Self-centered super graph of a graph and center number of a graph

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Abstract

In this paper, an algorithm for constructing self-centered graphs from trees and two more algorithms for constructing self-centered graphs from a given connected graph G, by adding edges are discussed. Motivated by this, a new graph theoretic parameter $sc_r(G)$, the minimum number of edges added to form a self-centered graph from G is defined. Bounds for this parameter are obtained and exact value of this parameter for several classes of graphs are also obtained.

1. Introduction

We consider only finite undirected graphs without loops and multiple edges. Let V(G) and E(G) denote the node set and edge set of G respectively. Let G be a connected graph with p nodes and q edges. The definitions and details not furnished here may be found in Buckley and Harary [2].

A *subgraph* of G is a graph having all of its nodes and edges in G. It is a *spanning subgraph* if it contains all the nodes of G. If H is a subgraph of G, then G is a *super graph* of H. A graph is *acyclic* if it has no cycles. A *tree* is a connected acyclic graph. A spanning subgraph of G, which is a tree, is called a *spanning tree* of G.

Let G be a connected graph and v be a node of G. The *eccentricity* e(v) of v is the distance to a node farthest from v. Thus, $e(v) = \max\{d(u, v) : u \in V\}$. The *radius* r(G) is the minimum eccentricity of the nodes, whereas the *diameter* diam(G) is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. v is a central node if e(v) = r(G). The *center C(G)* is the set of all central nodes. The central subgraph < C(G) > of a graph G is the subgraph induced by the center. v is a *peripheral node* if e(v) = diam(G). The *periphery*

$P(G)$ is the set of all such nodes. For a node v, each node at distance e(v) from v is an eccentric node of v.

The set E_k denote the set of nodes of G with eccentricity k. If $|E_k| = c_k$, the cardinality of E_k, then c_r _ 1 and $c_i > 1$ for i = r+1, r+2, ..., d.

Theorem 1.1 The center of a tree consists of either a single node or a pair of adjacent nodes.

A tree with one central node is called a *central* (or uni-central) *tree* and one with two central nodes is called *bi-central*. A graph is *self-centered* if every node is in the center. Thus, in a self-centered graph G all nodes have the same eccentricity, so r(G) = diam(G).

Buckley determined the extremal sizes of a connected self-centered graph having p nodes and radius r.

Theorem 1.2 (Buckley) [4] Let p _ 5 and p _ 2r >2. Then there exists a self-centered connected (p, q) graph with radius r if and only if $(pr-2r-1)/(r-1) \leq q \leq (p^2-4pr+5p+4r^2-6r)/2$. If p = 2r = 4, then q must be 4, where x denotes the smallest integer not less than x.

Schoone et.al [7] have proved the following theorem.

Theorem 1.3 (Schoone) For any connected graph G with diameter d, the removal of n edges will result in graph G with maximum diameter f(n, d) given by $f(n, d) \leq (n+1)d$.

Using Buckley [4] and Schoone [7], T.N. Janakiraman [6] has proved that the +ve roots D of the equation $0 \leq 4D^3-D^2(4p+6)+D(p^2+5p-2q+2)-2d$ and $(pD-2D-1)/(D-1) \leq q+n \leq (\_)(p^2-4pD+5p+4D^2-6D)$ gives the bounds for n for known values of p, q and d for a given graph G, where n is the minimum number of edges added to G to form G′, a self-centered graph with diameter D.

Li Hao and Lai Zaikand proved:

Theorem 1.4 [8] Let G be a self-centered graph with diameter d and |V(G)| _ 3. For any pair of nodes u and v such that d(u, v) = d, every minimum path between u and v is contained in a cycle with length not less than 2d.

A spanning tree T, which has the same radius as G, is called a *radius preserving spanning tree*. For any connected graph G, it is easy to generate a spanning tree T of G for which the distances from a fixed node v are preserved. One simply uses the well-known *'Breadth-first search' (BFS)* algorithm with root v. This algorithm begins at a node v and branches out to its neighbors u, including the edges uv in the tree. Next, edges joining those nodes at distance one from v with nodes at distance two from v are included so as not to form any cycle. This process continues until a spanning tree is formed. If one begins at a central node, the spanning tree T will have the same radius as G. So, it is a *radius preserving spanning tree*.

Definitions of some graphs, which are used in section 3 are given here.

Triangular cactus is a connected graph all of whose blocks are triangles. *A triangular snake* is a triangular cactus whose block cut node graph is a path.

The Helm H_n is the graph obtained from a wheel by attaching a pendant edge at each node of the n cycle. *Closed helm* is the graph obtained from a helm by joining each pendant node to form a cycle. *A flower* is a graph obtained from a helm by joining each pendant node to the central node of the helm.

Olive tree G is a rooted tree consisting of k branches, where the i^{th} branch is a path of length i. *P-star* is a graph obtained by joining p-disjoint paths of length k to a single node. *Firecracker* is a graph obtained from the concatenation of stars by linking one leaf from each.
A banana tree is a graph obtained by connecting a node v to one leaf of each of any number of stars (v is not in any of the stars).

For m _ 3 and k _ 1, where m and k are integers, *L(m, k) is the lollipop graph* of order (and size) n = m+k, obtained from a cycle C_m by attaching a path of length k to a node of the cycle. For m, n _ 3 and k _ 1, *dumb bell graph D(m, n, k)* is a graph of order p = m+n+(k−1) and size q = n+1 obtained from L(m, k) by attaching a cycle of length n to the end node of L(m, k). A *daisy* with m _ 2 petals is a connected graph with one node of degree 2m and all other nodes of degree two.

2. Forming self-centered graphs by adding edges

2.1 Forming self-centered graphs from a given tree

Algorithm I: Let T be a tree on p nodes with radius r and diameter d. Then d = 2r or 2r−1. Since T is a tree, eccentric nodes of nodes of T are its peripheral nodes only. From T form a new graph T^1 = SG(T), super graph of T, with the same node set as follows. Label the nodes of T as $v_1, v_2, \ldots, v_p$.
Step 1: Check whether all the pendant nodes are peripheral. If all are peripheral, go to step 1.1. Otherwise, go to step 2.
Step1.1: Consider all the pendant nodes of T. Join all those pendant nodes, which are at distance d to each other.
Step 2: Some of the pendant nodes are not peripheral. Let x be one such node. Let e(x) = k.

Step 2.1: Consider, $S = \{v_i \in V(G): e(v_i) = k+1$ and $d(x, v_i)$ is minimum$\}$. Let $y = v_i$ such that $v_i \in S$ with i minimum. (This is always possible in T). Join x to y.
Step 2.2: Do this repeatedly for every non-peripheral pendant nodes of T.
Step 2.3: Join all peripheral nodes, which are at distance d to each other.
Name the new graph as $T^1 = SG(T)$.

<u>**Claim:** SG(T) is self-centered with diameter r.</u> That is to prove the eccentricity of every node in SG(T) is r.

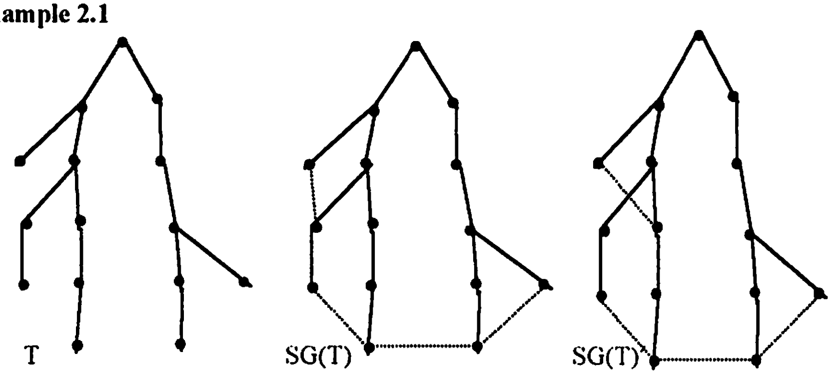
In SG(T), edges are added in such a way that $x, y \in V(T)$ at distance at least r in T lies on cycles of length 2r or 2r+1, which passes through the center of T. Let v be a central node of T and v^1 be its eccentric node in T. Then $d_T(v, v^1) = r$. In SG(T) also, $d(v, v^1) = r$. Therefore, $e(v) = r$ in SG(T).

Now, let $x \in V(T)$ such that $e_T(x) > r$. Let x^1 be an eccentric node of x in T. Therefore, $d(x, x^1) > r$ in T. Clearly, x^1 is a peripheral node of T, (since T is a tree). Consider the shortest path P from x to x^1 in T. $P : x \to v_1 \to v_2 \to \dots v \to v_k = x^1$. This path must contain a central node v_r of T. (If T is uni-central, P contains one central node otherwise two adjacent central nodes). Hence, $d(x^1, v_r) = r$ in T. In SG(T), $d(x^1, v_r) = r$ and by our construction, x, v_r and x^1 lie on an induced cycle of length 2r or 2r+1. Hence, there exists $y \in V(SG(T))$ such that $d(x, y) = r$ in SG(T). Also, in SG(T), distance between any two nodes is at most r, since by our construction any two nodes of SG(T) lie on a cycle of length at most 2r or 2r+1. This proves that eccentricity of every node of SG(T) is r. That is SG(T) is self-centered of diameter r.

Remark 2.1 Let P be the set of all peripheral nodes of T at distance d to each other. In the resulting graph SG(T), < P > is a clique.

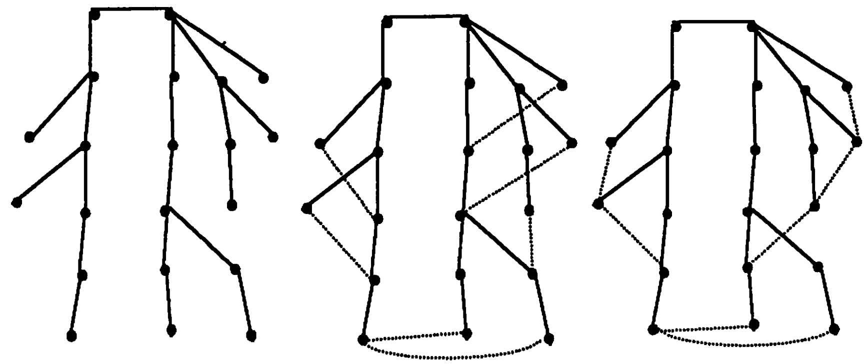
Remark 2.2 SG(T) is not unique. According to the labelling of nodes of T, SG(T) differs. In the examples, 2.1 and 2.2, SG(T) and SG(T)′ are two different super self-centered graphs of T.

Example 2.1



Remark 2.3 For a connected graph G, SG(T_u) ⌈ G may be self-centered with diameter r or bi-eccentric with diameter r, where T_u denotes a radius preserving spanning tree of G, rooted at u ⌊ C(G).

Example 2.2



T SG(T) SG(T)′

Theorem 2.1 Let G be a graph with radius r and diameter d. Let T be a radius preserving spanning tree of G with root node v, where v ⌊ C(G) (T is obtained from Breadth first search). Then H = SG(T) ⌈ G is self-centered with radius r if and only if for x ⌊ $N_k(v)$ in G, there exists a node x^1 in $N_k{}^1(v)$ in G such that $d_G(x, x^1)$ _ r in G, where $k+k^1$ = r or r+1. This is true for every x ⌊ V(G) with e(x) > r.

Proof: Let T be a radius preserving spanning tree of G with root node v, obtained by using BFS. Clearly SG(T) is self-centered with radius r.

<u>Case 1: Suppose T is a bi-central tree.</u> In this case, diam(T) = 2r−1 and in SG(T), the eccentric node of x ⌊ $N_k(v)$ lies in $N_k{}^1(v)$, where $k+k^1$ = r.

<u>Case 2: T is a uni-central tree</u>. In this case, diam (T) = 2r and in SG(T), the eccentric node of x ⌊ $N_k(v)$ lies in $N_k{}^1(v)$, where $k+k^1$ = r or r+1. Therefore, eccentricity of x in H is r if and only if there exists x^1 ⌊ $N_k{}^1(v)$ such that $d_G(x, x^1)$ _ r (suppose $d_G(x, x^1) < r$ in G for all x^1 in $N_k{}^1(v)$, then in H, eccentricity of x is less than r, which is a contradiction). Hence, H is self-centered with diameter r if and only if for x ⌊ $N_k(v)$ there exists a node x^1 in $N_k{}^1(v)$ with $d_G(x, x^1)$ _ r.

Corollary 2.1a If for every node x ⌊ V(G), there exists x^1 ⌊ V(G) such that $d_G(x, x^1)$ = r and there is a shortest path from x to x^1 passing through v, then H is self-centered with diameter r.

Corollary 2.1b If the root node v is such that it lies in some eccentric path of x, for all x ⌊ V(G), then H is self-centered with radius r.

Corollary 2.1c If every node of G, lies in some diametral path joining some peripheral nodes and v is also in that diametral path, then H is self-centered with diameter r.

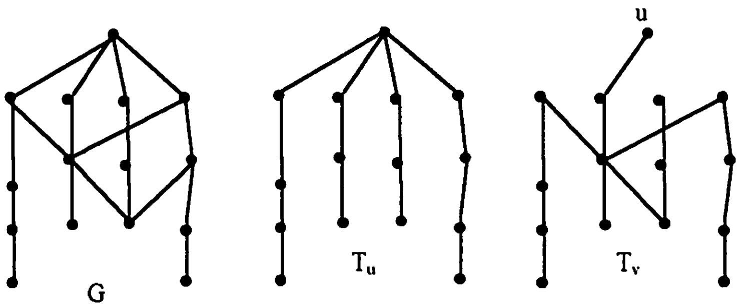
Theorem 2.2 Let G be a uni-central graph with v as center. H = SG(T) $\lceil$ G, where T is a radius preserving spanning tree obtained from BFS, having v as root. Then H is self-centered with diameter r.

Proof: Clearly, in H, e(v) = r. Now, consider any element v_1 in $N_1(v)$ in G.
(i) Suppose $d_G(v_1, x) \leq r-1$ for all x in $N_{r-1}(v)$. Then $d(v_1, y) \leq r$ for all y $\lfloor$ V(G). Hence, $e_G(v_1) = r$, which is a contradiction to G is uni-central.
(ii) Suppose $d_G(v, x) \leq r$ for all x in $N_r(v)$. Then again $d(v_1, y) \leq r$, for all y $\lfloor$ V(G). Hence, $e_G(v_1) = r$, which is again a contradiction. Therefore, for v_1 $\lfloor$ $N_1(v)$, there exists some element x in $N_{r-1}(v)$ or $N_r(v)$ such that $d_G(v_1, x)$ _ r. Now, consider v_2 $\lfloor$ $N_2(v)$. Suppose $d_G(v_2, x) < r$, for all x in $N_{r-2}(v)$ or $N_{r-1}(v)$. Then $e_G(v_2) = r$, which is again a contradiction. Therefore, there exists at least one x in $N_{r-2}(v)$ or $N_{r-1}(v)$ such that $d_G(v_2, x)$ _ r. This implies that $d(v_2, x) = r$ in H. Hence, $e_H(v_2) = r$. Similarly, one can prove that eccentricity of every element of H is r. Therefore, H is self-centered with diameter r.

Theorem 2.3 Let G be a bi-central graph with u, v $\lfloor$ V(G) as central nodes such that u and v are adjacent in G. Then H = SG(T_u) $\lceil$ G and SG(T_v) $\lceil$ G are self-centered with diameter r. (T_u and T_v are obtained from BFS).

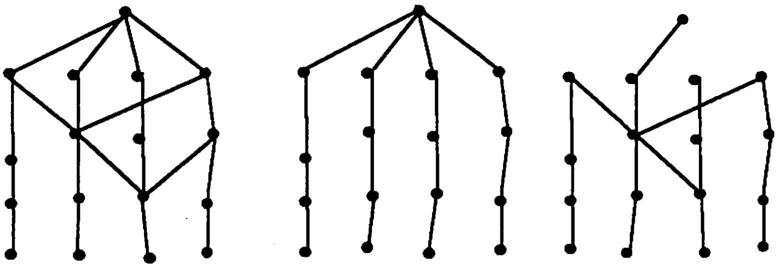
Proof: G is bi-central. Let u, v $\lfloor$ C(G) such that u and v are adjacent in G. As in the previous theorem, one can prove this theorem.

Example 2.3



Here, SG(T_u) $\lceil$ G is not self-centered with diameter four. SG(T_v) $\lceil$ G is not self-centered with diameter 4, it is bi-eccentric with diameter 4.

Example 2.4



G $\qquad$ T_u $\qquad$ T_v

Here u and v are not adjacent. $H = SG(T_v) \lceil G$ is self-centered with diameter 4. $SG(T_u) \lceil G$ is bi-eccentric with diameter 4.

Problem: Whether Converse of the above theorem 2.3 is true ?. That is, if $SG(T_u) \lceil G$ and $SG(T_v) \lceil G$ are self-centered with radius $r = r(G)$, where u and v are the only central nodes of G, then u and v are adjacent in G.

Remark 2.4 When $|C(G)| > 2$ or $|C(G)| = 2$ and u, v are not adjacent, there may not exist H, which is self-centered of diameter d. Therefore, for a given graph G, there may exist no self-centered graph, having a central node as root node using this algorithm I. So, it is necessary to modify the algorithm for a graph G, which is not a tree. (Since algorithm I always gives a self-centered graph with radius r, when G is a tree).

2.2 Forming radius preserving spanning trees of G using BFS:

Let G be a connected (p, q) graph with radius r and diameter d.

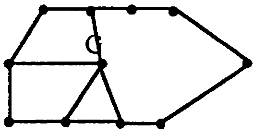
If $d = 2r$, consider, a diametral path passing through a central node u. If $d < 2r$, consider a path joining two peripheral nodes (at distance d) and passing through some central nodes. If the length of the path is 2r, it must have at least one central node of G. Consider that node as root node. If the length of the path is $2r-1$, consider those path containing two central nodes and take any one as root node. Let $u \mathrel{L} V(G)$ be such a root node of G. Consider a spanning tree rooted at u as follows:

Begin at u, find its neighbor and then their neighbors and so on, until the whole graph G is spanned with the condition that if $x \mathrel{L} N_i(u)$, $i = 1, 2, \ldots, r-1$, then x may be pendant in T_u or it must have some adjacent nodes with eccentricity _ $e_G(x)$ in $N_{i+1}(u)$ or some adjacent central nodes in $N_{i+1}(u)$. -----I.

T_u is a spanning tree under BFS with the above condition. If T_u exists, clearly T_u is radius preserving.

Remark 2.5 For a given u, there may not exist such a spanning tree. In this case, take another central node v, adjacent to u (in the path) as root node.

Example 2.5



In G, take u as a root, a spanning tree satisfying condition I does not exist. But, if v (or w) is taken as root, spanning tree satisfying condition I exists and is radius preserving.

2.3 Forming self-centered graphs from a given graph

Algorithm II

Let G be a graph with diameter d and radius r, where $d < 2r$. Let $v_1, v_2, \ldots, v_p \in V(G)$.

Step 1: Form a spanning tree using BFS with the condition I as in 2.2, Forming radius preserving spanning trees of G using BFS, with u as a root, where $u \in C(G)$. Denote this by T_u. T_u is clearly radius preserving.

Step 2: Consider all the pendant nodes of T_u such that they are not peripheral in G. Let v be such a pendant node.

Step 2.1: If $v \in C(G)$, then leave it as such.

Step 2.2: v is not in C(G):

Let $e_G(v) = k > r$. Consider a central node in G, which is nearer to v in G. Let it be v_o. Consider a peripheral node of T_u nearer to v in G. Let it be y. Let z be an eccentric node of y in G. Consider all paths passing through v_o from y to z in G. Among them consider a shortest path P.

Consider $S = \{v_i \in V(P) : e_G(v_i) = e_G(v)$ or $e_G(v)+1$, v_i is not a predecessor of v in $T_u\}$. Joining v to v_i, there exists another path P (passing through v, v_o, y and z) joining y and z in T_u. P_1 may be same as P. Let $W = \{v_i \in S :$ the new path P_1 is of length equal to that of P and $d(v, v_i)$ is minimum$\}$. Let $x = v_i \in W$ with i minimum. Join v to x.

Step 3: Consider all the pendant nodes of T_u such that they are in P(G). Let v be one such node.

Step 3.1: v is not peripheral in T_u.

$e_G(v) = d$ and $e_T(v) < diam(T_u)$.

Let $S = \{v_i \in V(G) : e_T(v_i) = e_T(v)+1 > e_T(v)$ and $e_G(v) = e_G(v_i) = d$ with $d_G(v, v_i)$ minimum$\}$. Join v to that x, where $x = v_i \in S$ with i minimum.

Step 3.2: v is peripheral in T_u that is $e_T(v) = diam(T_u)$

Join this v to all those x, which are pendant in T_u such that $e_G(x) = d$ and $d_G(v, x) = d$ or $d_T(v, x) = diam(T_u)$

Name the resultant graph obtained as T_u'. Define $SG_u(G) = T_u' \Gamma G$.

Claim: $SG_u(G)$ is self-centered with radius r.

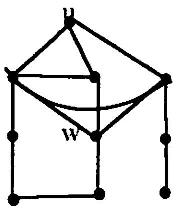
By our construction, for any node there exists another node such that they lie on a cycle of length 2r or 2r+1. (Since length of the path passing through the central node v_o, y, z and v must be of length 2r or 2r−1). Also, any two nodes of $SG_u(G)$ lie on a cycle of length at most 2r+1. Hence, eccentricity of each node in $SG_u(G)$ is r.

This proves that $SG_u(G)$ is self-centered with radius r.

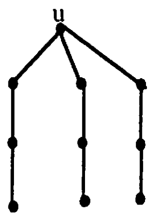
Remark 2.6 Some edges may be repeated in the construction.

Remark 2.7 If the condition on BFS is not given, then for some nodes, eccentricity in the constructed graph may be greater than r.

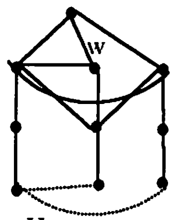
Example 2.6



G

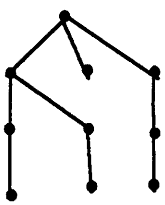


T_u

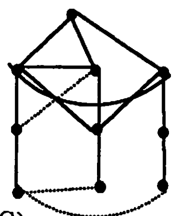


H

Here, eccentricity of w in H is 4. Hence H is not self-centered. H is bi-eccentric with diameter 4. But if T_u is taken as below, satisfying condition I on B.F.S., then $SG_u(G)$ is self-centered with diameter 3.



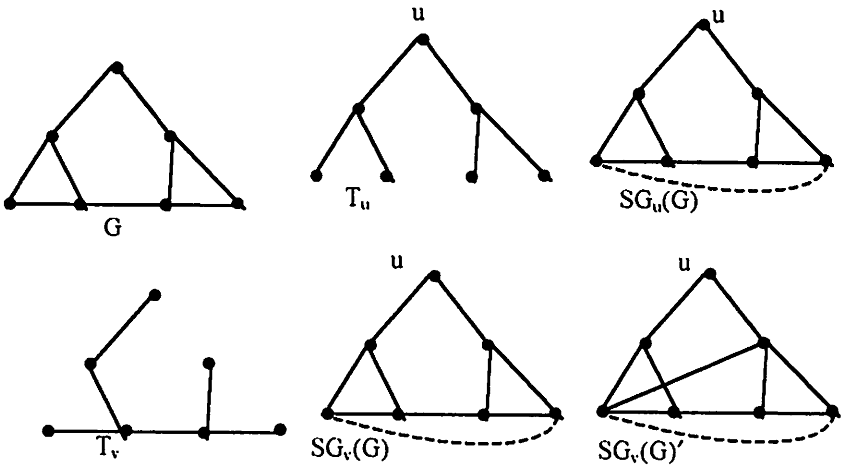
T_u



$SG_u(G)$

Remark 2.8 For central nodes u and v, $SG_u(G)$ and $SG_v(G)$ need not be isomorphic.

Example 2.7



Here, $SG_v(G)$ and $SG_v(G)'$ are two different self-centered graphs of G (they differ according to the labellings).

2.4 Forming self-centered graphs from a given graph

Algorithm III

Let G be a graph with radius r and diameter d = 2r. Label the nodes of G by v_1, v_2, ..., v_p.

Step 1: Let u be a central node of G. Form T_u as in 2.2.

Step2: Consider all the pendant nodes of T_u such that they are not peripheral in G. Let v be such a pendant node.

Step 2.1: If v is a central node of G, that is $v \in C(G)$, then leave it as such.

Step 2.2: v is not in C(G).

Let $e_G(v) = k > r$. Consider a central node in G, which is nearer to v in G. Let it be v_o. Consider a peripheral node nearer to v in G. Let it be y. Let z be an eccentric node of y in G. Consider all paths passing through v_o from y to z in G. Among them, consider the shortest path P. Consider $S = \{v_i \in V(P) : e_G(v_i) = e_G(v)+1, v_i \text{ is not a predecessor of } v \text{ in } T_u\}$.

Joining v to v_i, there exists another path P_1 (passing through v, v_o, y and z) joining y and z.

Let $W = \{v_i \in S: \text{the new path } P_1 \text{ is of length } 2r \text{ in } G\}$.

Let $x = v_i \in W$ with i minimum. Join v to x.

Step 3: Consider all pendant nodes of T_u such that they are in P(G). Let v be one such node.

Step 3.1: $e_G(v) = 2r$ and $e_T(v) = 2r$. Join this v to all those x which are pendant in T_u such that $e_G(x) = d$ or $d-1$ and $d_G(v, x) = d$ or $d-1$.

Name the resultant graph obtained as T_u'. Define $SG_u(G) = T_u' \cup G$.

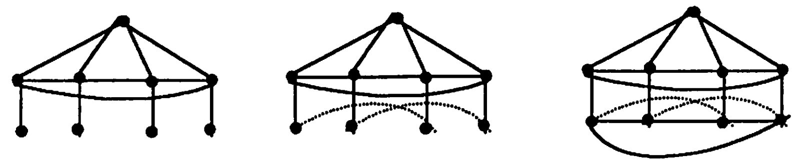
Claim: <u>$SG_u(G)$ is self-centered with radius r.</u>

By our construction, for any node there exists another node such that they lie on a cycle of length 2r or 2r+1. (Since length of the path passing through the central node v_o, y, z and v must be of length 2r). Also, any two nodes of $SG_u(G)$ lie on a cycle of length at most 2r+1. Hence, eccentricity of each node in $SG_u(G)$ is r.

This proves that $SG_u(G)$ is self-centered with radius r.

Remark 2.9 (1) T_u is a spanning tree obtained from G by using BFS, satisfying condition I in 2.2. (2) If v is not joined to nodes x of eccentricity $d-1$ such that $d_G(v, x) = d-1$, x is pendant in T_u, then the new graph constructed need not be self-centered with diameter r.

Example 2.8

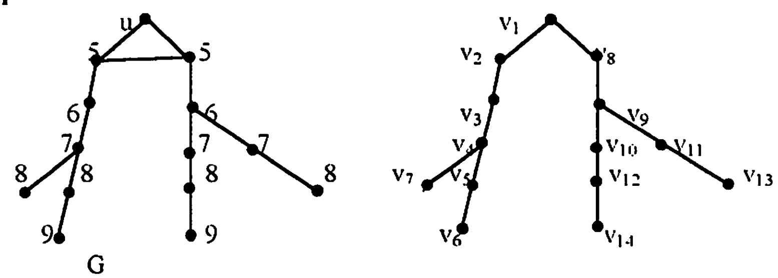


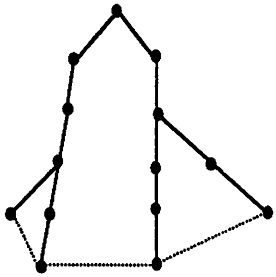
G H $SG_u(G)$

$SG_u(G)$ is self-centered with diameter two. H is not self-centered.

Remark 2.10 If the length of P_1 is greater than P, then the newly constructed graph need not be self-centered with diameter r.

Example 2.9

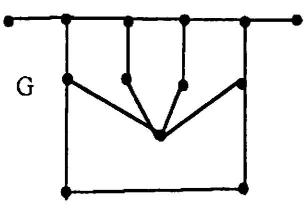




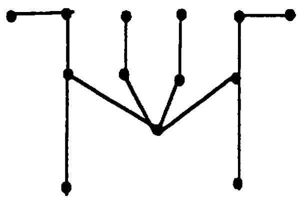
v_7 is pendant, $d_G(v_5, v_7) = 2$ and $d_G(v_6, v_7) = 3$, $e_G(v_5) = 8$, $e_G(v_7) = 8$, $e_G(v_6) = 9$. But if v_7 is joined to v_5 and v_{13} is joined to v_{12}, then the new graph is not self-centered with diameter 5. But $SG_u(G)$ is self-centered with diameter 5.

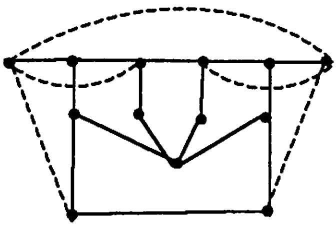
Example 2.10

G



T





Here, G is uni-central and there is no diametral path passing through the center. SG(G) is self-centered of diameter three.

3. Center number of a graph

From section two, it is obvious that by adding some more edges to a given graph G with radius r and diameter d, one can get a self-centered super graph with the same radius r. This motivates to study the minimum number of edges to be added to G to get a self-centered graph of diameter r.

Define $sc_r(G)$ (or sc(G)) be the minimum number of edges added to G to form a self-centered graph. If G is not self-centered, $sc_r(G)$ _ 1. In [6], bounds were obtained to form a self-centered graph with diameter D from a given (p, q) graph with radius r and diameter d. If G is a (p, q) graph with diameter d and n is the minimum number of edges added to get a self-centered graph with diameter D, then n _ ((p–q–2)d–1)/(d+1).

Bound for $sc_r(G)$:

Let $G'(p, q')$ be a graph obtained from G by adding minimum number of edges such that G' is self-centered with diameter r. By Buckley's theorem [Theorem1.2], $(pr-2r-1)/(r-1) \leq q' \leq (p^2-4pr+5p+4r^2-6r)/2$, $q' = q+sc_r(G)$. Therefore, $(pr-2r-1)/(r-1) -q \leq sc_r(G) \leq (p^2-4pr+5p+4r^2-6r)/2-q$. Also, $sc_r(G)$ _ 1, when G is not self-centered. Thus, $1 \leq sc_r(G) \leq (p^2-4pr+5p+4r^2-6r)/2-q$.

Now, let us proceed to find the exact values of $sc_r(G)$ for several graphs G.

Theorem 3.1 $sc_r(P_n) = 1$ for all n.
Proof: Let P_n be the path on n nodes, that is length of $P_n = n-1$. By joining the end nodes of P_n, it can be seen that the new graph is a cycle, which is self-centered of the same radius. Therefore, $sc_r(P_n) = 1$. This proves the proposition.

Theorem 3.2 If G is a graph with radius one, then $sc_r(G) = (p(p-1)/2)-q$.
Proof: Since G is a graph with radius one, G' is nothing but K_p. Therefore, $sc_r(G) = (p(p-1)/2)-q$.

Corollary 3.2 (1) $sc_r(K_{1,n}) = n(n-1)/2$. (2) $sc_r(W_n) = n(n-3)/2$.

Theorem 3.3 (1) If $G = \overline{K}_m+K_1+K_1+\overline{K}_n$ with m+n pendant nodes, then $sc_r(G) = m+n-1$.

(2) If $G = \overline{K}_m+K_1+K_1+K_1+\overline{K}_n$, then $sc_r(G) = m+n-1$.

Proof of (1): Let u, v ⌊ V(G) such that deg u = m+1 and deg v = n+1. By joining each pendant node adjacent to u with each pendant node adjacent to v, a two self-centered graph can be obtained. Thus, $sc_r(G) \leq mn$.

Join any two peripheral nodes. Therefore, remaining pendant nodes are m+n−2. Now, join a pendant node adjacent to u to v and vice versa. So, m+n−2 edges are needed. The resultant graph is self-centered. Thus, $sc_r(G) \leq \min\{mn, m+n-1\} = m+n-1$. Actually, $sc_r(G) = m+n-1$.
Proof of (2): Similar to proof of (1).

Theorem 3.4 If $G = \overline{K}_m+\underbrace{K_1+\ldots+K_1+}_{k\ \text{times}}\overline{K}_n$, where k > 3, then $sc_r(G)=\max\{m, n\}$.

Proof: Let $G = \overline{K}_m+K_1+\ldots+K_1+\overline{K}_n$. Then diam(G) = (k−1)+2 = k+1, and radius of G = {(k+1)/2 if k is odd, (k/2)+1 if k is even}. G has m+n pendant nodes such that each of these pendant nodes is peripheral. Let m _ n. Let u, v ⌊ V(G) such that deg u = m+1 and deg v = n+1. Join each pendant node adjacent to u to a pendant node (only-one) adjacent to v in such a way that there are no other pendant nodes. Thus m = max {m, n} edges are added. In the resultant graph, every node lies on a cycle of length k+2 or 6. Also, for any node u, there exists v such that u and v lie on a cycle of length k+2 only. Therefore, the resultant graph is self-centered with radius (k+2)/2 or (k+1)/2 and if at least one edge is removed from the added edges, then there is a pendant node and hence it is not self-centered. Thus, $sc_r(G) = \max\{m, n\}$.

Theorem 3.5 $sc_r(P_n^+) = \begin{cases} n-1 \text{ for } n = 3, 4, 5. \\ 4 \text{ for } n = 6. \\ (n+1)/2 \text{ if } n \text{ is odd and } n \_ 7. \\ n/2 \text{ if } n \_ 8 \text{ and is even.} \end{cases}$

Proof: It can be directly verified that $sc_r(P_n^+) = n-1$ for n = 3, 4, 5 and $sc_r(P_n^+) = 4$ for n = 6.

Consider n _ 7.

__Case 1: n is even.__

Let $v_1, v_2, ..., v_k, v_{k+1}, ..., v_{2k}$ be the nodes of P_n, where n = 2k. In P_n^+, let v_1' be adjacent to v_1, ..., v_{2k}' be adjacent to v_{2k}. Now, join v_k' to v_{2k-1}, v_{k-1}' to v_{2k-2}', ..., v_2' to v_{k+1}' and v_1' to v_{2k}'. Radius of P_n^+ = n/2+1 = (2k+2)/2 = k+1. In the resultant graph radius is k+1 and diameter is k+1. That is, it is self-centered with diameter k+1. Also, the number of edges added = n/2, and is minimum, since removal of any one edge gives a pendant node. Hence, $sc_r(P_n^+) = n/2$ if n _ 7 and is even.

__Case 2: n is odd.__

Let $v_1, v_2, ..., v_{2k+1}$ be the nodes of P_n, n = 2k+1. In P_n^+, let v_1' be adjacent to v_1, ..., v_{2k+1}' be adjacent to v_{2k+1}. Now join v_{k+1}' to v_{k-1}; v_k' to v_{2k}'; v_{k-1}' to v_{2k-1}', v_{k-2}' to v_{2k-2}'; ..., v_2' to v_{k+2}' and v_1' to v_{2k+1}'. Radius of P_n^+ is (n+1)/2 = k+1, and the resultant graph is self-centered with radius k+1. The number of edges added is minimum since there are n = 2k+1 pendant nodes. Therefore, $sc_r(P_n^+) = (n+1)/2$ if n _ 7 and is odd. This proves the theorem.

Theorem 3.6 $sc_r(C_n^+) = n-1$ for all n.

Proof: $G = C_n^+$. Radius of G = $\begin{cases} (n/2)+1 \text{ if } n \text{ is even.} \\ (n+1)/2 \text{ if } n \text{ is odd.} \end{cases}$

Diameter of G = $\begin{cases} (n/2)+2 \text{ if } n \text{ is even.} \\ (n+1)/2+1 \text{ if } n \text{ is odd.} \end{cases}$

Therefore, G is bi-eccentric. Consider a spanning tree T (radius preserving) of C_n^+. (Removing exactly one edge from C_n). It contains exactly n pendant nodes. Actually it is P_n^+. (Here, the procedure in the previous theorem cannot apply. If the resultant graph obtained from P_n^+ is G′, then G′ ⌈ E(G) is not self-centered. Actually the radius of the new graph decreases).

Using algorithm I a self-centered graph can be formed from T. T contains exactly n pendant nodes. Among them exactly two of them are peripheral nodes. Thus, (n−2)+1 edges could be added to get a self-centered graph from T. Let it be T′. T′ ⌈ G = G′ is a super self-centered graph of G. Therefore, $sc_r(C_n^+) \leq n-1$ for all n.

If one leaves any edge or joins any two pendant nodes of G in some other manner, then either eccentricity of some node decreases below the radius of G or increases from the radius of G. Hence, $sc_r(C_n^+) = n-1$ for all n.

Theorem 3.7 If T is a tree with p nodes and radius r, then $sc_r(T)$ _ $(p-r-2)/(r-1)$.

Proof: T is a tree. Hence, it has $p-1$ edges. By Buckley's theorem, $(pr-2r-1)/(r-1) \leq q' \leq (p^2-4pr+5p+4r^2-6r)/2$, where q′ is the minimum number of edges in a self-centered graph obtained from T having radius r. $q' = p-1+sc_r(T)$. Thus,

$$(pr-2r-1)/(r-1) \leq sc_r(G)+(p-1) \leq ((p^2-4pr+5p+4r^2-6r)/2).$$

That is, $(pr-2r-1)/(r-1) -(p-1) \leq sc_r(G) \leq ((p^2-4pr+5p+4r^2-6r)/2)-(p-1)$.

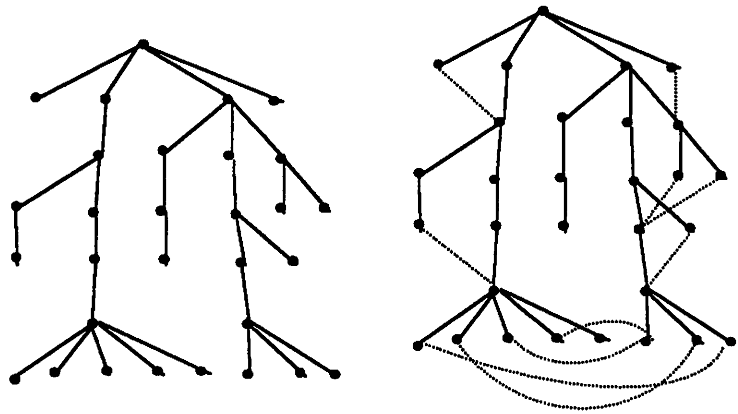
That is, $(p-r-2)/(r-1) \leq sc_r(T) \leq ((p^2-4pr+5p+4r^2-6r-2p+2)/2)$.

Therefore, $sc_r(T)$ _ $(p-r-2)/(r-1)$.

Theorem 3.8 If T is a tree with r(T) _ 3, $|E_{d-1}| = 2$ and $|P(T)| = m+n$, then $sc_r(G)$ = max {m, n} + number of pendant nodes, which are not peripheral.

Proof: Let t = number of pendant nodes, which are not peripheral. Let x be a pendant node such that $e(x) = k < r$. Join x to y such that $e(y) = k+1$ with d(x, y) minimum. Repeat this for these t nodes. Hence, t edges are needed for this. Now, consider the peripheral nodes. Let $E_{d-1} = \{u, v\}$. Let m peripheral nodes $x_1, x_2, ..., x_m$ are adjacent to u and others, namely $y_1, y_2, ..., y_n$ are adjacent to v in T. Let m _ n. Join x_1 to y_1, x_2 to y_2, ..., x_n to y_n. Join the remaining x_i's to y_1. Thus, max {m, n} edges are needed for this. Since r(T) _ 3, the resultant graph is self-centered with diameter r. Hence, $sc_r(T) \leq$ max {m, n}+t. But, by the construction, max {m, n}+t is the minimum number of edges added to form a super self-centered graph of T. This proves the theorem.

Example 3.1



Theorem 3.9 For a tree T, if s is the number of pendant nodes and $|P(T)|=2$, then $sc_r(T) = s-1$.

Proof: Only one edge is needed to join the peripheral nodes and s−2 edges are needed to join the other pendant nodes and these are the minimum number of edges needed to form the required self-centered graph. Hence, $sc_r(T) = s-1$.

Remark 3.1 For some trees, $sc_r(T) \leq s-1$ and for others $sc_r(T) > s-1$.

Theorem 3.10 For a tree T, if s is the number of pendant nodes and t is the number of peripheral nodes, then $sc_r(T) \leq (s-t)+C(t, 2)$.

Proof: At most C(t, 2) edges are needed to join the peripheral nodes at distance d to each to other. Hence, $sc_r(T) \leq (s-t)+C(t, 2)$.

Theorem 3.11 If G is a triangular snake t_n, then $sc_r(G) = n$ if n is even, n+1 if n is odd.

Proof: Triangular cactus is a connected graph all of whose blocks are triangles. A triangular snake is a triangular cactus whose block cut node graph is a path. A triangular snake is obtained from a path $v_1, v_2, ..., v_n$ by joining v_i and v_{i+1} to a new node w_i for i = 1, 2, ..., n−1. Clearly, every triangular snake has exactly four peripheral nodes. For a peripheral node u, there are exactly two eccentric nodes, and is easy to see that when n is even, $sc_r(G) = 4+(n-4) = n$ and when n is odd, $sc_r(G) = 4+(n-3) = n+1$.

Theorem 3.12 If G is a p-star, $sc_r(G) = C(p, 2)$.

Proof: P-star is a graph obtained by joining p-disjoint paths of length k to a single node. Thus, p-star is a tree of radius k with exactly one central node and p pendant nodes, which are also peripheral nodes at distance 2k. Thus, clearly

C(p, 2) edges are needed to find a self-centered super graph. Thus, $sc_r(G) = C(p, 2)$.

Theorem 3.13 For a grid $P_m \leftrightarrow P_n$, $sc_r(G) = 2$.

Proof: Let $G = P_m \leftrightarrow P_n$. Exactly two edges are needed to form a super self-centered graph (joining the peripheral nodes at distance d = m+n to each other). Hence, $sc_r(G) = 2$.

Theorem 3.14 (1.) If G is the helm H_n, then $sc_r(G) = n(n-3)/2$ for n > 4.
(2.) If G is the closed helm, then $sc_r(G) = n(n-3)/2$.
(3.) If G is a flower, then $sc_r(G) = n(2n-3)$.

Proof: (1) The Helm H_n is the graph obtained from a wheel by attaching a pendant edge at each node of the n cycle. H_n is of radius two and diameter four and has only one center and n peripheral nodes. Only the peripheral nodes are the pendant nodes. When n = 3, it can be easily seen that $sc_r(G) = 2$ and when n = 4, $sc_r(G) = 3$. Now, assume that n > 4. Let $v_1, v_2, ..., v_n$ form the circle in H_n, and $v_1', v_2', ..., v_n'$ be the corresponding pendant nodes. Join v_i' to v_k', where k _ i−1 or i+1 for all i = 2, 3, ..., n−1. Join v_1' to v_k', where k _ 2, n. Join v_n' to v_k', where k _ 1 and n−1. The new graph obtained is self-centered with diameter two. Hence, $sc_r(G) \leq n(n-3)/2$. If one remove, any edge from the new edges added or if add edges in some other manner, a self-centered graph of diameter two with minimum edges cannot be obtained. Hence,

$sc_r(G) \leq n(n-3)/2$ if n _ 4.

(2) Closed helm is the graph obtained from a helm by joining each pendant node to form a cycle. Thus, $sc_r(G) = n(n-3)/2$ as in (1).

(3) A flower is a graph obtained from a helm by joining each pendant node to the central node of the helm. Radius of a flower is one. Hence the theorem follows from Theorem 3.2.

Theorem 3.15 If G is an olive tree with k branches, then $sc_r(G) = k-1$.

Proof: G is a rooted tree consisting of k branches, where the i^{th} branch is a path of length i. Hence, G is a bi-central tree with radius k and diameter 2k−1. Thus, by algorithm 1, super self-centered graph can be formed and since G has exactly two peripheral nodes, the number of edges added is minimum. Hence, $sc_r(G) = k-1$.

Theorem 3.16 (1) If G is a fire cracker having k branches and $k_1+k_2+...+k_k$ pendant edges, then $sc_r(G) \leq (k-1)(k_1+k_2+...+k_n)$. (2) If G is a banana tree having $k_1+k_2+k_3+...+k_k$ pendant edges then $sc_r(G) \leq \sum\max\{k_i, k_j\}$, where i, j = 1, 2, ..., k.

Proof of (1): Firecracker is a graph obtained from the concatenation of stars by linking one leaf from each. It is a uni-central tree with radius two and diameter four. Suppose there are k branches. Assume that the branches have, $k_1, k_2, ..., k_k$

pendant edges respectively. Let $u_1, u_2, ..., u_k$ be the nodes with degrees $k_1+1, k_2+1, ..., k_k+1$ respectively. Then join the k_1 pendant nodes (adjacent to u_1) to $u_2, u_3, ..., u_k$; k_2 pendant nodes (adjacent to u_2) to $u_1, u_3, ..., u_k$ and so on. Thus a new graph, which is self-centered of diameter two, is obtained. Thus, $sc_r(G) \leq (k-1)(k_1+k_2+k_3+...+k_k)$.

Proof of (2): A banana tree is a graph obtained by connecting a node v to one leaf of each of any number of stars (v is not in any of the stars).

G is a uni-central tree with radius three and diameter six. Let $u_1, u_2, ..., u_k$ be the nodes with degrees $k_1+1, k_2+1, ..., k_k+1$. Assume k_1 _ k_2. Join each pendant node adjacent to u_1 to any one (only one)-pendant node adjacent to u_2, in such a way that there is no other pendant node adjacent to u_1 or u_2. Thus, max (k_1, k_2) edges are added. Repeat this procedure to pendant nodes adjacent to u_1 and u_3; u_1 and u_4; ...; u_1 and u_k. Similarly, repeat the procedure for the pendant nodes adjacent to u_2 and u_3; u_2 and u_4; ...; u_i and u_k. The resultant graph obtained is super self-centered graph of G with diameter three.

Hence, $sc_r(G) \leq \max\{k_1, k_2\}+\max\{k_1, k_3\}+...+\max\{k_1, k_k\}+\max\{k_2, k_3\}+\max\{k_2, k_4\}+...+\max\{k_{k-1}, k_k\} = \sum\max\{k_1, k_i\}+\sum\max\{k_2, k_i\}+...+\sum\max\{k_{k-1}, k_i\} = \sum\max\{k_i, k_j\}$.

Theorem 3.17 If G = L(m, k), the lollipop graph, then $sc_r(G)$ = 1 or 2.

Proof: For m _ 3 and k _ 1, where m and k are integers, L(m, k) is the lollipop graph of order (and size) n = m+k, obtained from a cycle C_m by attaching a path of length k to a node of the cycle.

<u>Case 1: m is even.</u>

When m is even, G has exactly two peripheral nodes and all other nodes lie on some diametral path passing through the center. Hence, by joining the peripheral nodes, a super self-centered graph can be obtained. Thus, $sc_r(G) = 1$.

<u>Case 2: m is odd.</u>

<u>Sub case 2.1: k is odd.</u> When k is odd and m is odd, G has exactly two adjacent central nodes and three peripheral nodes (two of them are adjacent). Also, diam(G) = 2r(G)−1 and every node lie on some diametral path passing through the center. Hence, by joining two peripheral nodes at distance diam(G) to each other, a super self-centered graph of G can be obtained. Hence, $sc_r(G) = 1$.

<u>Sub Case 2.2: k is even.</u> When k is even and m is odd, G has only one central node and three peripheral nodes (two of them are adjacent). Also, diam(G) = 2r(G) and each node lie on some diametral path passing through the center. Hence, joining the peripheral nodes at distance d to each other, a super self-centered graph of G with diameter = r(G) can be obtained. Hence, $sc_r(G) = 2$.

Theorem 3.18 If G = D(m, n, k), dumb bell graph, then $sc_r(G)$ = 1, 2 or 4.

Proof: For m, n _ 3 and k _ 1, dumb bell graph D(m, n, k) is a graph of order p = m+n+(k−1) and size q = n+1 obtained from L(m, k) by attaching a cycle of length n to the end node of L(m, k); that is D(m, n, k) is obtained from C_m Γ C_n

by joining a node of C_m to a node of C_n and subdividing this edge (k−1) times. In G, p = m+n+(k−1) and q = m+n+k.

If m and n are even, diameter of G = m/2+n/2+k.

If m and n is odd, diameter of G = (m−1)/2+(n−1)/2+k.

If m is even and n is odd, diameter of G = m/2+(n−1)/2+k.

If m is odd and n is even, diameter of G = (m−1)/2+n/2+k.

and radius of G = diam(G), if diam(G) is even; diam(G)+1, if diam(G) is odd. If both m, n are even, G has exactly two peripheral nodes and $sc_r(G)$ = 1 as in the previous theorem. (If k is odd, G has one central node. If k is even, G has two central nodes). If both m and n are odd, G has two central nodes and four peripheral nodes or one central node and four peripheral nodes. In the first case, diam(G) = 2r(G)−1 and in the second case, diam(G) = 2r(G). Therefore, when G has two central nodes, join one peripheral node u with a peripheral node w at distance diam(G), and other peripheral node v adjacent to u with a peripheral node x adjacent to w. The resultant graph is a super self-centered graph of G with diameter = r(G). Hence, $sc_r(G)$ = 2 if m and n are odd and G has two central nodes. If G has only one central node, diam(G) = 2r(G). Hence, joining peripheral nodes at distance diam(G) to each other, a super self-centered graph is obtained and hence $sc_r(G)$ = 4. If m is even and n is odd, G has three peripheral nodes and has one or two centers. In both cases, $sc_r(G)$ = 2. This proves the theorem.

Theorem 3.19 If G is a daisy with m _ 2 petals, then $sc_r(G) \leq 2m-2$.

Proof: A daisy with m _ 2 petals is a connected graph with one node of degree 2m and all other nodes of degree two. That is, a daisy with m _ 2 petals is constructed from m disjoint cycles by identifying a set of m nodes, one from each cycle, into one node. In G, radius = max $\{r(C_k)\}$ and diameter = max $\{r(C_i)+r(C_j)\}$. Also, $sc_r(G) \leq 2m-2$. This proves the theorem.

Forming self-centered graphs with diameter less than r

Let G be a (p, q) graph with radius r and diameter d. Consider a central node v_o L V(G). By adding minimum number of edges, make the eccentricity of v_o be r−1. Let the resultant graph be G^1. Radius of G^1 is r−1. Now, construct $SGv_o(G^1)$. $SGv_o(G^1)$ is self-centered with radius r−1. Define $sc_{r,s}(G)$ be the minimum number of edges added to G to form a self-centered graph with diameter s, where s < r. Bounds can be obtained for $sc_{r,s}(G)$.

Consider, k = min $\{|N_i(v)|\}$. At most k edges are needed to construct G^1. Hence, the maximum number of edges needed to find a self-centered super graph of diameter (r−1) can be found out. Similarly, bounds for $sc_{r,s}(G)$ can be found out for s < r.

Conclusion: The important application of facility location in networks is based on various types of graphical centrality. Self-centered graphs are very much useful in communication networks. Also, addition of new links (which doesn't exist in the system) reduces the cardinality of the dominating sets. So, forming self-centered graphs by adding minimum number of edges will be very much useful in facility location problems, domination problems and in the field of central location theory in communication networks.

Maximum connectivity graphs play an important role in the design of reliable networks. A reason for this is its relation to the reliability and vulnerability of large-scale computer and telecommunication networks. When designing a communication network, one not only wants to maximize the connectivity and edge-connectivity, but also to minimize the diameter as well as the number of edges. By minimizing the diameter, transmission times are kept small and the possibility of distortion due to a weak signal is avoided. Minimizing the number of edges will keep down the cost of building the network. In general one cannot simultaneously maximize _ and _ while minimizing |E(G)| and diam(G). Hence, SG(G) will be very much useful.

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