Graceful Lobsters Obtained from Diameter Four Trees Using Partitioning Technique

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April 12, 2005

Abstract

We view a lobster in this paper as below. A lobster with diameter at least five has a unique path $H = x_0, x_1, \ldots, x_m$ with the property that, besides the adjacencies in H, both x_0 and x_m are adjacent to the centers of at least one $K_{1,s}$ where $s > 0$, and each x_i, $1 \leq i \leq m-1$, is at most adjacent to the centers of some $K_{1,s}$ where $s \geq 0$. This unique path H is called the *central path* of the lobster. We call $K_{1,s}$ an *even branch* if s is nonzero even, an *odd branch* if s is odd, and a *pendant branch* if $s = 0$. In this paper we give graceful labelings to some new classes of lobsters with diameter at least five. In these lobsters the degree of each vertex x_i, $0 \leq i \leq m-1$, is even and the degree of x_m may be odd or even, and we have one of the following features.

1. For some t_1, t_2, t_3, $0 \leq t_1 < t_2 < t_3 \leq m$, each x_i, $0 \leq i \leq t_1$, is attached to two types (odd and pendant), or all three types, of branches; each x_i, $t_1 + 1 \leq i \leq t_2$, is attached to all three types of branches; each x_i, $t_2 + 1 \leq i \leq t_3$, is attached to two types of branches; and if $t_3 < m$ then each x_i, $t_3 + 1 \leq i \leq m$, is attached to one type (odd or even) of branch.
2. For some t_1, t_2, $0 \leq t_1 < t_2 \leq m$, each x_i, $0 \leq i \leq t_1$, is attached to two types (odd and pendant), or all three types, of branches; each x_i, $t_1 + 1 \leq i \leq t_2$, is attached to two, or all three types of branches;

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†The research is supported by financial grant, CSIR, New Delhi, India.

and if $t_2 < m$ then each x_i, $t_2 + 1 \le i \le m$, is attached to one type (odd or even) of branch.

3. For some t, $0 \le t \le m$, each x_i, $0 \le i \le t$, is attached to all three types of branches; and if $t < m$ then each x_i, $t+1 \le i \le m$, is attached to one type (odd or even) of branch.

Keywords: graceful labeling, lobster, odd and even branches, inverse transformation, component moving transformation.

AMS classification: 05C78

1 Introduction

Recall that a *graceful labeling* of a tree T with q edges is a bijection $f : V(T) \to \{0, 1, 2, \ldots, q\}$ such that $\{|f(u) - f(v)| : \{u, v\} \text{ is an edge of } T\} = \{1, 2, \ldots, q\}$. A tree which has a graceful labeling is called a *graceful tree.* A *lobster* is a tree having a path from which every vertex has distance at most two. It is easy to check that the following lemma holds.

Lemma 1.1 If L is a lobster with diameter at least five then there exists a unique path $H = x_0, x_1, x_2, \ldots, x_m$ in L such that, besides the adjacencies in H,

(i) x_0 and x_m are adjacent to the centers of at least one star $K_{1,s}$ where $s \ge 1$,

(ii) each x_i, $1 \le i \le m-1$, is at most adjacent to the centers of some stars $K_{1,s}$ where $s \ge 0$. □

We call the path H in Lemma 1.1 the *central path* of L. Throughout the paper we use H to denote the central path of a lobster with diameter at least five. Take $x_i \in V(H)$. If x_i is adjacent to the center of some $K_{1,s}$ then $K_{1,s}$ will be called an *even branch* if s is non-zero even, an *odd branch* if s is odd, and a pendant branch if $s = 0$. Therefore, according to Lemma 1.1, the branches incident on a vertex in the central path of a lobster can be divided into three types, i.e. even, odd, and pendant defined as above. Furthermore, whenever we say x_i, for some $0 \le i \le m$, is attached to an even number of branches we mean a "non-zero" even number of branches unless otherwise stated.

In 1979, Bermond [1] conjectured that all lobsters are graceful. This conjecture is a special case of the famous and unsolved "graceful tree conjecture" of Ringel and Kotzig (1964) [8], which states that all trees are graceful. Bermond's conjecture is also open and very few classes of lobsters are known to be graceful. Ng [7], Wang et al. [9], Chen et al. [2], Morgan [6] (see [3]), and Mishra and Panigrahi [5] have given graceful labeling to some classes of lobsters. Figures 1, 2, and 3 show the graceful lobsters due to Ng [7], Chen et al. [2], and Wang et al. [9], respectively. Morgan [6] has proved that a lobster is graceful if it has a perfect matching. The lobsters appear in [5] and [9] have one common feature that the degree of each x_i, $0 \leq i \leq m-1$, is even. However, in the lobsters of [5], the degree of x_m may be odd or even and the branches incident on each x_i, $0 \leq i \leq m$, need not be of the same type. In the lobsters of [5], the branches incident on x_0 may be of the same type, any two types, or all three types, whereas those incident on x_i, $1 \leq i \leq m$, may be of the same type (odd or even), or any two types in which the number of branches of each type is odd. The lobsters of this paper share one common feature with those in [5] that the degree of each x_i, $0 \leq i \leq m-1$, is even and the degree of x_m may be odd or even. In the lobsters of [5], at most the vertex x_0 is attached to a combination of all three types of branches, whereas in the lobsters of this paper, not only the vertex x_0 but also some (or all) $x_i, 1 \leq i \leq m$, may exhibit this property. None of the earlier papers covers any graceful lobster where more than one vertices on the central path are attached to all three types of branches. Moreover, in the lobsters of this paper if some x_i, $1 \leq i \leq m$, is attached to two different types of branches, then the number of branches of both types may be odd or even.

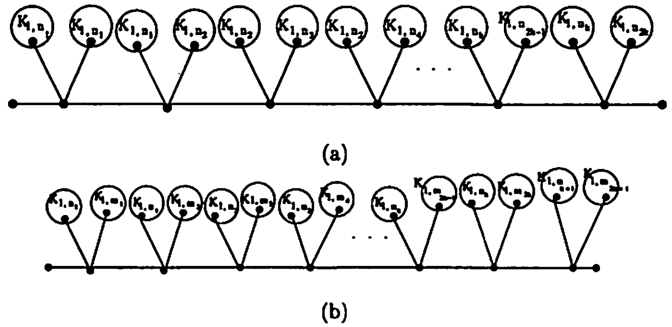

Figure 1: The graceful lobsters given by Ng [7].

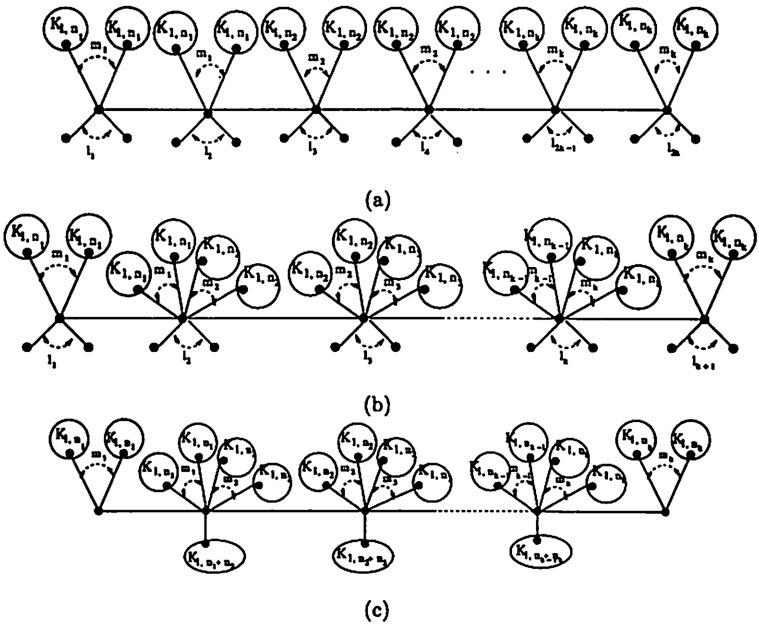

Figure 2: The graceful lobsters given by Chen et al. [2].

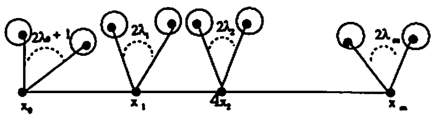

Figure 3: The graceful lobsters given by Wang et al. [9]($m \geq 1$, $\lambda_i > 0$, $i = 0, 1, 2, \ldots, m$, and the circles enclosing the vertices adjacent to the central path are either all odd or all even branches.)

In this paper, as in [5], for a given lobster L we first form a diameter four tree $T(L)$ by identifying all the vertices on the central path of L and give a graceful labeling to $T(L)$ by using the technique of [4]. Let A be the set of all the branches incident on the center of $T(L)$. In [5], the authors applied component moving transformation on A to get a graceful labeling of L, whereas here we partition A in an appropriate manner before applying component moving transformation on it.

2 Preliminaries

In order to prove the results of this paper we need some definitions, terminologies and existing results which are described in this section.

Lemma 2.1 [9], [4] If f is a graceful labeling of a tree T with n edges then the inverse transformation of f, defined as $f_n(v) = n - f(v)$, for all $v \in V(T)$, is also a graceful labeling of T.

Definition 2.2 For an edge $e = \{u, v\}$ of a tree T, we define $u(T)$ as that connected component of $T - e$ which contains the vertex u. Here we say $u(T)$ is a *component incident on* the vertex v. If a and b are vertices of a tree T, $u(T)$ is a component incident on a, and $b \notin u(T)$, then deleting the edge $\{a, u\}$ from T and making b and u adjacent is called *the component moving transformation.* Here we say the component $u(T)$ has been moved from a to b. Throughout the paper we write "the component u" instead of writing "the component $u(T)$"; whenever, we wish to refer to u as a vertex, we write "the vertex u". By the label of the component "$u(T)$" we mean the label of the vertex u. Moreover, we shall not distinguish between a vertex and its label.

Lemma 2.3 [4] Let f be a graceful labeling of a tree T; let a and b be two vertices of T; and let $u(T)$ and $v(T)$ be two components incident on a, where $b \notin u(T) \cup v(T)$. Then the following hold:

(*i*) if $f(u)+f(v) = f(a)+f(b)$ then the tree T^* obtained from T by moving the components $u(T)$ and $v(T)$ from a to b is also graceful.

(*ii*) if $2f(u) = f(a) + f(b)$ then the tree T^{**} obtained from T by moving the component $u(T)$ from a to b is also graceful.

Lemma 2.4 [4] Let T be a diameter four tree with q edges. If a_0 is the center vertex and the degree of a_0 is $2k+1$ then there exists a graceful labeling f of T such that

(a) $f(a_0) = 0$ and the labelings of the neighbours of a_0 are $1, 2, \ldots, k, q, q-1, \ldots, q-k$.

(b) if n_1, n_2, and n_3 are the number of odd, even, and pendant branches incident on a_0, then from the sequence $S = (q, 1, q-1, 2, q-2, 3, \ldots, q-k+1, k, q-k)$ of vertex labels, n_1 terms from the beginning are the labels of the centers of the odd branches, the next n_2 terms are the labels of the centers of the even branches, and the rest n_3 terms are the labels of the centers of the pendant branches.

(c) for any $i = 1, 2, 3$, the n_i labels of S which are the labels of the centers of the same type of branches may be assigned in any order. However, different arrangements of branches of the same type may give different graceful labelings of the same diameter four tree without disturbing (a) and (b).

Remark 2.5 In the graceful labeling f of the diameter four tree T in Lemma 2.4, the labelings of the pendant vertices adjacent to the centers of the odd and even branches can be given by using the technique of [4].

Lemma 2.6 [5] Let $S = (t_1, t_2, \ldots, t_{2p})$ be a finite sequence of natural numbers in which the sums of consecutive terms are alternately $l+1$ and l, beginning (and ending) with the sum $l+1$. Then the sums of consecutive terms in the sequence $S_1 = (\phi_{l+1}(t_{2k+2}), \phi_{l+1}(t_{2k+3}), \ldots, \phi_{l+1}(t_{2p-2k_1-1}))$, where $\phi_n(x) = n - x$, $0 \leq k, k_1 \leq p-2$, and $0 \leq k + k_1 \leq p-2$, are alternately $l+2$ and $l+1$, beginning (and ending) with $l+2$.

3 Results

The main results of this paper are contained in Theorems 3.3, 3.4, and 3.7. However, the result of Theorem 3.2 is more general and involves a technique to generate graceful trees from a given graceful tree of certain kind. We apply this technique to diameter four trees with center having odd degree to obtain graceful lobsters.

Construction 3.1 Let T be a graceful tree with q edges. Let a_0 be a non pendant vertex of T with degree $2k+1$ such that there exists a graceful labeling f of T in which a_0 gets the label 0 and the labels of the neighbours of a_0 are $1, 2, \ldots, k, q, q-1, q-2, \ldots, q-k$ (see Figure 4).

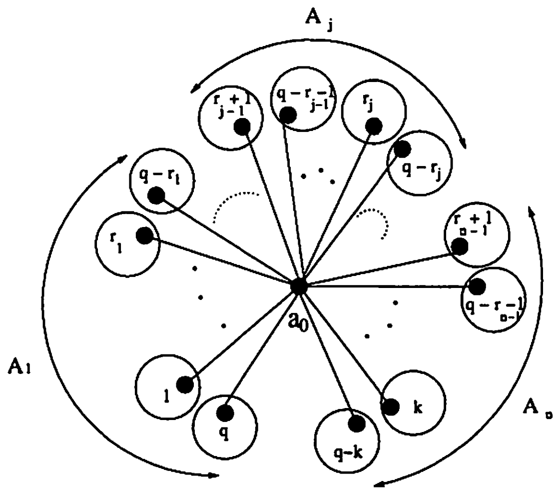

Figure 4: The tree T with vertex a_0 and its neighbours. The circles around the neighbouring vertices of a_0 represent the respective components incident on them.

Consider the sequence $S = (q, 1, q-1, 2, \ldots, k, q-k)$ of vertices adjacent to a_0 (recall that we do not distinguish between a vertex and its label). For any integer n, $n \geq 2$, if possible, we partition this sequence into n parts $A_1, A_2, \ldots, A_n$ (see Figure 4), where

$$A_1 = (q, 1, q-1, 2, \ldots, r_1, q-r_1)$$

and $$A_j = (r_{j-1}+1, q-r_{j-1}-1, r_{j-1}+2, q-r_{j-1}-2, \ldots, r_j, q-r_j),$$

$2 \leq j \leq n$, and $0 < r_1 < r_2 < \ldots < r_n = k$. We construct a tree T_1 (see Figure 5) from T by identifying the vertex y_0 of a path $H' = y_0, y_1, \ldots, y_m$, with a_0 and distributing the components (incident on the vertex a_0) in A_j, $j = 1, 2, \ldots, n$, to y_i, $i = 1, 2, \ldots, s_j$, where $0 \leq s_j \leq m$, in the following manner.

(1) The components in A_1 are distributed to the vertices $y_0, y_1, \ldots, y_{s_1}$, in the following way:

(*i*) At y_0 we retain $2\lambda_0^{(1)} + 1$ components from A_1, where $\lambda_0^{(1)} \geq 0$. In particular, we retain $2p_0$, $0 \leq p_0 \leq \lambda_0^{(1)}$, components whose labels appear consecutively from the beginning of A_1, namely $q, 1, q-1, 2, q-2, 3, \ldots, q-p_0+1, p_0$, and $2\lambda_0^{(1)}+1-2p_0$ components whose labels appear consecutively from the end of A_1, namely $q-k, k, q-k+1, k-1, \ldots, k-\lambda_0^{(1)}+p_0+1, q-k+\lambda_0^{(1)}-p_0$. If $s_1 \geq 1$ then we delete these components from A_1 which are kept at y_0 and name the remaining sequence as $A_1^{(1)}$.

(*ii*) If $s_1 \geq 1$ then we move $2\lambda_i^{(1)}$, $\lambda_i^{(1)} \geq 1$, components from A_1 to y_i, where $1 \leq i \leq s_1$. In particular, we move $2p_i+1$, $0 \leq p_i < \lambda_i^{(1)}$, components whose labels appear consecutively from the beginning of $A_1^{(i)}$, and $2\lambda_i^{(1)} - 2p_i - 1$ components whose labels appear consecutively from the end of $A_1^{(i)}$, where, for $i \geq 2$, $A_1^{(i)}$ is obtained from $A_1^{(i-1)}$ by deleting the components which are moved to y_{i-1}. The numbers $\lambda_i^{(1)}$, $i = 0, 1, 2, \ldots, s_1$, are chosen in such a way that $\sum_{i=0}^{s_1} \lambda_i^{(1)} = r_1$.

(2) The components in A_j, $j = 2, 3, \ldots, n$, are distributed to the vertices $y_0, y_1, \ldots, y_{s_j}$, in the following way:

(*i*) At y_0 we retain $2\lambda_0^{(j)}$ components from A_j, where $\lambda_0^{(j)} > 0$. In particular, we retain $2\lambda_0^{(j)} - 1$ components whose labels appear consecutively from the beginning of A_j, namely $r_{j-1}+1, q-r_{j-1}-1, r_{j-1}+2, \ldots, q-r_{j-1}-\lambda_0^{(j)}+1, r_{j-1}+\lambda_0^{(j)}$, and one component whose label is the last term of A_j, namely $q-r_j$. If $s_j \geq 1$ then we delete these components from A_j which are kept at y_0 and name the remaining sequence as $A_j^{(1)}$.

(*ii*) If $s_j > 0$, we move $2\lambda_i^{(j)}$, $\lambda_i^{(j)} \geq 1$, components from A_j to y_i, where $1 \leq i \leq s_j$. In particular, we move $2\lambda_i^{(j)}-1$ components whose labels appear consecutively from the beginning of $A_j^{(i)}$ and one component whose label is the last term of $A_j^{(i)}$, where, for $i \geq 2$, $A_j^{(i)}$ is obtained from $A_j^{(i-1)}$ by deleting the components which are moved to y_{i-1}. The numbers $\lambda_i^{(j)}$, $i = 0, 1, 2, \ldots, s_j$, are chosen in such a way that $\sum_{i=0}^{s_j} \lambda_i^{(j)} = r_j - r_{j-1}$. □

In the following theorem, for a graceful tree R with n edges and a grace-

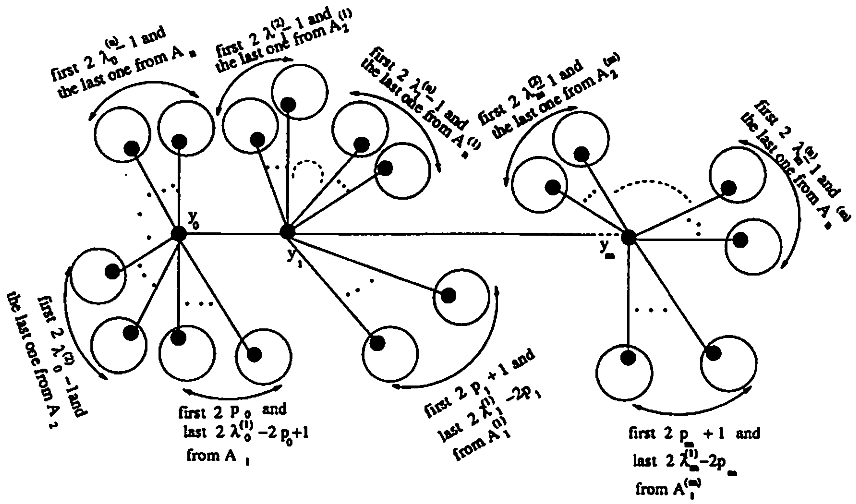

Figure 5: The tree T_1 obtained from T. Here we take $s_1 = s_2 = \ldots = s_n = m$.

ful labeling g we use the notation " $g(R)$" to denote the tree R with the graceful labeling g. Also, for any sequence $F = (a_1, a_2, \ldots, a_r)$, $g_n(F)$ is the sequence $(n - a_1, n - a_2, \ldots, n - a_r)$.

Theorem 3.2 The tree T_1 in Construction 3.1 is graceful.

Proof: We first consider the tree $T \cup \{y_0, y_1\}$, where the vertices a_0 and y_0 are identified and $\{y_0, y_1\}$ is an edge. We give the label $q+1$ to y_1. Clearly $T \cup \{y_0, y_1\}$ is graceful with a graceful labeling $f^{(1)}$, where $f^{(1)}$ is the same as f on T and it gives the label $q+1$ to y_1. Then we move all the components in $A_j^{(1)}$, $j = 1, 2, \ldots, n$, to y_1 and let the resultant tree be $T^{(1)}$. One can notice that each $A_j^{(1)}$, for $j = 1, 2, \ldots, n$, can be partitioned into pairs of labels whose sum is $q+1$ (consecutive terms). Therefore, the tree $T^{(1)}$ is graceful by Lemma 2.3(i).

Next we take the inverse transformation $f_{q+1}^{(1)}$ of the graceful labeling $f^{(1)}$ of $T^{(1)}$. So $f_{q+1}^{(1)}$ is a graceful labeling of $T^{(1)}$ by Lemma 2.1 and the label of y_1 in $f_{q+1}^{(1)}(T^{(1)})$ is 0. Next we make y_2 adjacent to y_1 in $f_{q+1}^{(1)}(T^{(1)})$ and give the label $q+2$ to y_2. Obviously, the tree $T^{(1)} \cup \{y_1, y_2\}$ is graceful with the graceful labeling $f^{(2)}$, where $f^{(2)}$ is the same as $f^{(1)}_{q+1}$ on $T^{(1)}$ and it gives the label $q+2$ to y_2.

For those j with $s_j \geq 2$, $j = 1, 2, \ldots, n$, we move all the components in $f^{(1)}_{q+1}(A^{(2)}_j)$, from y_1 to y_2 and let the resultant tree be $T^{(2)}$. Observe that the sums of consecutive terms in $A^{(1)}_j$ are alternately $q+1$ and q (beginning and ending with $q+1$) so by Lemma 2.6 the sums of consecutive terms in $f^{(1)}_{q+1}(A^{(2)}_j)$, are alternately $q+2$ and $q+1$ beginning and ending with $q+2$. One sees that each $f^{(1)}_{q+1}(A^{(2)}_j)$ can be partitioned into pairs of labels whose sum is $q+2$. By Lemma 2.3(i), $T^{(2)}$ is graceful.

Let $s^* = max\{s_1, s_2, \ldots s_n\}$. On repeating the above procedure for s^* times we get the graceful tree $T^{(s^*)}$ with vertex set $V(T) \cup \{y_1, \ldots, y_{s^*}\}$ in which the vertex y_{s^*} gets the label $q + s^*$. If $s^* = m$, then we stop; otherwise we proceed as follows.

We apply inverse transformation to the graceful tree $T^{(s^*)}$ so that the vertex y_{s^*} gets the label 0. Then make the vertex y_{s^*+1} adjacent to y_{s^*} and give the label $q+s^*+1$ to y_{s^*+1}. If $s^*+1 = m$ then we stop; otherwise we repeat this procedure until the vertex y_m gets a label. The graceful tree that is obtained on the vertex set $V(T) \cup V(H')$ is easily seen to be the tree T_1. □

The lobsters we consider here will be of diameter at least five, so we use the representation in Lemma 1.1. Given a lobster L of the type to which we give a graceful labeling, we construct a diameter four tree, say $T(L)$, from L by successively identifying the vertices x_i, $i = 1, 2, \ldots, m$, with x_0. The vertex x_0 is the center of $T(L)$ and its degree is odd, say $2k+1$. By Lemma 2.4, $T(L)$ has a graceful labeling in which x_0 gets the label 0 and the neighbours of x_0 get labels in S. However, we note that the manner in which we partition the sequence S and the order in which the centers of the branches incident on x_0 in $T(L)$ get labels from the sequence S plays an important role. To get back L and a graceful labeling of it we have to follow an appropriate partition and ordering, which will be clear from the proof of Theorems 3.3 and 3.4. Next we apply Theorem 3.2 to $T(L)$ and to the central path $H = x_0, x_1, \ldots, x_m$, so as to get a graceful labeling of L.

By using the techniques of this paper, if we take $n = 1$, i.e. when we do not partition the sequence S, in Construction 3.1 then we get the results that appear in [5] so we do not consider the case $n = 1$ in Theorems 3.3 and 3.4. We get graceful labelings of lobsters that appear in Theorem 3.3 (respectively, Theorem 3.4) by taking $n = 2$ (respectively, $n = 3$) in Construction 3.1.

Theorem 3.3 The lobsters in Tables 3.1, 3.2, 3.3, and 3.4 below are graceful.

Descriptions of Tables: In the column headings, the triple (x, y, z) represents the number of odd, even, and pendant branches, respectively, where e means any even number of branches (non-zero, unless otherwise stated), o means any odd number of branches, and 0 means no branch. For example, $(e, 0, o)$ means an even number of odd branches, no even branch and an odd number of pendant branches. If in a triple e or o appears more than once then it does not mean that the corresponding branches are equal in number, for example, (e, e, o) does not mean that the number of odd branches is equal to the number of even branches. The symbol o^* represents any odd number of branches greater than or equal to 3.

1st column: The notation $0(r)$, $r = 1, 2$, means that x_0 is attached to the combination of branches mentioned in the column heading in which r is the superscript. Simply, 0 means that x_0 is attached to any one of the mentioned combinations of branches.

Other columns: $i \rightarrow j$ (respectively, $i \rightarrow j$ (r), $r = 1, 2$) means that each $x_l, i \leq l \leq j$, is attached to the mentioned combination or any one of the two combinations of branches (respectively, the branches mentioned in the triple with superscript r).

Further, when some vertex x_i on the central path is attached to two combinations $(x, y, 0)$ and $(0, 0, e)$, we mean that x_i is attached to the combination (x, y, e). For example, in Table 3.1(a) x_{t_1+1} is attached to the combinations $(o, o, 0)$ and $(0, 0, e)$, which means that x_{t_1+1} is attached to the combination (o, o, e). In Table 3.3(b), $**$ means that the number of even branches incident on x_i, $1 \leq i \leq t$, is at least 3.

Table 3.1

Lobsters ↓	$(e,0,o^*)^1$ or $(0,e,o^*)^2$	$(o,0,o^*)$	$(o,o,0)$	$(e,e,0)$	$(0,o,o^*)$	$(e,0,0)^1$ or $(0,e,0)^2$	$(0,0,e)$
a	0 (1)	$1 \to t_1$, $t_1 < m-1$	$t_1+1 \to t_2$, $t_2 < m$	——	——	$t_2+1 \to m$	$t_1+1 \to s$, $s \leq m$
b	0 (1)	$1 \to t_1$, $t_1 < m-1$	——	——	$t_1+1 \to t_2$, $t_2 < m$	$t_2+1 \to m$ (2)	$t_2+1 \to s$, $s \leq m$
c	0 (1)	——	$1 \to t$, $t < m$	——	——	$t+1 \to m$	$1 \to s$, $s \leq m$
d	0	——	——	——	$1 \to t$, $t < m$	$t+1 \to m$ (2)	$t+1 \to s$, $s \leq m$
e	0 (1)	$1 \to t$, $t < m$	——	——	——	$t+1 \to m$	$t+1 \to s$, $s \leq m$
f	0 (1)	$1 \to t$, $t < m$	$t+1 \to t'$, $m-1 \leq t' \leq m$	m, if $t' = m-1$	——	——	$t+1 \to s$, $s \leq m$
g	0 (1)	——	——	——	——	$1 \to m$	$1 \to s$, $s \leq m$
h	0 (1)	——	$1 \to t'$, $m-1 \leq t' \leq m$	m, if $t' = m-1$	——	——	$1 \to s$, $s \leq m$
i	0 (2)	——	——	——	——	$1 \to m$ (2)	$1 \to s$, $s \leq m$

Table 3.2

Lobsters ↓	$(e,e,o^*)^1$ or $(o,o,o^*)^2$	$(o,o,0)$	$(0,o,o^*)^1$ or $(e,e,0)^2$	$(e,0,0)^1$ or $(0,e,0)^2$	$(0,0,e)$
a	0	——	$1 \to t$ (1), $t < m$	$t+1 \to m$(2)	$t+1 \to s$, $s \leq m$
b	0 (1)	$1 \to t$, $t < m$	——	$t+1 \to m$	$1 \to s$, $s \leq m$
c	0 (2)	——	——	$1 \to m$ (2)	$1 \to s$, $s \leq m$

1. We determine r_1, and hence A_1 and A_2. Let the number of pendant branches incident on x_i, $i = 0, 1, \ldots, t_1$, be $2\lambda_i + 1$ and those incident on x_i, $i = t_1 + 1, \ldots, s$, be $2\lambda_i$, where, for $0 \leq i \leq s$, $\lambda_i \geq 1$. Note that x_i, for $i \geq s + 1$, is not attached to any pendant branch. We will label $T(L)$ in such a way that the centers of the pendant branches incident on each x_i, $t_1 + 1 \leq i \leq s$, get labels from A_2, and among the pendant branches incident on each x_i, $0 \leq i \leq t_1$, the centers of $2\beta_i + 1$ branches get labels from A_1 and the centers of the rest of these branches get labels from A_2, where β_i, $0 \leq \beta_i < \lambda_i$, are arbitrary integers. Let $\sum_{i=0}^{t_1}(2\lambda_i - 2\beta_i) + \sum_{i=t_1+1}^{s} 2\lambda_i = 2r$. Therefore, A_2 contains the centers of $2r$ pendant branches. We choose A_2 in such a way that it will not contain the center of any other branch, so $|A_2| = 2r$. Since $|A_1| + |A_2| = 2k + 1$ and $|A_1| = 2r_1 + 1$, we get $r_1 = k - r$.

2. We label the centers of the branches incident on x_0 in $T(L)$ in the following manner:

(i) The centers of the odd branches incident on x_0 in L get labels from the beginning of A_1.

(ii) For $i = 1, 2, \ldots, t_2$, the centers of the odd branches incident on x_i in L get labels from the beginning of $A_1^{(i)}$.

(iii) For $i = t_1 + 1, t_1 + 2, \ldots, t_2$, the centers of the even branches incident on x_i in L get labels from the end of $A_1^{(i)}$.

(iv) For $i = t_2 + 1, \ldots, m$, among the odd (or even) branches incident on x_i in L, the centers of any odd number of branches get labels from the beginning of $A_1^{(i)}$ and the centers of the rest of these branches get labels from the end of $A_1^{(i)}$.

(v) For $i = 0, 1, \ldots, t_1$, the centers of $2\lambda_i + 1$ pendant branches incident on x_i in L get $2\beta_i + 1$ labels from the end of $A_1^{(i)}$, $2(\lambda_i - \beta_i) - 1$ labels from the beginning and the last label of $A_2^{(i)}$, where $A_1^{(0)} = A_1$ and $A_2^{(0)} = A_2$.

(vi) For $i = t_1 + 1, \ldots, s$, the centers of $2\lambda_i$ pendant branches incident on x_i in L get $2\lambda_i - 1$ labels from the beginning and the last label of $A_2^{(i)}$.

One can notice that the labeling of the centers of the branches incident on the center x_0 of $T(L)$ given in step 2 follows part (b) of Lemma 2.4. Therefore, by Lemma 2.4 there exists a graceful labeling of $T(L)$ with the above labels of the center x_0 and the centers of the branches incident on x_0. Finally, we apply Theorem 3.2, for $n = 2$, on $T(L)$ and the path $H = x_0, x_1, \ldots, x_m$, so as to get a graceful labeling of L (see example below). This approach will be the same for all the remaining cases of this theorem and hence we will just indicate the modification we make in steps 1 and 2.

Example: Consider the lobster presented in Figure 6 which is of the type (a) in Table 3.1. We construct the graceful diameter four tree $T(L)$ shown in Figure 7. Here, $|E(T(L))| = q = 85$ and $deg(x_0) = 2k+1 = 39$. So, $k = 19$ and $S = (85, 1, 84, 2, \ldots, 19, 66)$. Here $m = 6$, $t_1 = 2$, $t_2 = 4$, $s = 5, \lambda_0 = 2$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 1$, $\lambda_4 = 1$, and $\lambda_5 = 2$. We take $\beta_0 = 1$, $\beta_1 = 1$, and $\beta_2 = 0$. Therefore, $r = 7, r_1 = k - r = 19 - 7 = 12$, $A_1 = (85, 1, \ldots, 12, 73)$, and $A_2 = (13, 72, 14, \ldots, 19, 66)$. We obtain a graceful labeling of $T(L)$, as given in Figure 7, by assigning the label 0 to x_0, giving labelings to the neighbours of x_0 as per step 2, and finally, giving labelings to the remaining vertices of $T(L)$ by using the technique described in [4]. Then in Figure 8 we make x_1 adjacent to x_0, give the label 86 to x_1, and move all the components in $A_j^{(1)}$, $j = 1, 2$, to x_1. The tree in Figure 9 is obtained by applying inverse transformation to the lobster found in Figure 8, making x_2 adjacent to x_1, giving the label 87 to x_2, and moving all the components in $f_{86}^{(1)}(A_j^{(2)})$, $j = 1, 2$, to x_2. Continuing in this manner we finally get the graceful labeling of L presented in Figure 10.

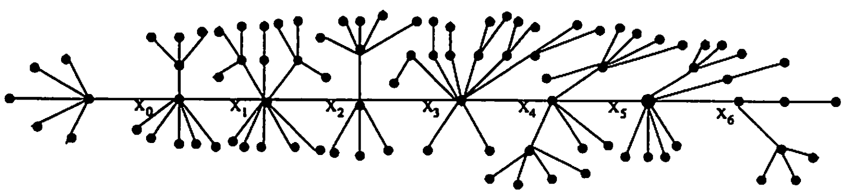

Figure 6: A lobster L of type (a) in Table 3.1. Here $m = 6$, $t_1 = 2$, $t_2 = 4$, and $s = 5$.

Continuation of Table 3.2

Lobsters ↓	$(e,e,o^*)^1$ or $(o,o,o^*)^2$	$(o,o,0)$	$(0,o,o^*)^1$ or $(e,e,0)^2$	$(e,0,0)^1$ or $(0,e,0)^2$	$(0,0,e)$
d	0 (1)	——	——	$1 \to m$	$1 \to s,\ s \le m$
e	0 (1)	$1 \to t'$, $m-1 \le t' \le m$	m (2) if $t' = m-1$	——	$1 \to s,\ s \le m$

Table 3.3

Lobsters ↓	$(e,o,0)$, $e \ge 0$	$(o,o,0)$	$(e,e,0)$	$(e,0,0)^1$ or $(0,e,0)^2$	$(0,0,e)$
a	0	$1 \to t,\ t < m$	——	$t+1 \to m$	$0 \to s,\ s \le m$
b	0 $(o \ge 3)$	$1 \to t,\ t < m$, $(_{**})$	$t+1 \to t',\ t' \le m$	$t'+1 \to m$ if $t' < m$	——
c	0	——	——	$1 \to m$	$0 \to s,\ s \le m$
d	0	$1 \to t',\ m-1 \le t' \le m$	m, if $t' = m-1$	——	$0 \to s,\ s \le m$
e	0 $(o \ge 3)$	——	$1 \to t',\ t' \le m$	$t'+1 \to m$ if $t' < m$	——

Table 3.4

Lobsters ↓	$(o,e,0)$	$(o,0,0)^1$ or $(0,o,0)^2$	$(e,e,0)$	$(e,0,0)^1$ or $(0,e,0)^2$	$(0,0,e)$
a	0	——	$1 \to t',\ t \le m$	$t'+1 \to m$ if $t' < m$	——
b	——	0 (1)	——	$1 \to m$ (1)	$0 \to s,\ s \le m$
c	——	0 (2)	——	$1 \to m$ (2)	$0 \to s,\ s \le m$

Proof: For every lobster L of this theorem we first construct the diameter four tree $T(L)$ corresponding to L. Let $|E(T(L))| = q$ and $deg(x_0) = 2k+1$. We give the label 0 to x_0. We partition the sequence S in Construction 3.1 into two parts, i.e. we take $n = 2$, in Construction 3.1.

Let L be a lobster of type (a) in Table 3.1. We follow the two steps given below.

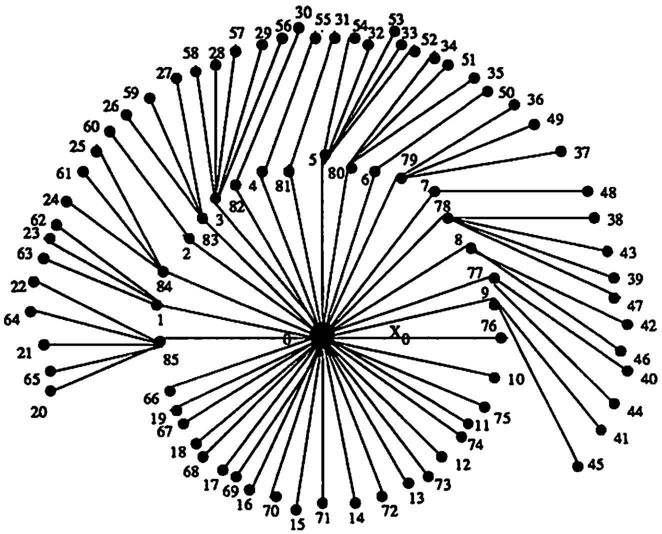

Figure 7: The tree $T(L)$ corresponding to the lobster in Figure 6.

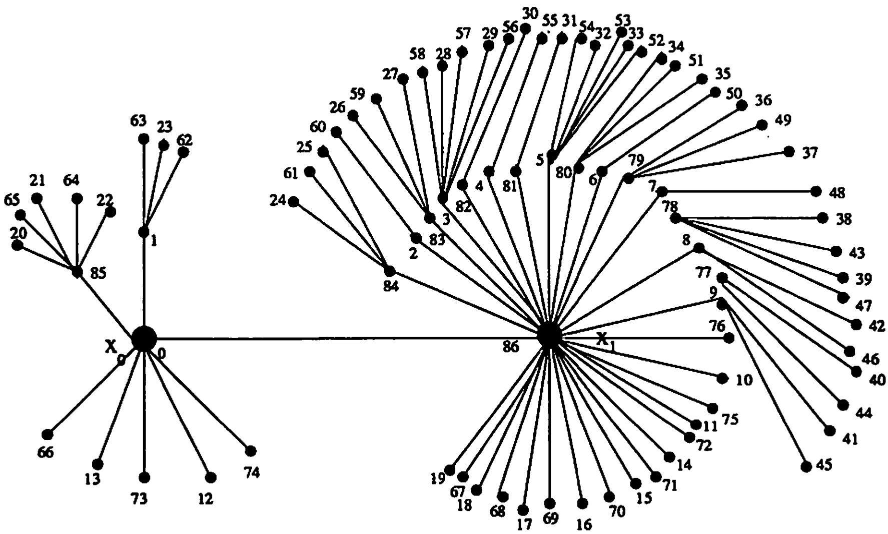

Figure 8: The graceful lobster obtained by making x_1 adjacent to x_0, giving the label 86 to x_1, and moving all of the branches in $A_j^{(1)}$, $j = 1, 2$, to x_1.

by even branches in step 2(ii), and odd (or even) branches by even branches only in step 2(iv). For lobsters of type (e), we set $t_1 = t$ and $t_2 = t$ in steps 1, 2(i), 2(ii), 2(iv), 2(v), and 2(vi). For lobsters of type (f), we set $t_1 = t$ and $t_2 = m$ in steps 1, 2(i), 2(ii), 2(iii), 2(v), and 2(vi). For lobsters of type (g), we set $t_1 = 0$ and $t_2 = 0$ in steps 1, 2(i), 2(iv), 2(v), and 2(vi). For lobsters of type (h), we set $t_1 = 0$ and $t_2 = m$, in steps 1, 2(i), 2(ii), 2(iii), 2(v), and 2(vi). For lobsters of type (i), we set $t_1 = 0$ and $t_2 = 0$ in steps 1, 2(i), 2(iv), 2(v), and 2(vi), replace odd branches by even branches in step 2(i) and odd (or even) branches by even branches in step 2(iv).

For lobsters of types (a), (b), (c), (d), and (e) in Table 3.2, the proof follows if we proceed as the proof involving the lobsters of types (d), (c), (i), (g), and (h), respectively, in Table 3.1 by repeating steps 1 and 2 by replacing step 2(i) with " The centers of the odd branches followed by the even branches incident on x_0 in L get labels from the beginning of A_1".

For lobsters L of type (a) in Table 3.3, the proof follows if we proceed as the proof involving the lobsters of type (a) in Table 3.1 by modifying steps 1 and 2 in the following manner.

1. We determine r_1, and hence A_1 and A_2. Let the number of pendant branches incident on x_i, $i = 0, 1, \ldots, s$, be $2\lambda_i$, where $\lambda_i \geq 1$. The terms of A_2 are the labels given to the centers of the pendant branches only, so $|A_2| = 2\sum_{i=0}^{s} \lambda_i$. Let this value be $2r$, so we get $r_1 = k - r$.

2. (i) For $i = 0, 1, \ldots, t$, the centers of the odd (respectively, even) branches incident on x_i in L get labels from the beginning (respectively, end) of $A_1^{(i)}$, where $A_1^{(0)} = A_1$.
(ii) Set $t_2 = t$ in step 2(iv).
(iii) For $i = 0, \ldots, s$, the centers of $2\lambda_i$ pendant branches incident on x_i in L get $2\lambda_i - 1$ labels from the beginning and the last label of $A_2^{(i)}$, where $A_2^{(0)} = A_2$.

Next consider the lobsters of type (b) in Table 3.3. Let p be an integer defined as $p = m$ if either $t' = m$ or $t' < m$ with each x_i, $i = t'+1, \ldots, m$, is attached to an even number of even branches and $p = t'$ if $t' < m$ with each x_i, $i = t'+1, \ldots, m$, is attached to an even number of odd branches.

The proof follows if we proceed as the proof involving the lobsters of type (a) in Table 3.1 by modifying steps 1 and 2 in the following manner.

1. Set $t_1 = t$ and $s = p$, and replace pendant branches by even branches in step 1.
2.(i) Same as step 2(i).
(ii) Set $t_2 = t$ in step 2(ii).
(iii) Set $t_2 = t$ and $m = m + t' - p$, and replace odd (or even) branches by odd branches in step 2(iv).
(iv) Set $t_1 = t$ and $s = p$, and replace pendant branches by even branches in steps 2(v) and 2(vi).

If we proceed as the proof involving the lobsters of type (a) in Table 3.3 by setting $t = 0$ in steps 1 and 2 we get a proof for the lobsters of type (c) in Table 3.3; by setting $t = m$ in steps 1, 2(i), and 2(iii) we get a proof for lobsters of type (d) in Table 3.3. A proof follows for lobsters of type (e) in Table 3.3 if we proceed as the proof involving the lobsters of type (b) in Table 3.3 by setting $t = 0$ in steps 1, 2(i), 2(iii), and 2(iv).

For lobsters L of type (a) in Table 3.4, the proof follows if we proceed as the proof involving the lobsters of type (a) in Table 3.1 with the following changes in step 1 and step 2 below.

1. Here A_1 will consist of the labels given to the centers of the odd branches only and A_2 will consist of the labels given to the centers of the even branches only so $|A_1| = 2r_1 + 1$ is the number of odd branches of L.

2. (i) Among the odd branches incident on x_0, the centers of any even number of branches get labels from the beginning of A_1 and the centers of the rest of these branches get labels from the end of A_1.

(ii) For $i = 1, 2, \ldots, m + t' - p$, among the odd branches incident on x_i, the centers of any odd number of branches get labels from the beginning of $A_1^{(i)}$ and the centers of the rest of these branches get labels from the end of $A_1^{(i)}$, where p is an integer defined as in the proof involving the lobsters of type (b) in Table 3.3.

(iii) For $i = 0, 1, 2, \ldots, p$, among the even branches incident on x_i, the center of one branch gets label from the end of $A_2^{(i)}$ and the centers of the rest of these branches get labels from the beginning of $A_2^{(i)}$.

For lobsters of types (b) and (c) in Table 3.4, we replace $m + t' - p$ with m, p with s, and even branches with pendant branches in steps 1 and 2 in the proof involving the lobsters of type (a) in Table 3.4. Further, in case of lobsters of type (c) we replace odd branches with even branches in both the steps. □

In the previous theorem we have taken $n = 2$, i.e. we have partitioned the sequence S in Construction 3.1 into two parts. In the theorem below (Theorem 3.4) we take $n = 3$ and improve some of the results in Table 3.3 and Table 3.4.

Theorem 3.4 The lobsters in Table 3.3(b), 3.3(e), or 3.4(a), with the additional condition that for a fixed integer s, $0 \leq s \leq m$, each x_i, $i = 0, 1, \ldots, s$, is attached to an even number of pendant branches, are graceful.

Proof: As in the proof of Theorem 3.3, for every lobster L of this theorem we first construct the diameter four tree $T(L)$ corresponding to L. Let $|E(T(L))| = q$ and $deg(x_0) = 2k + 1$. We give the label 0 to x_0. We partition the sequence S in Construction 3.1 into three parts, i.e. we take $n = 3$, in Construction 3.1. For any lobster L of this theorem we follow the two steps given below.

1. We determine r_1 and r_2, and hence A_1, A_2, and A_3. Since each x_i, $i = 0, 1, \ldots, s$, is attached to an even number of pendant branches, the sum total of all the pendant branches incident on the central path of L is an even number, say $2p$. The terms of A_3 will be the labels given to the centers of the pendant branches only, i.e. $|A_3| = 2p$. Since $|S| = 2k + 1$, $|A_1| + |A_2| = 2(k - p) + 1$. Now we determine A_1 and A_2 by repeating step 1 in the respective proofs involving the lobsters described in Tables 3.3(b), 3.3(e), and 3.4(a), by setting $k = k - p$.

2. We label the centers of the branches incident on the x_0 in $T(L)$ in the following manner:

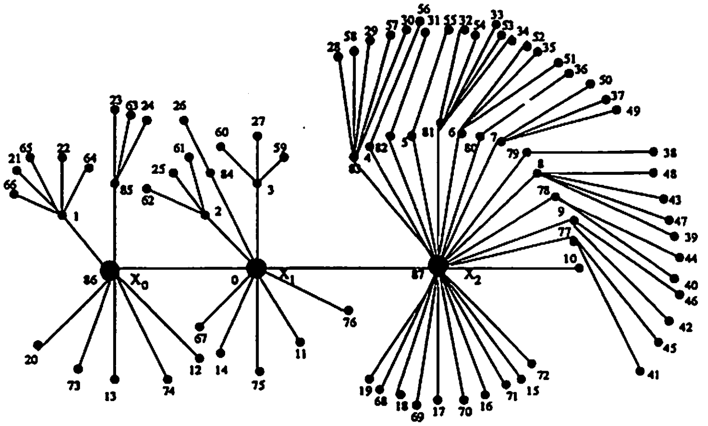

Figure 9: The graceful lobster obtained by applying inverse transformation to the lobster in Figure 8, making x_2 adjacent to x_1, giving the label 87 to x_2, and moving all the components in $f_{86}^{(1)}(A_j^{(2)})$, $j = 1, 2$, to x_2.

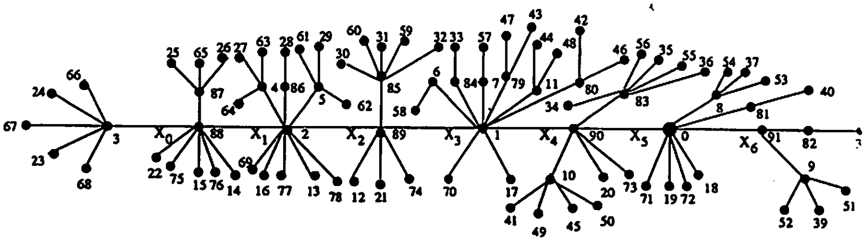

Figure 10: The lobster L with a graceful labeling.

For lobsters of type (x), $x = b, c, \ldots, i$, in Table 3.1, the proof follows if we proceed as the proof involving the lobsters of type (a) in Table 3.1 by modifying steps 1 and 2. For lobsters of type (b), we set $t_1 = t_2$ in step 1, repeat step 2(i), set $t_2 = t_1$ in step 2(ii), substitute step 2(iii) with "for $i = t_1 + 1, t_1 + 2, \ldots, t_2$, the centers of the even branches incident on x_i get labels from the beginning of $A_1^{(i)}$", replace odd (or even) branches by even branches in step 2(iv), and set $t_1 = t_2$ in steps 2(v) and 2(vi). For lobsters of type (c), we set $t_1 = 0$ and $t_2 = t$ in steps 1 and 2. For lobsters of type (d), we set $t_1 = t$ and $t_2 = t$ in steps 1, 2(i), 2(ii), 2(iv), 2(v), and 2(vi), replace odd branches by odd (or even) branches in step 2(i), odd branches

(i) Repeat step 2 of the respective proofs involving the lobsters in Tables 3.3(b), 3.3(e), and 3.4(a), by setting $k = k - p$.
(ii) For $i = 0, 1, \ldots, s$, among the pendant branches incident on x_i in L, the center of one branch gets the last label of $A_3^{(i)}$ and the centers of the rest of these branches get labels from the beginning of $A_3^{(i)}$, where $A_3^{(0)} = A_3$.

We notice that the labeling of the centers of the branches incident on the center x_0 of $T(L)$ given in step 2 follows part (b) of Lemma 2.4. Therefore, by Lemma 2.4, there exists a graceful labeling of $T(L)$ with the above labels of the center x_0 and the centers of the branches incident on x_0. Finally, we apply Theorem 3.2, for $n = 3$, on $T(L)$ and the path $H = x_0, x_1, \ldots, x_m$, so as to get a graceful labeling of L. □

Next we show that by partitioning the sequence S of Theorem 3.2 in a slightly different manner we get graceful lobsters in which the combination of odd, even, and pendant branches incident on x_0 will be different from those in Theorems 3.3 and 3.4. For this we modify Construction 3.1 as given below.

Construction 3.5 Let the tree T, the path H', the labeling f, and the sequence S be the same as in Construction 3.1. In (a), (b), (c), and (d) below, we construct the trees T_2, T_3, T_4, and T_5, respectively. For (a), (b), and (c), we take $n = 3$ in Construction 3.1, and use the notations A_1, A_2, and A_3, whereas for (d) we take $n = 2$ in the same construction, and use the notations A_1 and A_2.

(a) We partition S as $S = \bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3$, where $\bar{A}_1 = A_1 \setminus \{q - r_1\}$, $\bar{A}_2 = \{q - r_1\} \cup A_2$ (where $q - r_1$ occurs as the first term of $\bar{A}_2$), and $\bar{A}_3 = A_3$. We construct the tree T_2 from T by identifying the vertex y_0 of path H' with a_0 and moving the components (incident on the vertex a_0) in $\bar{A}_j$, $j = 1, 2, 3$, to y_i, $i = 1, 2, \ldots, s_j$, $0 \le s_j \le m$, in the following manner:

(i) We retain $2\lambda_0^{(1)}$, $\lambda_0^{(1)} \ge 0$, components from $\bar{A}_1$ at y_0. In particular, we retain $2p_0$, $0 \le p_0 \le \lambda_0^{(1)}$, components whose labels are from the beginning of $\bar{A}_1$ and $2\lambda_0^{(1)} - 2p_0$ components whose labels are from the end of $\bar{A}_1$. We retain $2\lambda_0^{(2)} + 1$, $\lambda_0^{(2)} \ge 0$, components from $\bar{A}_2$ at y_0. In particular, we retain $2\lambda_0^{(2)}$ components whose labels are from the beginning of $\bar{A}_2$ and one component whose label is the last term of $\bar{A}_2$. We retain $2\lambda_0^{(3)}$, $\lambda_0^{(3)} \geq 1$, components from $\bar{A}_3$ at y_0 in the same manner in which we have retained $2\lambda_0^{(3)}$ components of A_3 at y_0 in Construction 3.1. After deleting the components to be kept at y_0, let the sequences obtained from $\bar{A}_j$, $j = 1,2,3$, be $\bar{A}_j^{(1)}$.

(*ii*) If $s_j \geq 1$, then, for $1 \leq i \leq s_j$, to y_i we move $2\lambda_i^{(j)}$, $\lambda_i^{(j)} \geq 1$, components from $\bar{A}_j$ in the same way as we have moved the components from A_j to y_i in Construction 3.1.

(b) We partition S as $S = \bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3$, where $\bar{A}_1 = A_1$, $\bar{A}_2 = A_2 \backslash \{q - r_2\}$, and $\bar{A}_3 = \{q - r_2\} \cup A_3$ (where $q - r_2$ occurs as the first term of $\bar{A}_3$). We construct the tree T_3 from T by identifying the vertex y_0 of path H' with a_0 and moving the components in $\bar{A}_j$, $j = 1,2,3$, to y_i, $i = 1,2,\ldots,s_j$, $0 \leq s_j \leq m$, in the following manner:

(*i*) We retain $2\lambda_0^{(1)} + 1$, $\lambda_0^{(1)} \geq 1$, components from $\bar{A}_1$ at y_0 in the same manner in which we have retained $2\lambda_0^{(1)} + 1$ components of A_1 at y_0 in Construction 3.1. We retain $2\lambda_0^{(2)}+1$, $\lambda_0^{(2)} \geq 0$, components whose labels are from the beginning of $\bar{A}_2$ at y_0. We retain $2\lambda_0^{(3)} + 1$, $\lambda_0^{(3)} \geq 0$, components from $\bar{A}_3$ at y_0, among which $2\lambda_0^{(3)}$ components get labels from the beginning of $\bar{A}_3$ and the remaining one gets the label from the end of $\bar{A}_3$. After deleting the components to be kept at y_0, let the sequences obtained from $\bar{A}_j$, $j = 1,2,3$, be $\bar{A}_j^{(1)}$.

(*ii*) Same as (*ii*) in (*a*).

(c) We partition S as $S = \bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3$, where $\bar{A}_1 = A_1 \setminus \{q - r_1\}$, $\bar{A}_2 = \{q - r_1\} \cup A_2 \setminus \{q - r_2\}$, and $\bar{A}_3 = \{q - r_2\} \cup A_3$ (where, $q - r_1$ occurs as the first term of $\bar{A}_2$ and $q - r_2$ occurs as the first term of $\bar{A}_3$). We construct the tree T_4 from T by identifying the vertex y_0 of path H' with a_0 and moving the components in $\bar{A}_j$, $j = 1,2,3$, to y_i, $i = 1,2,\ldots,s_j$, $0 \leq s_j \leq m$, in the following manner:

(*i*) We retain $2\lambda_0^{(1)}$, $\lambda_0^{(1)} \geq 0$, components from $\bar{A}_1$ at y_0 in the same manner as we have done in (*a*). We retain $2\lambda_0^{(2)}$, $\lambda_0^{(2)} \geq 0$, components whose labels are from the beginning of $\bar{A}_2$ at y_0. We retain $2\lambda_0^{(3)}+1$, $\lambda_0^{(3)} \geq 0$, components from $\bar{A}_3$ at y_0 in the same manner as we have done in (c). After deleting the components to be kept at y_0, let the sequences obtained from $\bar{A}_j$, $j=$ 1, 2, 3, be $\bar{A}_j^{(1)}$.

(ii) Same as (ii) in (a).

(d) Here we partition S as $S = \bar{A}_1 \cup \bar{A}_2$, where $\bar{A}_1$ and $\bar{A}_2$ are as defined in (a) with $r_2 = k$. We construct a tree T_5 from T by identifying the vertex y_0 of H' with a_0 and moving the components in $\bar{A}_j$, $j=$ 1, 2, to y_i, $i= 1,2,\ldots,s_j$, $0 \leq s_j \leq m$, in the same manner as in (a). □

Theorem 3.6 The trees T_2, T_3, T_4, and T_5 in Construction 3.5 are graceful.

Proof: The proof follows if we proceed exactly as the proof of Theorem 3.2 involving the tree T_1 in Construction 3.1.

Theorem 3.7 The lobsters in Table 3.5 below are graceful.

Table 3.5

Lobsters ↓	x_0 is attached to	Combination of branches incident on x_i, $1 \leq i \leq m$
(a)	(e,o,e) or $(0,o,e)$	same as the lobsters in Theorem 3.4.
(b)	(o,o,o)	same as the lobsters of type (a) in Table 3.4 and having the additional condition mentioned in Theorem 3.4.
(c)	(e,e,o) or $(0,e,o)$ or $(e,0,o)$	same as the lobsters in Theorem 3.4.
(d)	$(e,0,o)$	same as the lobsters of type (a), (b), (c), (e), (f), (g), or (h) in Table 3.1.
(e)	$(e,0,o)$ or $(0,e,o)$	same as the lobsters of type (d) in Table 3.1.
(f)	$(0,e,o)$	same as the lobsters of type (i) in Table 3.1.
(g)	$(e,o,0)$	same as the lobsters of type (b) or (e) in Table 3.3.
(h)	same as (c)	same as the lobsters of type (a), (b), (d), or (e) in Table 3.2.

Continuation of Table 3.5

Lob-sters ↓	x_0 is attached to	Combination of branches incident on x_i, $1 \le i \le m$
(i)	same as (b)	same as the lobsters of type (a) or (c) in Table 3.2.
(j)	same as (c)	same as the lobsters of type (a), (c), or (d) in Table 3.3.
(k)	same as (d)	same as the lobsters of type (b) in Table 3.4.
(l)	same as (f)	same as the lobsters of type (c) in Table 3.4.
(m)	same as (g)	same as the lobsters of type (a) in Table 3.4.

Proof: As before, for every lobster L of this theorem, we first construct the diameter four tree $T(L)$ corresponding to L.

Let L be a lobster of type (a) in Table 3.5. We follow the two steps given below.

1. We partition S into three parts as in Construction 3.5(a). We determine r_1 and r_2, and hence $\bar{A}_1$, $\bar{A}_2$, and $\bar{A}_3$. The total number of pendant branches incident on the central path of L is an even number, say $2p$. The terms of $\bar{A}_3$ will be the labels given to the centers of the pendant branches only, i.e. $|\bar{A}_3| = 2p$. Then, $|\bar{A}_1| + |\bar{A}_2| = 2(k-p)+1 = 2r_2+1$. Let the combination of branches incident on x_i, $1 \le i \le m$, be the same as those in the lobsters of Theorem 3.4 derived from the lobsters of type (b) or (e) in Table 3.3. Let the number of even branches incident on x_0 be $2\lambda_0+1$, $\lambda_0 \ge 0$. We label $T(L)$ such that the centers of the even branches incident on x_0 get $2\beta_0$, labels from $\bar{A}_1$, $0 \le \beta_0 \le \lambda_0$, and $2(\lambda_0-\beta_0)+1$ labels from $\bar{A}_2$. Let the number of even branches incident on the rest of the x_i, and the labels that they get, be the same as in the proof involving the lobsters in Theorem 3.4 derived from the lobsters of type (b) or (e) in Table 3.3. We have $|\bar{A}_2| = 2(\lambda_0-\beta_0)+1+\sum_{i=1}^{t}(2\lambda_i-2\beta_i)+\sum_{i=t+1}^{p}2\lambda_i$. Let this number be $2r+1$. Therefore, $|\bar{A}_1| = 2r_1 = (2r_2+1)-(2r+1) = 2(r_2-r)$. If the combination of branches incident on x_i, $1 \le i \le m$, are the same as those in the lobsters of Theorem 3.4 derived from the lobsters of type (a) in Table 3.4, then the terms of $\bar{A}_1$ will be the labels of the centers of all the odd branches only.

2. We label the centers of the branches incident on x_0 in $T(L)$ in the following manner:

(i) The centers of the odd branches incident on x_0 get labels from the beginning of $\bar{A}_1$. If the combination of branches incident on x_i, $1 \leq i \leq m$, are the same as those in the lobsters of Theorem 3.4 derived from the lobsters of type (b) or (e) in Table 3.3, then the centers of the even branches incident on x_0 in L get $2\beta_0$ labels from the end of $\bar{A}_1$, $2(\lambda_0 - \beta_0)$ labels from the beginning and the last label of $\bar{A}_2$. If the combination of branches incident on x_i, $1 \leq i \leq m$, are the same as those in the lobsters of Theorem 3.4 derived from the lobsters of type (a) in Table 3.4, then among the even branches incident on x_0, the center of one even branch gets the last label from $\bar{A}_2$ and the centers of the rest of these branches get labels from the beginning of $\bar{A}_2$.

(ii) The centers of all the odd and even branches incident on x_i, $1 \leq i \leq m$, and all the pendant branches incident on x_i, $0 \leq i \leq s$, get labels in the same manner as described in step 2 of the proof involving the lobsters described in Theorem 3.4.

We notice that the labeling of the centers of the branches incident on the center x_0 of $T(L)$ given in step 2 follows part (b) of Lemma 2.4. Therefore, by Lemma 2.4 there exists a graceful labeling of $T(L)$ with the above labels of the center x_0 and the centers of the branches incident on x_0. Finally, we apply Theorem 3.6, for $n = 3$, on $T(L)$ and the path $H = x_0, x_1, \ldots, x_m$, so as to get a graceful labeling of L.

For the rest of the lobsters in Table 3.5 the proof follows if we proceed as the proof involving the lobsters of type (a) in Table 3.5 by modifying steps 1 and 2 only.

For lobsters of type (b) in Table 3.5:

1. We partition S into three parts as in Construction 3.5(b). Here $\bar{A}_1$, $\bar{A}_2$, and $\bar{A}_3$ will consist of the labels given to the centers of the odd, even, and pendant branches, respectively.

2. (i) The centers of the even branches incident on x_0 get labels from the beginning of $\bar{A}_2$. Among the pendant branches incident on x_0, the center of one branch gets the last label of $\bar{A}_3$, whereas, the centers of the rest of these branches get labels from the beginning of $\bar{A}_3$.

(ii) The centers of all the branches incident on x_i, $1 \leq i \leq m$, and all the odd branches incident on x_0 get labels in the same manner as described in step 2 of the proof involving the lobsters in Theorem 3.4.

For lobsters of type (c) in Table 3.5:

1. We partition S into three parts as in Construction 3.5(c). The terms of $\bar{A}_3$ will be the labels given to the centers of all the pendant branches only, so $|\bar{A}_3|$ is the number of pendant branches incident on the central path, say $2p+1$. Therefore, $|\bar{A}_1| + |\bar{A}_2| = 2(k-p) = 2r_2$. Let the combination of branches incident on x_i, $1 \leq i \leq m$, and the labels of the centers of these branches be the same as those in the lobsters of Theorem 3.4 derived from the lobsters of type (b) or (e) in Table 3.3. Let the number of even branches incident on x_0 be $2\lambda_0$, $\lambda_0 \geq 0$. We give a labeling to $T(L)$ such that, among the even branches incident on x_0, the centers of $2\beta_0$, of them get labels from $\bar{A}_1$, $0 \leq \beta_0 \leq \lambda_0$, and the centers of the rest of these branches get labels from $\bar{A}_2$. We have $|\bar{A}_2| = 2(\lambda_0 - \beta_0) + \sum_{i=1}^{t}(2\lambda_i - 2\beta_i) + \sum_{i=t+1}^{p} 2\lambda_i$. Let this number be $2r$ so $|\bar{A}_1| = 2r_1 = 2r_2 - 2r$. If the combination of branches incident on x_i, $1 \leq i \leq m$, are the same as those in the lobsters of Theorem 3.4 derived from the lobsters of type (a) in Table 3.4, then the terms of $\bar{A}_1$ will be the labels given to the centers of all the odd branches only.

2. (i) The centers of the odd branches incident on x_0 get labels from the beginning of $\bar{A}_1$, and among the pendant branches incident on x_0 the center of one branch gets the last label of $\bar{A}_3$ and the centers of the rest of them get labels from the beginning of $\bar{A}_3$. If the combination of branches incident on x_i, $1 \leq i \leq m$, are the same as those in the lobsters of Theorem 3.4 derived from the lobsters of type (b) or (e) in Table 3.3, then the centers of the even branches incident on x_0 get $2\beta_0$ labels from the end of $\bar{A}_1$ and $2(\lambda_0 - \beta_0)$ labels from the beginning of $\bar{A}_2$. If the combination of branches

incident on x_i, $1 \leq i \leq m$, are the same as those in the lobsters of Theorem 3.4 derived from the lobsters of type (a) in Table 3.4, then the centers of the even branches incident on x_0 get labels from the beginning of $\bar{A}_2$.

(ii) The centers of the branches incident on x_i, $1 \leq i \leq m$, get labels in the same manner as described in step 2 of the proof involving the lobsters in Theorem 3.4.

For lobsters L of types $(d) - (m)$ of Table 3.5:

1. We partition S into two parts as in Construction 3.5(d), and determine $\bar{A}_1$ and $\bar{A}_2$. For lobsters of type (d), we proceed as step 1 in the proof involving the lobsters of type (a) derived from those of type (b) in Table 3.3, with the changes given below. We first define a positive integer p_1 as $p_1 = t_1, t_2, 0, t, t, 0$, and 0, if the combination of branches incident on x_i, $1 \leq i \leq m$, are the same as those in the lobsters of types (a), (b), (c), (e), (f), (g), and (h), respectively, in Table 3.1. Then we replace r_2 with k, p with s, t with p_1, and even branches with pendant branches. If L is of type (e) (respectively, (f)), then we proceed as the proof involving the lobsters of type (d) by setting $p_1 = t$ (respectively, $p_1 = 0$). If L is of type (g), with the combination of branches incident on on x_i, $1 \leq i \leq m$, are the same as those in the lobsters of type (b) (respectively, (e)) in Table 3.3, then we repeat step 1 of the proof involving the lobsters in (a) derived from those of type (b) in Table 3.3, by setting $r_2 = k$ (respectively, $r_2 = k$ and $t = 0$). If L is of type (h), then we repeat the procedure of the proof involving the lobsters of type (d), by setting $p_1 = t$ (respectively, $p_1 = 0$) if the combination of branches incident on x_i, $1 \leq i \leq m$, are the same as those in the lobsters of type (a) (respectively, (b), (d), or (e)), in Table 3.2. If L is of type (i), then we repeat the procedure of the proof involving the lobsters of type (d), by setting $p_1 = t$ (respectively, $p_1 = 0$) if the combination of branches incident on x_i, $1 \leq i \leq m$, are the same as those in the lobsters of type (a) (respectively, (c)), in Table 3.2. If L is of type (j), (k), and (l), then the terms of $\bar{A}_2$ will be the labels given to the centers of all the pendant branches only. If L is of type (m), then the terms of $\bar{A}_2$ will be the labels given to the centers of all the even branches only.

2. (i) If L is of type (d), then we repeat step $2(i)$ in the proof involving the lobsters of type (a) derived from those of type (b) in Table 3.3, by replacing even branches with pendant branches. If L is of type (e) (respectively, (f)), then we repeat the procedure of the proof of the lobsters of type (d) by replacing odd branches with odd (or even) (respectively, even) branches. If L is of type (g), then we simply repeat step $2(i)$ in the proof involving the lobsters in (a) derived from the lobsters of type (b) in Table 3.3. If L is of type (h) and (i), then we repeat the procedure of the proof involving the lobsters of type (d) by replacing odd branches with odd branches followed by even branches. If L is of type (j), then the centers of the odd (respectively, even) branches incident on x_0 get labels from the beginning (respectively, end) of the sequence $\bar{A}_1$, and among the pendant branches incident on x_0, the center of one branch gets the last label of $\bar{A}_2$ while the centers of the rest of these branches get labels from the beginning of $\bar{A}_2$. If L is of type (k) (respectively, (l)) , then we replace even branches with pendant branches (respectively, even branches with pendant branches and odd branches with even branches) in the proof for the lobsters of type (d). If L is of type (m), then we repeat step $2(i)$ in the proof involving the lobsters in (a) derived from those of type (a) in Table 3.4.

(ii) The centers of the branches incident on x_i, $1 \leq i \leq m$, in L get labels as described in step 2 in the proof involving the respective lobsters mentioned in the second column of Table 3.5. □

Remark 3.8 In all the lobsters to which we give graceful labelings in this paper, the vertex x_m gets the largest label and x_{m-1} gets the label 0. Therefore, we get more graceful lobsters by attaching a caterpillar to the vertex x_m, or by attaching a suitable caterpillar (any number of pendant branches or an odd (or even) branch, or a combination of both) to the vertex x_{m-1} in any of the lobsters discussed in Theorem 3.3, 3.4, and 3.6.

4 Conclusion

In this paper we give graceful labelings to several types of lobsters. The lobsters which appear in Theorems 3.3, 3.4, and 3.7 have the following properties.

1. The degree of x_0 is at least two, the degree of each x_i, $1 \leq i \leq m-1$, is at least four, and the degree of x_m is at least three.

2. If some (or all) x_i, $1 \leq i \leq m$, is attached to a combination of all three types of branches, then this combination is either of the type (o, o, e) or (e, e, e). However, if x_0 is attached to all three types of branches then the combination may be from all possible combinations in which the total number of branches is odd.

3. We can partition the vertices of the central path into at most five parts such that the vertices in each part appear consecutively in the central path and are attached to the same combination (x, y, z) of branches. For example, in the lobsters of type (a) in Table 3.1 the central path have been partitioned into exactly five parts.

In this paper we also cover the graceful lobsters (as discussed in Remark 3.8) in which for some t, $1 \leq t < m$, the vertices x_i, $t+1 \leq i \leq m$, and the branches incident on them induce a caterpillar, while the vertices x_i, $0 \leq i \leq t$, exhibit the properties 1, 2, and 3 above. Bermond's conjecture will be resolved if one can give graceful labeling to lobsters in which any vertex of the central path is attached to an arbitrary combination (x, y, z) of branches or no branch.

Acknowledgement: We are thankful to the referee for their valuable comments and suggestions.

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