only the computation of CT_1 (ACT_1) needs r steps. The computations of other formulas just require constant steps for one fixed x_k. So shrinking a fan F is also $O(|V(F)|)$ time.

The algorithm to search Min HP_u is similar with the one to search Min $HP_{u,v}$ in Section 5. Besides the differences mentioned above, the remaining ones are given as follows:

The selection of "Root": if the vertex u lies in C, let Root:=the centre of the fan containing u and TreeType:=1; otherwise, let Root:=u and TreeType:=2.

The enumeration of HP_u in different selections: let w_u be the vertex u when TreeType==1; otherwise, let w_u be the pseudo-vertex containing u. Let k be the edge joining w_u and Root, j and l be the other two edges joining w_u such that the direction of j, w_u, k is clockwise. See Fig. 5.

We discuss the calculation when TreeType==1, first. Shrink the fan $H_w - w_u$ to the pseudo-vertex w_v. So $Cost := \min\{\alpha_k + UT(w_v), \alpha_j + CT(w_v), \alpha_l + ACT(w_v)\}$.

When TreeType==2, no "Shrinking Fan" operation is required. Let the vertices on the cycle of H_w be $x_1, x_2, \ldots, x_r$. See Fig. 5.

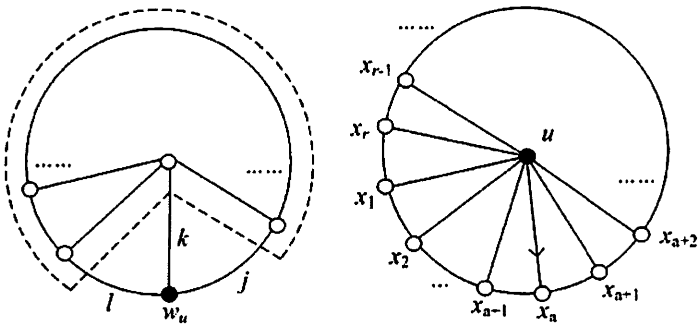

FIGURE 5. Different selections of Root

We assume that $\alpha(x_0,x_1)=\alpha(x_r,x_{r+1})=\alpha(x_r,x_1)$ and $\sum_{i=j}^{j-1} C_{jl}(x_i)=0$ for all values of j. We have

$$Cost := \min_{1\leqslant a\leqslant r}\left\{\begin{array}{l}\alpha(u,x_a)+C_{kj}(x_a)+\sum_{i=1}^{a-1}C_{jl}(x_i)+\sum_{i=a+2}^{r}C_{jl}(x_i)\\ +ACT(x_{a+1})+\sum_{i=1}^{r}\alpha(x_i,x_{i+1})-\alpha(x_a,x_{a+1}),\\ \alpha(u,x_a)+C_{kl}(x_a)+\sum_{i=a+1}^{r}C_{jl}(x_i)+\sum_{i=1}^{a-2}C_{jl}(x_i)\\ +CT(x_{a-1})+\sum_{i=1}^{r}\alpha(x_i,x_{i+1})-\alpha(x_{a-1},x_a),\\ \alpha(u,x_a)+dACT(x_a)+\sum_{i=1}^{a-1}C_{jl}(x_i)+\sum_{i=a+1}^{r}C_{jl}(x_i)+\sum_{i=1}^{r}\alpha(x_i,x_{i+1}),\\ \alpha(u,x_a)+dCT(x_a)+\sum_{i=a+1}^{r}C_{jl}(x_i)+\sum_{i=1}^{a-1}C_{jl}(x_i)+\sum_{i=1}^{r}\alpha(x_i,x_{i+1})\end{array}\right\}$$

The amended part above can be computed in $O(|H_w|)$ time which is bounded by $O(|V|)$. Using the same analysis method in Section 7, the total time of the extended algorithm to search Min HP_u is also $O(|V|)$.

8.2. **Min HP in Halin graphs.** Since no endvertex is specified in Min HP problem, a fan might contain two endvertices. Let F be a fan in Halin graph H and $\{k,j,l\}$ be $EC_3(F)$. See Fig. 2. Assume that there are two endvertices of HP in F. It can be concluded that the traversal of F is not consecutive; otherwise, the vertex in $H-F$ cannot lie in HP. By Theorem 3, HP covers exactly two of $\{k,j,l\}$. The traversal type of HP containing $\{k,j\}$ is denoted by *discrete* C_{kj} (dC_{kj} for short). Suppose F is shrunk to v_F. Use function $dC_{kj}(v_F)$ to store the minimum cost of all traversals of the type dC_{kj}. The definitions of dC_{kl} and dC_{jl} are similar.

The functions (such as C_{kj}, CT) which indicate F contains less than two endvertices have no relation to the ones indicating two endvertices. So the formulas (3) $\sim$ (4) in Subsection 8.1 remain the same in Min HP problem. Before presenting the formulas of dC_{kj}, dC_{kl} and dC_{jl}, we introduce the definition of "pseudo-fan". Suppose x_c be the centre of F and let $x_1, x_2, \ldots, x_r$ for $r \geqslant 2$ be the vertices of F, which belong to C. Given a k ($2 \leqslant k \leqslant r$), we denote the induced subgraph $F[x_c, x_a, x_{a+1}, \ldots, x_b]$ ($1 \leqslant a \leqslant b \leqslant r$) by pseudo-fan $PF_{a,b}$ of F. Figure 6 shows two pseudo-fans, $PF_{1,k}$ and $PF_{1,k+1}$, which are indicated by dotted lines.

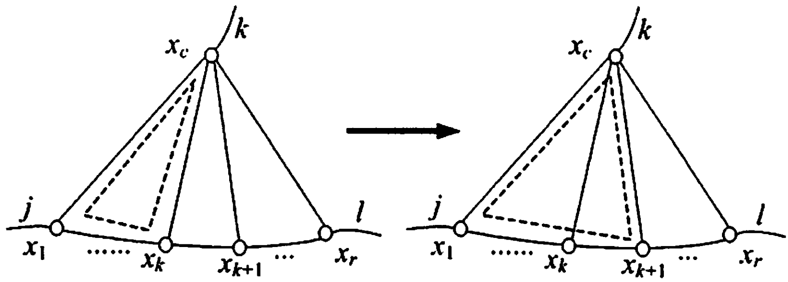

FIGURE 6. Pseudo-fans

Similarly with a fan, we calculate the functions C_{kj}, C_{kl}, C_{kj}, CT, ACT, dCT, $dACT$, UT and UDT to pseudo-fans $PF_{1,k}$ and $PF_{k,r}$ for all $1 \leqslant k \leqslant r$. We introduce the method to calculate the functions of $PF_{1,k+1}$ when the functions of $PF_{1,k}$ have been worked out. The method from $PF_{k,r}$ to $PF_{k-1,r}$ is symmetric.

We assume that

$$C_{kj}(PF_{1,1}) = \alpha(x_c, x_1) + C_{kj}(x_1), \qquad C_{kl}(PF_{1,1}) = \alpha(x_c, x_1) + C_{kl}(x_1);$$
$$C_{jl}(PF_{1,1}) = +\infty, \qquad CT(PF_{1,1}) = \alpha(x_c, x_1) + C_{kj}(x_1);$$
$$ACT(PF_{1,1}) = \alpha(x_c, x_1) + C_{kl}(x_1), \qquad dCT(PF_{1,1}) = dCT(x_1) + \alpha(x_c, x_1);$$
$$dACT(PF_{1,1}) = dACT(x_1) + \alpha(x_c, x_1), \qquad UT(PF_{1,1}) = UT(x_1) + \alpha(x_c, x_1);$$
$$UDT(PF_{1,1}) = \min\{UDT(x_1) + \alpha(x_c, x_1), C_{jl}(x_1)\}.$$

Once the functions of $PF_{1,k}$ are known, the functions of $PF_{1,k+1}$ can be calculated in the following:

$$C_{kj}(PF_{1,k+1}) = \alpha(x_c, x_{k+1}) + C_{kj}(x_{k+1}) + \alpha(x_k, x_{k+1}) + D_{1,k}$$
$$C_{kl}(PF_{1,k+1}) = C_{kl}(PF_{1,k}) + \alpha(x_k, x_{k+1}) + C_{jl}(x_{k+1})$$
$$C_{jl}(PF_{1,k+1}) = \min\left\{ \begin{array}{l} C_{kj}(PF_{1,k}) + \alpha(x_c, x_{k+1}) + C_{kl}(x_{k+1}), \\ C_{jl}(PF_{1,k}) + \alpha(x_k, x_{k+1}) + C_{jl}(x_{k+1}) \end{array} \right\}$$
$$CT(PF_{1,k+1}) = \min\left\{ \begin{array}{l} \alpha(x_c, x_{k+1}) + C_{kj}(x_{k+1}) + \alpha(x_k, x_{k+1}) + D_{1,k}, \\ UT(x_{k+1}) + \alpha(x_c, x_{k+1}) + C_{kj}(PF_{1,k}), \\ CT(x_{k+1}) + \alpha(x_k, x_{k+1}) + C_{jl}(PF_{1,k}), \\ dACT(PF_{1,k}) + \alpha(x_k, x_{k+1}) + C_{kj}(x_{k+1}) + \alpha(x_c, x_{k+1}), \\ UDT(PF_{1,k}) + \alpha(x_c, x_{k+1}) + C_{kj}(x_{k+1}) + \alpha(x_k, x_{k+1}) \end{array} \right\}$$

$$ACT(PF_{1,k+1}) = \min \left\{ \begin{array}{l} UT(PF_{1,k}) + \alpha(x_c, x_{k+1}) + C_{kl}(x_{k+1}), \\ ACT(PF_{1,k}) + \alpha(x_k, x_{k+1}) + C_{jl}(x_{k+1}), \\ dCT(x_{k+1}) + \alpha(x_k, x_{k+1}) + C_{kl}(PF_{1,k}) + \alpha(x_c, x_{k+1}), \\ UDT(x_{k+1}) + \alpha(x_c, x_{k+1}) + C_{kl}(PF_{1,k}) + \alpha(x_k, x_{k+1}) \end{array} \right\}$$

$$dCT(PF_{1,k+1}) = \min \left\{ \begin{array}{l} dCT(x_{k+1}) + \alpha(x_k, x_{k+1}) + D_{1,k} + \alpha(x_c, x_{k+1}), \\ CT(x_k) + \alpha(x_{k-1}, x_k) + D_{1,k-1} + \alpha(x_c, x_{k+1}) + C_{kj}(x_{k+1}), \\ dCT(PF_{1,k}) + \alpha(x_k, x_{k+1}) + C_{jl}(x_{k+1}) \end{array} \right\}$$

$$dACT(PF_{1,k+1}) = \min \left\{ \begin{array}{l} dACT(PF_{1,k}) + \alpha(x_k, x_{k+1}) + C_{jl}(x_{k+1}), \\ ACT(x_{k+1}) + C_{kj}(PF_{1,k}), \\ dACT(x_{k+1}) + \alpha(x_c, x_{k+1}) + \alpha(x_k, x_{k+1}) + D_{1,k} \end{array} \right\}$$

$$UT(PF_{1,k+1}) = \min \left\{ \begin{array}{l} CT(x_{k+1}) + \alpha(x_k, x_{k+1}) + C_{kl}(PF_{1,k}),\ ACT(x_1) + D_{1,k} \\ -C_{jl}(x_1) + \alpha(x_k, x_{k+1}) + C_{kj}(x_{k+1}) + \alpha(x_c, x_{k+1}) \end{array} \right\}$$

$$UDT(PF_{1,k+1}) = \min \left\{ \begin{array}{l} UDT(PF_{1,k}) + \alpha(x_k, x_{k+1}) + C_{jl}(x_{k+1}), \\ UDT(x_{k+1}) + \alpha(x_c, x_{k+1}) + D_{1,k} + \alpha(x_k, x_{k+1}) \end{array} \right\}$$

Now, investigate the traversal types dC_{kj}, dC_{kl} and dC_{jl}. To an original vertex w, let $dC_{kj}(w) = dC_{kl}(w) = dC_{jl}(w) = +\infty$. If the centre of fan F is one endvertex, this situation is denoted by Case 1; the other situation is denoted by Case 2. Suppose the endvertices of HP are u, v. See Fig. 7.

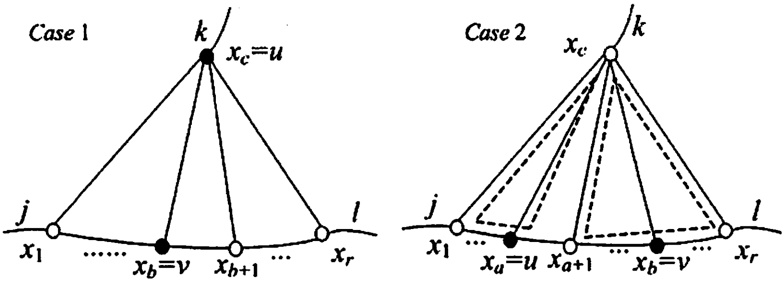

FIGURE 7. Fans containing two endvertices

In Case 1, we assume that $x_b = v$ where $1 \leqslant b \leqslant r$.

(1) If HP uses k and j, then v must be x_r and HP is x_c, k, …, j, x_1, (x_1, x_2), …, (x_{r-1}, x_r), $CT(x_r)$.

(2) If HP uses k and l, then v must be x_1 and HP is x_c, k, …, l, x_r, (x_{r-1}, x_r), …, (x_1, x_2), $ACT(x_1)$.

(3) If HP uses j and l, then there are many situations given in the following:
If u meets j earlier than l, then HP is x_c, (x_c, x_{b-1}), x_{b-1}, (x_{b-2}, x_{b-1}), …, x_1, j, …, l, x_r, (x_{r-1}, x_r), …, (x_b, x_{b+1}), $ACT(x_b)$
or x_c, (x_c, x_b), $S_2(dACT(x_b))$, (x_{b-1}, x_b), …, x_1, j, …, l, x_r, (x_{r-1}, x_r), …, (x_b, x_{b+1}), $S_1(dACT(x_b))$.
If u meets l earlier than j, then HP is x_c, (x_c, x_{b+1}), x_{b+1}, (x_{b+1}, x_{b+2}), …, x_r, l, …, j, x_1, (x_1, x_2), …, (x_{b-1}, x_b), $CT(x_b)$

or x_c, (x_c, x_b), $S_2(dCT(x_b))$, (x_b, x_{b+1}), $\ldots$, x_r, l, $\ldots$, j, x_1, (x_1, x_2), $\ldots$, (x_{b-1}, x_b), $S_1(dCT(x_b))$.

In Case 2, we suppose that $x_a = u$ and $x_b = v$ where $1 \leqslant a < b \leqslant r$.

(1) HP uses k and j:

If u meets k earlier than j, then v must be x_r. HP is $S_1(UDT(PF_{1,r-1}))$, k, $\ldots$, j, $S_2(UDT(PF_{1,r-1}))$, (x_{r-1}, x_r), $CT(x_r)$.

If u meets j earlier than k, then HP is $S_1(dCT(PF_{1,a}))$, j, $\ldots$, k, $S_2(dCT(PF_{1,a}))$, (x_a, x_{a+1}), x_{a+1}, $\ldots$, (x_{r-1}, x_r), $CT(x_r)$ when $x_b = x_r$ or $CT(x_a)$, (x_{a-1}, x_a), $\ldots$, x_1, j, $\ldots$, k, $UT(PF_{a+1,r})$.

(2) HP uses k and l:

If v meets k earlier than l, then u must be x_1. HP is $S_1(UDT(PF_{2,r}))$, k, $\ldots$, l, $S_2(UDT(PF_{2,r}))$, (x_1, x_2), $ACT(x_1)$.

If v meets l earlier than k, then HP is $S_1(dACT(PF_{b,r}))$, l, $\ldots$, k, $S_2(dACT(PF_{b,r}))$, (x_{b-1}, x_b), x_{b-1}, $\ldots$, (x_1, x_2), $ACT(x_1)$ when $x_a = x_1$ or $ACT(x_b)$, (x_b, x_{b+1}), $\ldots$, x_r, l, $\ldots$, k, $UT(PF_{1,b-1})$.

(3) HP uses j and l: Suppose the edges e, f are adjacent to x_c in HP.

(i) $\{e, f\} \bigcap E(PF_{1,a}) = \emptyset$:
HP is $CT(x_a)$, (x_{a-1}, x_a), $\ldots$, x_1, j, $\ldots$, l, $ACT(PF_{a+1,r})$.

(ii) $|\{e, f\} \bigcap E(PF_{1,a})| = 1$:

If u meets j earlier than l, then HP is $S_1(dCT(PF_{1,a}))$, j, $\ldots$, l, $S_2(dCT(PF_{a+1,r}))$, $S_2(dCT(PF_{1,a}))$, (x_a, x_{a+1}), $S_1(dCT(PF_{a+1,r}))$ or $S_1(dACT(PF_{1,a}))$, (x_a, x_{a+1}), $S_2(dACT(PF_{a+1,r}))$, $S_2(dACT(PF_{1,a}))$, j, $\ldots$, l, $S_1(dACT(PF_{a+1,r}))$;

If u meets l earlier than j, HP is $S_1(UDT(PF_{1,a}))$, $S_2(dCT(PF_{a+1,r}))$, l, $\ldots$, j, $S_2(UDT(PF_{1,a}))$, (x_a, x_{a+1}), $S_1(dCT(PF_{a+1,r}))$ or $S_1(dACT(PF_{1,a}))$, (x_a, x_{a+1}), $S_2(UDT(PF_{a+1,r}))$, l, $\ldots$, j, $S_2(dACT(PF_{1,a}))$, $S_1(UDT(PF_{a+1,r}))$.

(iii) $|\{e, f\} \bigcap E(PF_{1,a})| = 2$:
The vertex v must be x_{a+1} and HP is $CT(PF_{1,a})$, j, $\ldots$, l, x_r, (x_{r-1}, x_r), $\ldots$, (x_{a+1}, x_{a+2}), $ACT(x_{a+1})$.

The calculations of dC_{kj}, dC_{kl} and dC_{jl} from CT, ACT, dCT, $dACT$, UT and UDT, which indicate the pseudo-vertices containing an endvertex of HP, are examined in above. The calculations of dC_{kj}, dC_{kl} and dC_{jl} from themselves are simple. We present them within the following formulas.

$$JL_1(v_F) = \min_{2 \leqslant k \leqslant r} \left\{ \begin{array}{l} D_{1,k-2} + \alpha(x_{k-2}, x_{k-1}) + C_{kj}(x_{k-1}) + \alpha(x_c, x_{k-1}) \\ +\alpha(x_c, x_k) + dC_{kl}(x_k) + \alpha(x_k, x_{k+1}) + D_{k+1,r}, \\ C_{jl}(PF_{1,k-1}) + \alpha(x_{k-1}, x_k) + dC_{jl}(x_k) + \alpha(x_k, x_{k+1}) + D_{k+1,r}, \\ \alpha(x_c, x_{k-1}) + C_{kj}(x_{k-1}) + \alpha(x_{k-2}, x_{k-1}) + D_{1,k-2} \\ +D_{k+1,r} + \alpha(x_k, x_{k+1}) + ACT(x_k) \end{array} \right\}$$

$$JL_2(v_F) = \min_{1 \leqslant k \leqslant r} \left\{ \begin{array}{l} \alpha(x_c, x_k) + dACT(x_k) + \alpha(x_{k-1}, x_k) + D_{1,k-1} + D_{k+1,r} + \alpha(x_k, x_{k+1}), \\ \alpha(x_c, x_k) + dCT(x_k) + \alpha(x_k, x_{k+1}) + D_{k+1,r} + D_{1,k-1} + \alpha(x_{k-1}, x_k) \end{array} \right\}$$

$$JL_3(v_F) = \min_{1\leqslant k<r} \left\{ \begin{array}{l} D_{1,k-1} + \alpha(x_{k-1}, x_k) + dC_{kj}(x_k) + \alpha(x_c, x_k) \\ +\alpha(x_c, x_{k+1}) + C_{kl}(x_{k+1}) + \alpha(x_{k+1}, x_{k+2}) + D_{k+2,r}, \\ D_{1,k-1} + \alpha(x_{k-1}, x_k) + dC_{jl}(x_k) + \alpha(x_k, x_{k+1}) + C_{jl}(PF_{k+1,r}), \\ \alpha(x_c, x_{k+1}) + C_{kl}(x_{k+1}) + \alpha(x_{k+1}, x_{k+2}) + D_{k+2,r} \\ +D_{1,k-1} + \alpha(x_{k-1}, x_k) + CT(x_k), \\ dCT(PF_{1,k}) + dCT(PF_{k+1,r}) + \alpha(x_k, x_{k+1}), \\ dACT(PF_{1,k}) + \alpha(x_k, x_{k+1}) + dACT(PF_{k+1,r}), \\ UDT(PF_{1,k}) + dCT(PF_{k+1,r}) + \alpha(x_k, x_{k+1}), \\ dACT(PF_{1,k}) + UDT(PF_{k+1,r}) + \alpha(x_k, x_{k+1}), \\ CT(PF_{1,k}) + D_{k+2,r} + \alpha(x_{k+1}, x_{k+2}) + ACT(x_{k+1}), \\ CT(x_k) + \alpha(x_{k-1}, x_k) + D_{1,k-1} + ACT(PF_{k+1,r}) \end{array} \right\}$$

$$dC_{jl}(v_F) = \min\{JL_1(v_F), JL_2(v_F), JL_3(v_F)\}$$

$$KJ_1(v_F) = \min_{1\leqslant k<r} \left\{ \begin{array}{l} D_{1,k-1} + \alpha(x_{k-1}, x_k) + dC_{jl}(x_k) + \alpha(x_k, x_{k+1}) + C_{kj}(PF_{k+1,r}), \\ dCT(PF_{1,k}) + \alpha(x_k, x_{k+1}) + D_{k+1,r} - C_{jl}(x_r) + CT(x_r), \\ CT(x_k) + \alpha(x_{k-1}, x_k) + D_{1,k-1} + UT(PF_{k+1,r}) \end{array} \right\}$$

$$dC_{kj}(v_F) = \min \left\{ \begin{array}{l} KJ_1(v_F), \\ D_{1,r-1} + \alpha(x_{r-1}, x_r) + dC_{kj}(x_r) + \alpha(x_c, x_r), \\ D_{1,r-1} + \alpha(x_{r-1}, x_r) + CT(x_r), \\ UDT(PF_{1,r-1}) + \alpha(x_{r-1}, x_r) + CT(x_r) \end{array} \right\}$$

$$KL_1(v_F) = \min_{2\leqslant k\leqslant r} \left\{ \begin{array}{l} C_{kl}(PF_{1,k-1}) + \alpha(x_{k-1}, x_k) + dC_{jl}(x_k) + \alpha(x_k, x_{k+1}) + D_{k+1,r}, \\ dACT(PF_{k,r}) + \alpha(x_{k-1}, x_k) + D_{1,k-1} - C_{jl}(x_1) + ACT(x_1), \\ ACT(x_k) + \alpha(x_k, x_{k+1}) + D_{k+1,r} + UT(PF_{1,k-1}) \end{array} \right\}$$

$$dC_{kl}(v_F) = \min \left\{ \begin{array}{l} KL_1(v_F), \\ \alpha(x_c, x_1) + dC_{kl}(x_1) + \alpha(x_1, x_2) + D_{2,r}, \\ D_{2,r} + \alpha(x_1, x_2) + ACT(x_1), \\ UDT(PF_{2,r}) + \alpha(x_1, x_2) + ACT(x_1) \end{array} \right\}$$

For a fan F which contains $r-1$ vertices, the total time of the calculations of $PF_{1,k}$ and $PF_{k,r}$ ($1 \leqslant k \leqslant r$) is bounded by $O(r)$. So is the time of computing dC_{kj}, dC_{kl} and dC_{jl}. Thus, the complexity to shrink a fan in our algorithm to search Min HP is also $O(r)$.

The "Root" selection of the algorithm to search Min HP is: choose a vertex w_u which lies in C, let Root:=the centre of the fan containing w_u. Suppose the final wheel we get is H_w. Shrink the fan $H_w - w_u$ to the pseudo-vertex w_v. See Fig. 5. And the formula of "Cost" is

$$Cost := \min \left\{ \begin{array}{lll} UT(w_v) + \alpha_k, & CT(w_v) + \alpha_j, & ACT(w_v) + \alpha_l, \\ dC_{kj}(w_v) + \alpha_k + \alpha_j, & dC_{kl}(w_v) + \alpha_k + \alpha_l, & dC_{jl}(w_v) + \alpha_j + \alpha_l \end{array} \right\}$$

Similarly with the algorithm in Subsection 8.1, the total time of the extended algorithm to search Min HP is $O(|V|)$.

8.3. An example of extended algorithms. We present an example in the following. The vertex l is chosen to be w_u.

The main procedures are shown in Fig. 8. And the values of the functions are illustrated in Table. 2. Since $dC_{jl}(q) + 1 + 1 = 24$ gets minimum, the cost of Min HP equals 24 and one solution is Min HP= $u, p, m, e, d, c, b, a, l, k, v, q, n, f, g, h, i, j, o$. Moreover, $ACT(q) + 1 = 28$ obtains minimum, one solution is Min HP_l= $l, k, v, o, j, i, h, g, f, n, q, p, m, e, d, c, b, a, b, u$.

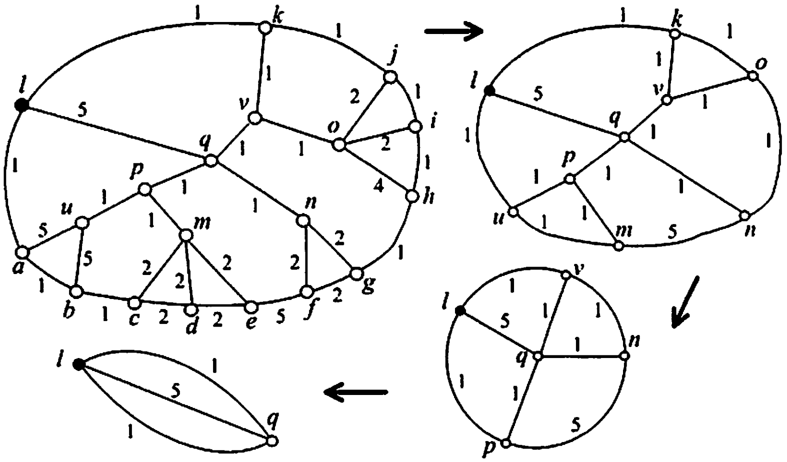

FIGURE 8. Example of extended algorithms

TABLE 2. The values of the functions

	C_{kj}	C_{kl}	C_{jl}	CT	ACT	dCT	$dACT$	UT	UDT	dC_{kj}	dC_{kl}	dC_{kl}
o	4	6	5	4	5	3	3	4	2	2	2	3
n	4	4	4	4	4	2	2	4	2	2	2	2
m	6	6	6	6	6	4	4	6	4	4	4	4
u	6	6	10	6	6	5	5	6	1	1	1	5
v	7	8	6	6	6	5	5	7	4	4	4	4
p	18	14	14	10	13	13	11	14	9	9	9	8
q	32	31	33	28	27	22	29	31	26	22	26	22

9. CONCLUSIONS

NP-hard (NP-complete) problems probably are not NP-hard (NP-complete) in Halin graphs. For example, the minimum TSP is solved in $O(V)$, in [3]. Our algorithm indicates that construction of optimal hamiltonian path (any of the three versions) belongs to these problems. The bottleneck travelling salesman problem was investigated in [8]. Our approach can be amended to search the bottleneck of hamiltonian paths. We believe that the approach is applicable to more hard problems in Halin graphs. And it might work in special cases of TSP [2]. However, not all NP-hard (NP-complete) problems can be solved in polynomial time in Halin graphs. The problem of isomorphism with respect to subgraphs and supergraphs, which is NP-complete in general graphs, is also NP-complete in Halin graphs. It is proved in [5].

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