Comment on "The Expectation Of Independent Domination Number Over Random Binary Trees"

L.H. Clark and J.P. McSorley
Department of Mathematics
Southern Illinois University Carbondale
Carbondale, IL 62901-4408

Lee [3] purportedly derives an asymptotic formula for the expected independent domination number of a uniformly random binary tree. We review the derivation in [3] of an asymptotic formula for the expectation using the notation therein, then we point out and correct several errors in the derivation.

The number of binary trees with $2n+1$ vertices is

$$y_{2n+1} = \frac{\binom{2n}{n}}{n+1}$$

Let $\mu(2n+1)$ denote the expected value of the independent domination number of a binary tree chosen uniformly at random. The ordinary generating function for $\{\mu(2n+1)\, y_{2n+1}\}$ is $M = M(x) = \sum_{n=0}^{\infty} \mu(2n+1)\, y_{2n+1}\, x^{2n+1}$. Then

$$M(x) = \frac{2x}{\sqrt{1-4x^2}\,(1+\sqrt{1-4x^2})\,(2-\sqrt{1-4x^2})},$$

hence,

$$\begin{aligned} M_*(u) &:= \sum_{n=0}^{\infty} \mu(2n+1)\, y_{2n+1}\, u^n \\ &= \frac{2}{\sqrt{1-4u}\,(1+\sqrt{1-4u})\,(2-\sqrt{1-4u})}. \end{aligned}$$

Then

$$A(u) = \frac{2}{(1+\sqrt{1-4u})\,(2-\sqrt{1-4u})}$$

has power series in u with radius of convergence $\rho_1 = 1/4$ which converges absolutely at $u = 1/4$, and,

$$B(u) = \sum_{n=0}^{\infty} b_n u^n = \frac{1}{\sqrt{1-4u}} = \sum_{n=0}^{\infty} (-4)^n \binom{-\frac{1}{2}}{n} u^n$$

has radius of convergence $\rho_2 = 1/4$, $b_n > 0$ for all n, and $\lim_{n\to\infty} b_{n-1}/b_n = 1/4$. At this point the following result in [3] is used.

"To determine the asymptotic behavior of $\mu(2n+1)/(2n+1)$, we need the following lemma, which is a slight modification of Theorem 2 in [1]; we omit the proof.

Lemma 5. Let $A(u) = \sum_{n=0}^{\infty} a_n u^n$ and $B(u) = \sum_{n=0}^{\infty} b_n u^n$ be power series with radii of convergence $\rho_1 \geq \rho_2$, respectively. Suppose that $A(u)$ converges absolutely at $u = \rho_1$. Suppose that $b_n > 0$ for all n and that b_{n-1}/b_n approaches a limit b as $n \to \infty$. If $\sum_{n=0}^{\infty} c_n u^n = A(u)\, B(u)$, then $c_n \sim A(b)\, b_n$."

The author then applies Lemma 5 to $M_*(u) = A(u)\, B(u)$ with $\rho_1 = \rho_2 = 1/4$ to find an asymptotic formula for $\mu(2n+1)\, y_{2n+1}$, hence, for $\mu(2n+1)$.

Unfortunately Lemma 5, as we will demonstrate, is false in general for any $\rho_1 = \rho_2 > 0$: the condition "$\rho_1 \geq \rho_2$" must be replaced with "$\rho_1 > \rho_2$" and the condition "$A(b) \neq 0$" must be added in which case the conditions "$A(u)$ converges absolutely at $u = \rho_1$" and "$b_n > 0$ for all n" may be omitted. See Bender [1; Theorem 2] for a correct statement and a very brief indication of a proof or see Odlyzko [4; Theorem 7.1] for a correct statement without proof. Consequently, the derivation in [3] of an asymptotic formula for $\mu(2n+1)$ is not valid.

Counter-examples to Lemma 5 for any $\rho_1 = \rho_2 = r > 0$ are readily found.

Fix $r > 0$. Let

$$A(u) = \sum_{n=0}^{\infty} \frac{u^n}{r^n\,(n+1)^2} = B(u)$$

which have radius of convergence r. Then $A(u)$ converges absolutely on the circle of convergence $|u| = r$ and $A(r) = \zeta(2) = \pi^2/6$. In addition,

$b_n = 1/r^n\,(n+1)^2 > 0$ for all n and $\lim_{n\to\infty} b_{n-1}/b_n = r$. Here

$$A(u)\,B(u) = \sum_{n=0}^{\infty} \left\{ \frac{1}{r^n} \sum_{k=0}^{n} \frac{1}{(k+1)^2(n-k+1)^2} \right\} u^n = \sum_{n=0}^{\infty} c_n\, u^n.$$

Further

$$\begin{aligned}\sum_{k=0}^{n} \frac{(n+2)^2}{(k+1)^2(n-k+1)^2} &= \sum_{k=0}^{n} \left\{ \frac{1}{k+1} + \frac{1}{n-k+1} \right\}^2 \\ &= 2 \sum_{k=0}^{n} \frac{1}{(k+1)^2} + 2 \sum_{k=0}^{n} \frac{1}{(k+1)\,(n-k+1)}.\end{aligned}$$

Now $f(x) = 1/(x+1)(n-x+1)$ decreases on $[0, n/2]$ and increases on $[n/2, n]$. For integer $\Delta \in [1, n/2]$,

$$\begin{aligned}&\sum_{k=0}^{n} \frac{1}{(k+1)\,(n-k+1)} \\ &= 2 \sum_{k=0}^{\Delta-1} \frac{1}{(k+1)\,(n-k+1)} + \sum_{k=\Delta}^{n-\Delta} \frac{1}{(k+1)\,(n-k+1)} \\ &\le \frac{2\,\Delta}{n+1} + \frac{n-2\,\Delta+1}{(\Delta+1)(n-\Delta+1)}.\end{aligned}$$

Setting $\Delta = \lceil \sqrt{n}\, \rceil$, for example, gives

$$0 \le \sum_{k=0}^{n} \frac{1}{(k+1)\,(n-k+1)} \le \frac{2\sqrt{n}+2}{n+1} + \frac{n-2\sqrt{n}+1}{(\sqrt{n}+1)(n-\sqrt{n})} \to 0 \text{ as } n \to \infty.$$

Consequently,

$$\begin{aligned}r^n\,(n+2)^2\, c_n &= \sum_{k=0}^{n} \frac{(n+2)^2}{(k+1)^2(n-k+1)^2} \\ &= 2 \sum_{k=0}^{n} \frac{1}{(k+1)^2} + 2 \sum_{k=0}^{n} \frac{1}{(k+1)\,(n-k+1)} \\ &\to \frac{\pi^2}{3} \text{ as } n \to \infty,\end{aligned}$$

which implies

$$r^n\,(n+1)^2\, c_n = \frac{(n+1)^2}{(n+2)^2}\, r^n\,(n+2)^2\, c_n \to \frac{\pi^2}{3} \text{ as } n \to \infty,$$

i.e.,

$$c_n \sim 2\,A(r)\,b_n \text{ as } n \to \infty$$

and not

$$c_n \sim \ A(r)\,b_n \text{ as } n \to \infty$$

as claimed in Lemma 5 in [3] ($r = b$ here). Further counter-examples are given by

$$A(u) = \sum_{n=0}^{\infty} \frac{u^n}{r^n\,(n+1)^s} = B(u) \qquad (s-1 \in \mathbb{P}).$$

We now give a correct derivation of an asymptotic formula for $\mu(2n+1)$. Darboux's Theorem (cf. Odlyzko [4; Theorem 11.7]) evidently does not apply since $A(u)$ in [3] is not analytic in a neighborhood of $u = 1/4$ for any branch of $\sqrt{1-4u}$. We use a transfer theorem of Flajolet and Odlyzko [2; Theorem 5] (cf. Odlyzko [4; Section 11.1] for definitions, notation and statement of Theorem 11.4).

Consider the closed domain $\Delta = \Delta(1, \pi/8, 1)$ and the function $L(u) = 1$ of slow variation at ∞. Then

$$M_*\left(\frac{u}{4}\right) = \frac{2}{\sqrt{1-u}\,(1+\sqrt{1-u})\,(2-\sqrt{1-u})}$$

is analytic on $\Delta - \{1\}$ where we take the principal branch of the square root. Consequently,

$$M_*\left(\frac{u}{4}\right) \sim \frac{1}{\sqrt{1-u}} = (1-u)^{-1/2}\,L\left(\frac{1}{1-u}\right)$$

uniformly as $u \to 1$ on $\Delta - \{1\}$. Then Theorem 11.4 (C) of [4] implies

$$\frac{\mu(2n+1)\,y_{2n+1}}{4^n} = [u^n]\,M_*\left(\frac{u}{4}\right) \sim \frac{n^{-1/2}}{\Gamma(1/2)}\,L(n) = \frac{n^{-1/2}}{\sqrt{\pi}} \text{ as } n \to \infty.$$

Stirling's Formula implies

$$\binom{2n}{n} = \frac{n^{-1/2}\,4^n}{\sqrt{\pi}}\,(1+o(1)) \text{ as } n \to \infty,$$

hence,

$$\mu(2n+1) \sim n+1 \sim \frac{2n+1}{2} \text{ as } n \to \infty.$$

References

1. E.A. Bender, Asymptotic Methods in Enumeration, *SIAM Rev.* **16** (1974), 485-515.
2. P. Flajolet and A.M. Odlyzko, Singularity Analysis of Generating Functions, *SIAM J. Discrete Math.* **3** (1990), 216-240.
3. C. Lee, The Expectation of Independent Domination Number over Random Binary Trees, *Ars Combinatoria* **56** (2000), 201-209.
4. A.M. Odlyzko, Asymptotic Enumeration Methods, in : *Handbook of Combinatorics* **V.2**, (R.L. Graham, M. Grötshel and L. Lovász Eds.), Elsevier Science B.V., New York, NY, 1995, pps. 1063–1229.