Q-ANALOGUE OF THE PASCAL MATRIX

DE-YIN ZHENG

[1] *Department of Mathematics, Hangzhou Normal University, Hangzhou 310012, P. R. China*
[2] *Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, P. R. China*
Email: deyinzheng@yahoo. com. cn

ABSTRACT. In this paper q-analogues of the Pascal matrix and the symmetric Pascal matrix are studied. It is shown that the q-Pascal matrix $\mathcal{P}_n$ can be factorized by special matrices and the symmetric q-Pascal matrix $\mathcal{Q}_n$ has the LDU-factorization and the Cholesky factorization. As by products, some q-binomial identities are produced by linear algebra. Furthermore these matrices are generalized in one or two variables where a short formula for all powers of q-Pascal functional matrix $\mathcal{P}_n[x]$ is given. Finally, it is similar to Pascal functional matrix, we have the exponential form for q-Pascal functional matrix.

1. INTRODUCTION

Pascal triangle, Pascal matrix are an ancient topic [3]. Nevertheless, it has carefully been studied only recently; see[1, 2, 4, 5, 6, 7, 9, 14, 15]. Bayat and Teimoori [2] studied the generalized Pascal matrix by defining the polynomials " Factorial Binomial ". El-Mikkawy and Cheon [7, 9] investigated the generalized Pascal matrix associated with the hypergeometric function. In [6], Factorizations of the Pascal-type matrices obtained from the two kinds Stirling numbers have been obtained.

The $(n+1)\times(n+1)$ Pascal matrix P_n and symmetric Pascal matrix Q_n are defined by $P_n(i,j) := \binom{i}{j}$ and $Q_n(i,j) := \binom{i+j}{j}$, respectively. In [4], Brawer and Pirovino have shown Pascal matrix to be factorized by special summation matrices and symmetric Pascal matrix to have the Cholesky factorization by the Gaussian elimination method.

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More generally, for a nonzero real variable x, the Pascal matrix was generalized in $P_n[x]$, $Q_n[x]$ and $R_n[x]$, respectively which are defined in [14], and these generalized Pascal matrices were also extended in $\Phi_n[x,y]$ and $\Psi_n[x,y]$ (see [15]) for any two nonzero real variables x and y where

$$\Phi_n(x,y;i,j) = x^{i-j}y^{i+j}\binom{i}{j}, \; i,j=0,1,\ldots,n, \quad \text{with}\binom{i}{j} = 0, \text{ if } i<j,$$

and

$$\Psi_n(x,y;i,j) = x^{i-j}y^{i+j}\binom{i+j}{j}, \quad i,j=0,1,\ldots,n,$$

respectively. In [14] and [15], the factorizations of $P_n[x]$, $Q_n[x]$, $R_n[x]$, $\Phi_n[x,y]$ and $\Psi_n[x,y]$ are obtained, respectively.

In Sections 2 and 3, we study the $(n+1)\times(n+1)$ q-Pascal matrix and symmetric q-Pascal matrix whose elements are related to q-binomial coefficients. As a consequence it is shown that q-Pascal matrix and symmetric q-Pascal matrix have analogous factorization of Pascal matrix and symmetric Pascal matrix. Furthermore, in Section 4, q-Pascal matrix and symmetric q-Pascal matrix are generalized in one or two variables. Similarly, their factorizations are obtained. Moreover, in Section 5, we give a simple formula for Powers of the q-Pascal functional matrix. Finally, it is similar to Pascal functional matrix, we have the exponential form for q-Pascal functional matrix.

2. Factorization of the q-Pascal matrix

Definition 2.1. We define the $(n+1)\times(n+1)$ *q-Pascal matrix* $\mathcal{P}_n = \mathcal{P}_{n,q}$ by

$$\mathcal{P}_n(i,j) := \begin{bmatrix} i \\ j \end{bmatrix}, \; i,j=0,1,\ldots,n,$$

where $\begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} i \\ j \end{bmatrix}_q$ are the *Gaussian polynomials*, or *q-binomial coefficients*:

$$\begin{bmatrix} i \\ j \end{bmatrix} := 0 \text{ if } i<j \quad \text{and} \quad \begin{bmatrix} i \\ j \end{bmatrix} = \frac{[i]!}{[j]![i-j]!} \text{ if } i \geq j \text{ for } i,j \in \mathbb{N},$$

$$[i]! = [i]_q! = [i][i-1]\cdots[1], \; [0]! = 1, \; [i] = [i]_q = (1-q^i)/(1-q).$$

The q-Pascal matrix $\mathcal{P}_n$ is characterized by its construction rule:

$$\mathcal{P}_n(i,i) := \mathcal{P}_n(i,0) := 1 \text{ for } i=0,1,\ldots,n, \quad \mathcal{P}_n(i,j) := 0 \text{ if } i<j,$$

$$\mathcal{P}_n(i,j) := \mathcal{P}_n(i-1,j) + q^{i-j}\mathcal{P}_n(i-1,j-1) \quad \text{for } i,j=1,2,\ldots,n. \quad (2.1)$$

As in [4], we list several definitions which will be required in the development of this paper. For any nonnegative integers n and k, the $(n+1)\times(n+1)$ matrices I_n, $\mathcal{S}_n^{(k)}$, $\mathcal{D}_n^{(k)}$, and $\mathcal{P}_n^{(k)}$ are defined by

$$\begin{aligned}
I_n &:= \operatorname{diag}(1,1,\dots,1),\\
\mathcal{S}_n^{(k)}(i,j) &:= \begin{cases} q^{(i-j)k} & \text{if } i \geq j,\\ 0 & \text{if } i < j,\end{cases} \qquad i,j = 0,1,\dots,n,\\
\mathcal{D}_n^{(k)}(i,i) &:= 1, \qquad \text{for} \quad i = 0,1,\dots,n,\\
\mathcal{D}_n^{(k)}(i+1,i) &:= -q^k, \quad \text{for} \quad i = 0,1,\dots,n-1,\\
\mathcal{D}_n^{(k)}(i,j) &:= 0, \qquad \text{for} \quad i<j \text{ or } j<i-1,\\
\mathcal{P}_n^{(k)}(i,j) &:= \mathcal{P}_n(i,j)q^{(i-j)k},
\end{aligned}$$

Clearly, $\mathcal{P}_n^{(0)} = \mathcal{P}_n$. Furthermore we need the $(n+1)\times(n+1)$ matrices

$$\bar{\mathcal{P}}_n^{(k)} := \begin{bmatrix} 1 & 0^T \\ 0 & \mathcal{P}_{n-1}^{(k)} \end{bmatrix},$$

$$\mathcal{F}_k := \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & \mathcal{D}_k^{(n-k)} \end{bmatrix}, \quad k = 1,2,\dots,n-1, \quad \text{and} \quad \mathcal{F}_n := \mathcal{D}_n^{(0)},$$

$$\mathcal{G}_k := \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & \mathcal{S}_k^{(n-k)} \end{bmatrix}, \quad k = 1,2,\dots,n-1, \quad \text{and} \quad \mathcal{G}_n := \mathcal{S}_n^{(0)}.$$

It is easy to see that

$$(\mathcal{D}_n^{(k)})^{-1} = \mathcal{S}_n^{(k)} \quad \text{and} \quad \mathcal{F}_k^{-1} = \mathcal{G}_k.$$

Lemma 2.2. *For any nonnegative integer k, one has*

$$\mathcal{D}_n^{(k)}\mathcal{P}_n^{(k)} = \bar{\mathcal{P}}_n^{(k+1)}. \tag{2.2}$$

Proof: It is clear that the (i,j)-entry of $\mathcal{D}_n^{(k)}\mathcal{P}_n^{(k)}$ equals 0 if $i<j$, and equals 1 if $i=j$, and it is easily verified that $(\mathcal{D}_n^{(k)}\mathcal{P}_n^{(k)})(i,0) = 0$, for $i = 0,1,\dots,n$. Now suppose $i>j$. By the definition of the matrix product and the recurrence (2.1) we get that for $i>j>0$:

$$\begin{aligned}
(\mathcal{D}_n^{(k)}\mathcal{P}_n^{(k)})(i,j) &= \mathcal{P}_n^{(k)}(i,j) - q^k\mathcal{P}_n^{(k)}(i-1,j)\\
&= \mathcal{P}_n(i,j)q^{(i-j)k} - q^k\mathcal{P}_n(i-1,j)q^{(i-j-1)k} \text{ (by (2.1))}\\
&= \mathcal{P}_n(i-1,j-1)q^{(i-j)(k+1)}\\
&= \bar{\mathcal{P}}_n^{(k+1)}(i,j).
\end{aligned}$$

This completes the proof. □

Remark 2.3. In fact, the matrix $\mathcal{D}_n^{(k)}$ performs the first Gaussian elimination step for the matrix $\mathcal{P}_n^{(k)}$.

By Lemma 2.2 and the definition of the $\mathcal{F}_k$'s, it follows immediately that

$$\mathcal{F}_1\mathcal{F}_2\cdots\mathcal{F}_n\mathcal{P}_n = I_n \quad \text{or} \quad \mathcal{P}_n = \mathcal{F}_n^{-1}\mathcal{F}_{n-1}^{-1}\cdots\mathcal{F}_1^{-1}. \tag{2.3}$$

Therefore, we have

Theorem 2.4. *The q-Pascal matrix $\mathcal{P}_n$ can be factorized by the matrices $\mathcal{G}_k$'s:*

$$\mathcal{P}_n = \mathcal{G}_n\mathcal{G}_{n-1}\cdots\mathcal{G}_1. \tag{2.4}$$

For example, when n=3, we have the following factorization

$$\mathcal{P}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & q^2 & q & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & q^2 & 1 \end{bmatrix}.$$

From (2.3) we immediately get factorization of the inverse of q-Pascal matrix:

$$\mathcal{P}_n^{-1} = \mathcal{F}_1\mathcal{F}_2\cdots\mathcal{F}_n = \widetilde{\mathcal{P}}_n, \tag{2.5}$$

where $\widetilde{\mathcal{P}}_n(i,j) := (-1)^{i-j}\mathcal{P}_n(i,j)q^{\binom{i-j}{2}}$.

In fact, it follows (see also [8], P118.) from $\mathcal{F}_1\mathcal{F}_2\cdots\mathcal{F}_n = \widetilde{\mathcal{P}}_n$ or $\mathcal{P}_n^{-1} = \widetilde{\mathcal{P}}_n$ by means of a simple computation.

3. The Cholesky factorization of the symmetric q-Pascal matrix

Definition 3.1. We define the *symmetric q-Pascal matrix* $\mathcal{Q}_n$ as

$$\mathcal{Q}_n(i,j) := \begin{bmatrix} i+j \\ j \end{bmatrix}, \qquad i,j = 0,1,\ldots,n.$$

Obviously, $\mathcal{Q}_n(i,j) = \mathcal{Q}_n(j,i)$, $\mathcal{Q}_n(i,j) = \mathcal{P}_n(i+j,j)$. Thus, by the recursion of the q-Pascal matrix $\mathcal{P}_n$, the elements of $\mathcal{Q}_n$ obey the following construction rule:

$$\mathcal{Q}_n(0,j) := \mathcal{Q}_n(i,0) := 1 \qquad \text{for} \quad i,j = 0,1,\ldots,n,$$
$$\mathcal{Q}_n(i,j) := \mathcal{Q}_n(i-1,j) + q^i\mathcal{Q}_n(i,j-1) \quad \text{for} \quad i,j = 1,2,\ldots,n. \tag{3.1}$$

By Lemma 2.2, the matrix equation $\mathcal{F}_1\mathcal{F}_2\cdots\mathcal{F}_n\mathcal{P}_n = I_n$ shows that the q-Pascal matrix $\mathcal{P}_n$ is changed into the unit matrix I_n by Gaussian elimination method. Similarly, $\mathcal{Q}_n$ can be changed into upper triangular by the matrices $\mathcal{F}_k$, $k = 1,2,\ldots,n$.

Lemma 3.2. *One has*

$$\mathcal{F}_1\mathcal{F}_2\cdots\mathcal{F}_n\mathcal{Q}_n = \check{\mathcal{P}}_n^T, \tag{3.2}$$

where $\check{\mathcal{P}}_n(i,j) := \mathcal{P}_n(i,j)q^{j^2}, \quad i,j = 0,1,\ldots,n.$

Proof: First assume $i \geq j$. Applying the *q-Vandermonde convolution formula*

$$\begin{bmatrix} a+b \\ n \end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix} a \\ k \end{bmatrix} \begin{bmatrix} b \\ n-k \end{bmatrix} q^{k(b-n+k)} \tag{3.3}$$

we obtain

$$\begin{aligned} (\mathcal{P}_n\check{\mathcal{P}}_n^T)(i,j) &= \sum_{k=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} q^{k^2} \\ &= \sum_{k=0}^{j} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j \\ j-k \end{bmatrix} q^{k(j-j+k)} \\ &= \begin{bmatrix} i+j \\ j \end{bmatrix}. \end{aligned}$$

For the case $i < j$ we have similar result. It implies that $\mathcal{P}_n\check{\mathcal{P}}_n^T = \mathcal{Q}_n$, and from (2.3) we obtain (3.2). □

From (3.2) and (2.3), it is easy to deduce the following result:

$$\mathcal{Q}_n = \mathcal{P}_n\check{\mathcal{P}}_n^T. \tag{3.4}$$

Furthermore,

$$\begin{aligned} \check{\mathcal{P}}_n &= \mathcal{P}_n\mathrm{diag}(1,q,q^4,\ldots,q^{n^2}) \\ &= (\mathcal{P}_n\mathrm{diag}(1,q^{\frac{1}{2}},q^2,\ldots,q^{\frac{1}{2}n^2}))\mathrm{diag}(1,q^{\frac{1}{2}},q^2,\ldots,q^{\frac{1}{2}n^2}). \end{aligned}$$

So we can immediately get the LDU-factorization and the Cholesky factorization of $\mathcal{Q}_n$.

Theorem 3.3. *The LDU-factorization and Cholesky factorization for* $\mathcal{Q}_n$ *is given by*

$$\mathcal{Q}_n = \mathcal{P}_n\widetilde{\mathcal{I}}_n\mathcal{P}_n^T \tag{3.5}$$

and

$$\mathcal{Q}_n = \widehat{\mathcal{P}}_n\widehat{\mathcal{P}}_n^T, \tag{3.6}$$

respectively, where $\widetilde{\mathcal{I}}_n := \mathrm{diag}(1,q,q^4,\ldots,q^{n^2})$, $\widehat{\mathcal{P}}_n(i,j) := \mathcal{P}_n(i,j)q^{j^2/2}$, $i,j = 0,1,\ldots,n$.

From (3.5) and (2.5) we have factorization of the inverse of symmetric q-Pascal matrix:

$$\mathcal{Q}_n^{-1} = \widetilde{\mathcal{P}}_n^T\widetilde{\mathcal{I}}_n^{-1}\widetilde{\mathcal{P}}_n. \tag{3.7}$$

By carrying out the multiplication of the equation (3.4) we obtain an identity for the q-binomial coefficients:

Corollary 3.4. *For $i, j = 0, 1, \ldots, n$, one has*

$$\begin{bmatrix} i+j \\ j \end{bmatrix} = \sum_{l=0}^{j} \begin{bmatrix} i \\ l \end{bmatrix} \begin{bmatrix} j \\ l \end{bmatrix} q^{l^2}.$$

We notice that this corollary follows also from (3.3) with $a \to i$, $b \to j$ and $n \to j$.

Remark 3.5. The diagonal entries of the matrix $\mathcal{Q}_n$ are essentially the q-Catalan numbers [13], which are defined as

$$C_k(q) := \frac{1}{[k+1]} \begin{bmatrix} 2k \\ k \end{bmatrix}.$$

Therefore we have

$$\mathcal{Q}_n(k,k) = \sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix}^2 q^{l^2} = \begin{bmatrix} 2k \\ k \end{bmatrix} = [k+1] C_k(q), \quad k \geq 0.$$

From the matrix equation (3.7) or $I_n = \mathcal{Q}_n \widetilde{\mathcal{P}}_n^T \widetilde{\mathcal{I}}_n^{-1} \widetilde{\mathcal{P}}_n$, we find

Corollary 3.6. *For $i, j = 0, 1, \ldots, n$, one has*

$$\sum_{k=0}^{n} \sum_{l=0}^{n} (-1)^{k+j} \begin{bmatrix} i+k \\ k \end{bmatrix} \begin{bmatrix} l \\ k \end{bmatrix} \begin{bmatrix} l \\ j \end{bmatrix} q^{\binom{l-k}{2} + \binom{l-j}{2} - l^2} = \delta_{i,j}.$$

Corollary 3.4 and 3.6 yield

Corollary 3.7. *For $i, j = 0, 1, \ldots, n$, one has*

$$\sum_{k=0}^{n} \sum_{l=0}^{n} \sum_{m=0}^{n} (-1)^{k+j} \begin{bmatrix} i \\ m \end{bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} \begin{bmatrix} l \\ k \end{bmatrix} \begin{bmatrix} l \\ j \end{bmatrix} q^{\binom{l-k}{2} + \binom{l-j}{2} + m^2 - l^2} = \delta_{i,j}.$$

4. Factorization of the q-Pascal functional matrix

We generalize q-Pascal matrix in one variable as follows:

Definition 4.1. Let x be any real number. *The q-Pascal functional matrix of the first kind, $\mathcal{P}_n[x]$,* is defined for $i, j = 0, 1, \ldots, n$ as

$$\mathcal{P}_n(x; i, j) = \begin{cases} \begin{bmatrix} i \\ j \end{bmatrix} x^{i-j} & \text{if } i \geq j\,, \\ 0 & \text{otherwise.} \end{cases}$$

Here and in the sequel to this paper, for convenience, we set $0^0 := 1$. Then $\mathcal{P}_n[0]$ equals the identity matrix. Obviously, $\mathcal{P}_n[1] = \mathcal{P}_n$ and $\mathcal{P}_n[x]=P_n[x]$ if $q = 1$; see [5, 14].

Define $(n+1) \times (n+1)$ matrix $\mathcal{S}_n^{(k)}[x]$ and $\mathcal{D}_n^{(k)}[x]$ by

$$\mathcal{S}_n^{(k)}(x; i,j) = \mathcal{S}_n^{(k)}(i,j)x^{i-j} \quad \text{and} \quad \mathcal{D}_n^{(k)}(x;i,j) = \mathcal{D}_n^{(k)}(i,j)x^{i-j},$$

respectively. Then for $k = 1, 2, \ldots, n-1$

$$\mathcal{F}_k[x] := \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & \mathcal{D}_k^{(n-k)}[x] \end{bmatrix}, \quad \text{and} \quad \mathcal{F}_n[x] := \mathcal{D}_n^{(0)}[x];$$

$$\mathcal{G}_k[x] := \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & \mathcal{S}_k^{(n-k)}[x] \end{bmatrix}, \quad \text{and} \quad \mathcal{G}_n[x] := \mathcal{S}_n^{(0)}[x].$$

Clearly, $\mathcal{F}_k[x] = \mathcal{G}_k^{-1}[x]$, $k = 1, 2, \ldots, n$. We need again the $(n+1) \times (n+1)$ matrices $\mathcal{W}_n[x]$, $\mathcal{U}_n[x]$, $\mathcal{J}_n[x]$, $\mathcal{W}_n[x,y]$, $\mathcal{U}_n[x,y]$:

$$\begin{aligned}
&\mathcal{J}_n[x] := \operatorname{diag}(1, x, \ldots, x^n), \\
&\mathcal{W}_n(x, y; i, j) := \mathcal{S}_n^{(0)}(i,j)x^{i-j}y^{i+j}, \\
&\mathcal{U}_n(x, y; i, j) := \mathcal{D}_n^{(0)}(i,j)x^{i-j}y^{-i-j}, \\
&\mathcal{W}_n[x] := \mathcal{W}_n[1, x], \\
&\mathcal{U}_n[x] := \mathcal{U}_n[1, x].
\end{aligned}$$

By definition, we see that

$$\begin{aligned}
\mathcal{S}_n^{(k)}[x] &= \mathcal{J}_n[x]\mathcal{S}_n^{(k)}\mathcal{J}_n^{-1}[x], \\
\mathcal{G}_k[x] &= \mathcal{J}_n[x]\mathcal{G}_k\mathcal{J}_n^{-1}[x], \\
\mathcal{P}_n[x] &= \mathcal{J}_n[x]\mathcal{P}_n\mathcal{J}_n^{-1}[x].
\end{aligned}$$

Hence, by Theorem 2.4 we have

$$\begin{aligned}
\mathcal{P}_n[x] &= \mathcal{J}_n[x]\mathcal{G}_n\mathcal{G}_{n-1}\cdots\mathcal{G}_1\mathcal{J}_n^{-1}[x] \\
&= (\mathcal{J}_n[x]\mathcal{G}_n\mathcal{J}_n^{-1}[x])(\mathcal{J}_n[x]\mathcal{G}_{n-1}\mathcal{J}_n^{-1}[x])\cdots(\mathcal{J}_n[x]\mathcal{G}_1\mathcal{J}_n^{-1}[x]) \\
&= \mathcal{G}_n[x]\mathcal{G}_{n-1}[x]\cdots\mathcal{G}_1[x].
\end{aligned}$$

Theorem 4.2. *Let x be any nonzero real number. The q-Pascal functional matrix of the first kind $\mathcal{P}_n[x]$ can be factorized by the matrices $\mathcal{G}_k[x]$:*

$$\mathcal{P}_n[x] = \mathcal{G}_n[x]\mathcal{G}_{n-1}[x]\cdots\mathcal{G}_1[x]. \tag{4.1}$$

For the inverse of $\mathcal{P}_n[x]$, we get

$$\mathcal{P}_n^{-1}[x] = \mathcal{G}_1^{-1}[x]\mathcal{G}_2^{-1}[x]\cdots\mathcal{G}_n^{-1}[x] = \mathcal{F}_1[x]\mathcal{F}_2[x]\cdots\mathcal{F}_n[x],$$

which implies, together with (2.5), that the following relation holds

Theorem 4.3. *One has*

$$\mathcal{P}_n^{-1}[x] = \mathcal{F}_1[x]\mathcal{F}_2[x]\cdots\mathcal{F}_n[x] = \widetilde{\mathcal{P}}_n[x], \tag{4.2}$$

where $\widetilde{\mathcal{P}}_n(x;i,j) := \mathcal{P}_n(i,j)q^{\binom{i-j}{2}}(-x)^{i-j}$.

Definition 4.4. Let x be any real number. *The q-Pascal functional matrix of the second kind,* $\mathcal{R}_n[x]$, is defined for $i,j = 0,1,\ldots,n$ as

$$\mathcal{R}_n(x;i,j) = \begin{cases} \left[{i \atop j}\right] x^{i+j} & \text{if } i \geq j\,, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\mathcal{R}_n[x] = \mathcal{J}_n[x]\mathcal{P}_n\mathcal{J}_n[x] \quad \text{or} \quad \mathcal{R}_n[x] = \mathcal{P}_n[x]\mathcal{J}_n[x^2],$$

where $\mathcal{J}_n[x^2] = \mathrm{diag}(1, x^2, \ldots, x^{2n})$. Thus, by (2.4) and (4.1), we have

$$\mathcal{R}_n[x] = \mathcal{W}_n[x]\mathcal{G}_{n-1}[x^{-1}]\cdots\mathcal{G}_1[x^{-1}]$$

or

$$\mathcal{R}_n[x] = \mathcal{G}_n[x]\mathcal{G}_{n-1}[x]\cdots\mathcal{G}_1[x]\mathcal{J}_n[x^2].$$

Using (2.5) and (4.2), we get

$$\mathcal{R}_n^{-1}[x] = \mathcal{J}_n^{-1}[x]\mathcal{P}_n^{-1}\mathcal{J}_n^{-1}[x] = \mathcal{F}_1[x^{-1}]\mathcal{F}_2[x^{-1}]\cdots\mathcal{F}_{n-1}[x^{-1}]\mathcal{U}_n[x]$$

or

$$\mathcal{R}_n^{-1}[x] = \mathcal{J}_n^{-1}[x^2]\mathcal{P}_n^{-1}[x] = \mathcal{J}_n[x^{-2}]\mathcal{F}_1[x]\mathcal{F}_2[x]\cdots\mathcal{F}_n[x].$$

Theorem 4.5. *Let x be any nonzero real number. Then*

(i) *The q-Pascal functional matrix of the second kind $\mathcal{R}_n[x]$ can be factorized by the matrices $\mathcal{G}_k[x]$ and the diagonal matrix $\mathcal{J}_n[x^2]$*

$$\mathcal{R}_n[x] = \mathcal{W}_n[x]\mathcal{G}_{n-1}[x^{-1}]\cdots\mathcal{G}_1[x^{-1}] \tag{4.3}$$

or

$$\mathcal{R}_n[x] = \mathcal{G}_n[x]\mathcal{G}_{n-1}[x]\cdots\mathcal{G}_1[x]\mathcal{J}_n[x^2]. \tag{4.4}$$

(ii) *The inverse of the q-Pascal functional matrix of the second kind can be factorized by the diagonal matrix $\mathcal{J}_n[x^{-2}]$ and the matrices $\mathcal{F}_k[x]$*

$$\mathcal{R}_n^{-1}[x] = \mathcal{F}_1[x^{-1}]\mathcal{F}_2[x^{-1}]\cdots\mathcal{F}_{n-1}[x^{-1}]\mathcal{U}_n[x] \tag{4.5}$$

or

$$\mathcal{R}_n^{-1}[x] = \mathcal{J}_n[x^{-2}]\mathcal{F}_1[x]\mathcal{F}_2[x]\cdots\mathcal{F}_n[x]. \tag{4.6}$$

Definition 4.6. Let x be any nonzero real number. *The symmetric q-Pascal functional matrix,* $\mathcal{Q}_n[x]$, is defined for $i,j = 0,1,\ldots,n$ as

$$\mathcal{Q}_n(x;i,j) = \begin{cases} \left[{i+j \atop j}\right] x^{i+j} & \text{if } i \geq j\,, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently, $\mathcal{Q}_n[x] = \mathcal{J}_n[x]\mathcal{Q}_n\mathcal{J}_n[x]$. By Theorem 3.3 we have

$$\begin{aligned}\mathcal{Q}_n[x] &= \mathcal{J}_n[x]\mathcal{Q}_n\mathcal{J}_n[x] = \mathcal{J}_n[x]\mathcal{P}_n\widetilde{\mathcal{I}}_n\mathcal{P}_n^T\mathcal{J}_n[x] \\ &= (\mathcal{J}_n[x]\mathcal{P}_n\mathcal{J}_n^{-1}[x])\widetilde{\mathcal{I}}_n(\mathcal{J}_n[x]\mathcal{P}_n^T\mathcal{J}_n[x]) = \mathcal{P}_n[x]\widetilde{\mathcal{I}}_n\mathcal{R}_n^T[x].\end{aligned}$$

Theorem 4.7. *Let x be any nonzero real number. The LDU-factorization and Cholesky factorization for symmetric q-Pascal functional matrix $\mathcal{Q}_n[x]$ are given by*

$$\mathcal{Q}_n[x] = \mathcal{P}_n[x]\widetilde{\mathcal{I}}_n\mathcal{R}_n^T[x] \tag{4.7}$$

and

$$\mathcal{Q}_n[x] = \widehat{\mathcal{Q}}_n[x]\widehat{\mathcal{Q}}_n^T[x], \tag{4.8}$$

respectively, where $\widehat{\mathcal{Q}}_n(x;i,j) := \mathcal{P}_n(i,j)q^{j^2/2}x^i, \quad i,j = 0,1,\ldots,n.$

As seen in Definition 4.1, 4.4, 4.6, the q-Pascal functional matrices and the symmetric q-Pascal functional matrix have been defined for one variable x. Now we generalize these definitions for two variables x and y.

Definition 4.8. Let x and y be any two nonzero real numbers. We define the $(n+1)\times(n+1)$ matrices $\mathcal{P}_n[x,y]$ and $\mathcal{Q}_n[x,y]$ for $i,j = 0,1,\ldots,n$ by

$$\mathcal{P}_n(x,y;i,j) = \begin{cases} \begin{bmatrix} i \\ j \end{bmatrix} x^{i-j}y^{i+j} & \text{if } i \geq j\,, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathcal{Q}_n(x,y;i,j) = \begin{cases} \begin{bmatrix} i+j \\ j \end{bmatrix} x^{i-j}y^{i+j} & \text{if } i \geq j\,, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, we see that

$$\mathcal{P}_n[x,1] = \mathcal{P}_n[x], \quad \mathcal{P}_n[1,y] = \mathcal{R}_n[y], \quad \mathcal{Q}_n[1,y] = \mathcal{Q}_n[y].$$

It is easy to see that the following theorems hold by the similar arguments for $\mathcal{P}_n[x]$ and $\mathcal{Q}_n[x]$.

Theorem 4.9. *Let x and y be any two nonzero real numbers. Then the following results hold*

(a) $\mathcal{P}_n[-x,y] = \mathcal{P}_n[x,-y]$,

(b) $\mathcal{Q}_n[-x,y] = \mathcal{Q}_n[x,-y]$,

(c) $\mathcal{P}_n^{-1}[x,y] = \widehat{\mathcal{P}}_n[-x,y^{-1}] = \widehat{\mathcal{P}}_n[x,-y^{-1}]$,

(d) $\mathcal{P}_n[x,y] = \mathcal{W}_n[x,y]\mathcal{G}_{n-1}[xy^{-1}]\cdots\mathcal{G}_1[xy^{-1}]$,

(e) $\mathcal{P}_n^{-1}[x,y] = \mathcal{F}_1[xy^{-1}]\mathcal{F}_2[xy^{-1}]\cdots\mathcal{F}_{n-1}[xy^{-1}]\mathcal{U}_n[x,y]$,

(f) $\mathcal{Q}_n[x,y] = \mathcal{P}_n[x,y]\widetilde{\mathcal{I}}_n\mathcal{P}_n^T[x^{-1}y] = \mathcal{P}_n[xy]\widetilde{\mathcal{I}}_n\mathcal{P}_n^T[x^{-1},y]$.

For the previous several kinds of q-Pascal matrix, we also can get

Theorem 4.10.

$$\begin{aligned}
&\det\mathcal{P}_n == \det\widetilde{\mathcal{P}}_n = 1, \\
&\det\mathcal{Q}_n = \det\check{\mathcal{P}}_n = (\det\widehat{\mathcal{P}}_n)^2 = q^{\frac{1}{6}n(n+1)(2n+1)}, \\
&\det\mathcal{P}_n[x] = \det\widetilde{\mathcal{P}}_n[x] = 1, \\
&\det\mathcal{R}_n[x] = x^{n(n+1)}, \\
&\det\mathcal{Q}_n[x] = (\det\widehat{\mathcal{Q}}_n[x])^2 = q^{n(n+1)(2n+1)/6}x^{n(n+1)}, \\
&\det\mathcal{P}_n[x,y] = \det\mathcal{Q}_n[x,y] = y^{n(n+1)}.
\end{aligned}$$

5. Powers of the q-Pascal functional matrix

Let us consider again the q-Pascal matrix $\mathcal{P}_n$. It turns out that there is a short formula for the elements of all powers of $\mathcal{P}_n$. To do so we let $S_k^{(n)} = S_{k,q}^{(n)}$ denote the q-numbers

$$S_k^{(n)} := \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k\geq 0}} \begin{bmatrix} n \\ i_1,\cdots,i_k \end{bmatrix}. \tag{5.1}$$

Here, $\left[{n \atop i_1,\cdots,i_k}\right]$ is called a q-multinomial coefficient defined by $\left[{n \atop i_1,\cdots,i_k}\right] = \left[{n \atop i_1,\cdots,i_k}\right]_q = [n]!/([i_1]!\cdots[i_k]!)$. For instance, $S_k^{(0)} = 1$, $S_k^{(1)} = k$, $S_k^{(2)} = k + \binom{k}{2}(1+q)$, $S_1^{(n)} = 1$, $S_2^{(n)} = \sum_{l=0}^{n} \left[{n \atop l}\right]$. The q-numbers $S_2^{(n)}$ are studied by Goldman and Rota in [11], where $S_2^{(n)}$, i.e. G_n in [11], are called the *Galois numbers.* They satisfy the following recurrence:

$$G_{n+1} = 2G_n + (q^n - 1)G_{n-1}, \quad G_0 = 1,\ G_1 = 2.$$

Theorem 5.1. *For any positive integer k, one has*

$$\mathcal{P}_n^k[x] = \mathcal{P}_n[xS_k] \tag{5.2}$$

where $\mathcal{P}_n[xS_k] = \left(\left[{i \atop j}\right] S_k^{(i-j)} x^{i-j}\right)_{i,j}$.

Proof (by induction on k): It clearly holds for $k = 1$. Suppose that it holds for a certain $k > 1$, and we want to prove it for $k + 1$. With the definition of the matrix product and the inductive assumption, we find

$$\begin{aligned}
(\mathcal{P}_n^{k+1}[x])_{i,j} &= (\mathcal{P}_n[x]\mathcal{P}_n^k[x])_{i,j} = (\mathcal{P}_n[x]\mathcal{P}_n[xS_k])_{i,j} \\
&= \sum_{l=j}^{i} \begin{bmatrix} i \\ l \end{bmatrix} x^{i-l} \begin{bmatrix} l \\ j \end{bmatrix} S_k^{(l-j)} x^{l-j} = \begin{bmatrix} i \\ j \end{bmatrix} x^{i-j} \sum_{r=0}^{i-j} \begin{bmatrix} i-j \\ r \end{bmatrix} S_k^{(r)} \\
&= \begin{bmatrix} i \\ j \end{bmatrix} x^{i-j} \sum_{r=0}^{i-j} \sum_{\substack{i_1+\cdots+i_k=r \\ i_1,\ldots,i_k\geq 0}} \begin{bmatrix} i-j \\ r \end{bmatrix} \begin{bmatrix} r \\ i_1,\cdots,i_k \end{bmatrix} \\
&= \begin{bmatrix} i \\ j \end{bmatrix} x^{i-j} \sum_{\substack{i_1+\cdots+i_{k+1}=i-j \\ i_1,\ldots,i_{k+1}\geq 0}} \begin{bmatrix} i-j \\ i_1,\cdots,i_k,i_{k+1} \end{bmatrix} \\
&= \begin{bmatrix} i \\ j \end{bmatrix} (xS_{k+1})^{i-j},
\end{aligned}$$

for $i \geq j$ and it follows that $(\mathcal{P}_n^{k+1}[x])_{i,j} = 0$ for $i < j$. This completes the proof. □

For n=3, we write explicitly the following:

$$\begin{aligned}
\mathcal{P}_3^3[x] &= \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & 0 & 0 & 0 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}x & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & 0 & 0 \\ \begin{bmatrix} 2 \\ 0 \end{bmatrix}x^2 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}x & \begin{bmatrix} 2 \\ 2 \end{bmatrix} & 0 \\ \begin{bmatrix} 3 \\ 0 \end{bmatrix}x^3 & \begin{bmatrix} 3 \\ 1 \end{bmatrix}x^2 & \begin{bmatrix} 3 \\ 2 \end{bmatrix}x & \begin{bmatrix} 3 \\ 3 \end{bmatrix} \end{bmatrix}^3 \\
&= \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & 0 & 0 & 0 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}S_3^{(1)}x & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & 0 & 0 \\ \begin{bmatrix} 2 \\ 0 \end{bmatrix}S_3^{(2)}x^2 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}S_3^{(1)}x & \begin{bmatrix} 2 \\ 2 \end{bmatrix} & 0 \\ \begin{bmatrix} 3 \\ 0 \end{bmatrix}S_3^{(3)}x^3 & \begin{bmatrix} 3 \\ 1 \end{bmatrix}S_3^{(2)}x^2 & \begin{bmatrix} 3 \\ 2 \end{bmatrix}S_3^{(1)}x & \begin{bmatrix} 3 \\ 3 \end{bmatrix} \end{bmatrix},
\end{aligned}$$

where $S_3^{(1)} = 3$, $S_3^{(2)} = 3 + 3[2] = 6 + 3q$, $S_3^{(3)} = 3 + 6[3] + [3][2] = 10 + 8q + 8q^2 + q^3$.

Now we consider negative integer powers of $\mathcal{P}_n[x]$. To do so we first consider the following lemma.

Lemma 5.2. *For any positive integer k, there holds*

$$\widetilde{\mathcal{P}}_n^k[x] = \widehat{\mathcal{I}}_n \mathcal{P}_{n,q^{-1}}^k[x] \widehat{\mathcal{I}}_n^{-1}, \tag{5.3}$$

where $\widehat{\mathcal{I}}_n := \mathrm{diag}(1, -1, q, -q^3, \ldots, (-1)^n q^{\binom{n}{2}})$.

Proof: By the definition of $\widetilde{\mathcal{P}}_n[x]$ (noticing $\begin{bmatrix} i \\ j \end{bmatrix} = 0$, if $i < j$), we have

$$\widetilde{\mathcal{P}}_n(x; i, j) = \begin{bmatrix} i \\ j \end{bmatrix}_q q^{\binom{i-j}{2}} (-x)^{i-j} = \begin{bmatrix} i \\ j \end{bmatrix}_{q^{-1}} q^{\binom{i}{2}} q^{-\binom{j}{2}} (-x)^{i-j},$$

or

$$\widetilde{\mathcal{P}}_n[x] = \widehat{\mathcal{I}}_n \mathcal{P}_{n,q^{-1}}[x]\widehat{\mathcal{I}}_n^{-1}.$$

Hence

$$\widetilde{\mathcal{P}}_n^k[x] = \left(\widehat{\mathcal{I}}_n \mathcal{P}_{n,q^{-1}}[x]\widehat{\mathcal{I}}_n^{-1}\right)^k = \widehat{\mathcal{I}}_n \mathcal{P}_{n,q^{-1}}^k[x]\widehat{\mathcal{I}}_n^{-1}. \qquad \square$$

Theorem 5.3. *For any positive integer k, one has*

$$\mathcal{P}_n^{-k}[x] = \widetilde{\mathcal{P}}_n[xS_{k,q^{-1}}] \tag{5.4}$$

where $\widetilde{\mathcal{P}}_n[xS_{k,q^{-1}}] = \left(\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right] q^{\binom{i-j}{2}} S_{k,q^{-1}}^{(i-j)} (-x)^{i-j}\right)_{i,j}$.

Proof: By (4.2), (5.3) and (5.2), we get

$$\mathcal{P}_n^{-k}[x] = \widetilde{\mathcal{P}}_n^k[x] = \widehat{\mathcal{I}}_n \mathcal{P}_{n,q^{-1}}^k[x]\widehat{\mathcal{I}}_n^{-1} = \widehat{\mathcal{I}}_n \mathcal{P}_{n,q^{-1}}[xS_{k,q^{-1}}]\widehat{\mathcal{I}}_n^{-1} = \widetilde{\mathcal{P}}_n[xS_{k,q^{-1}}].$$

This completes the proof. $\square$

Now we consider the problem of the calculation for the q-numbers $S_k^{(n)}$. First of all, we adopt the following conventions. From now on we let e_i be the ith unit vector in $\mathbb{R}^{n+1}, i = 0, 1, \ldots, n$, and $e := (1, \ldots, 1)^T \in \mathbb{R}^{n+1}$ the summation vector. Then, we have the following lemma.

Lemma 5.4. *For any positive integer k, one has*

$$e_i^T \mathcal{P}_n^k e = \sum_{j=0}^{i} \begin{bmatrix} i \\ j \end{bmatrix} S_k^{(i-j)} = S_{k+1}^{(i)}, \quad i = 0, 1, \ldots, n. \tag{5.5}$$

Proof: This result has been implied by the procedure of the proof of Theorem 5.1.

Theorem 5.5. *The q-numbers $S_k^{(n)}$ satisfy the following recurrence relation:*

$$S_{k+1}^{(n)} = 1 + \sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix} \sum_{j=1}^{k} S_j^{(i)}, \quad S_1^{(n)} = 1, S_2^{(n)} = G_n. \tag{5.6}$$

Proof: By Lemma 5.4, we have

$$S_{k+1}^{(n)} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} S_k^{(n-j)} = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} S_k^{(i)},$$

namely,

$$S_{k+1}^{(n)} - S_k^{(n)} = \sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix} S_k^{(i)}.$$

Hence

$$S_{k+1}^{(n)} - S_1^{(n)} = \sum_{j=1}^{k} \sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix} S_j^{(i)}.$$

$$S_{k+1}^{(n)} = 1 + \sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix} \sum_{j=1}^{k} S_j^{(i)}. \quad \square$$

For example,

$$S_3^{(n)} = 1 + \sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix} (1 + G_n), \qquad S_4^{(n)} = 1 + \sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix} (1 + G_n + S_3^{(i)}).$$

Lemma 5.6. *For any positive integer m, one has*

$$e_m^T (\mathcal{P}_n - I_n)^p e = 0 \quad or \quad \sum_{l=0}^{p} (-1)^{p-l} \binom{p}{l} S_{l+1}^{(m)} = 0, \quad if\, p > m. \tag{5.7}$$

Proof: First we state that, for any square matrix A having nonzero entries under the diagonal, the first m rows of A^m are always zero. Thus, if $m < p$, the Lemma 5.4 yields for every $n \geq p$

$$\begin{aligned} 0 &= e_m^T (\mathcal{P}_n - I_n)^p e = \sum_{l=0}^{p} \binom{p}{l} (-1)^{p-l} e_m^T \mathcal{P}_n^l e \\ &= \sum_{l=0}^{p} (-1)^{p-l} \binom{p}{l} S_{l+1}^{(m)}. \qquad \square \end{aligned}$$

Theorem 5.7. *For any positive integers m and n, one has*

$$\sum_{k=1}^{n} S_k^{(m)} = \sum_{l=0}^{m} \sum_{p=l}^{m} (-1)^{p-l} \binom{n}{p+1} \binom{p}{l} S_{l+1}^{(m)}. \tag{5.8}$$

Proof: By Lemma 5.4 and 5.6, we have

$$\begin{aligned} \sum_{k=1}^{n} S_k^{(m)} &= \sum_{l=0}^{n-1} e_m^T \mathcal{P}_n^l e = \sum_{l=0}^{n-1} e_m^T (\mathcal{P}_n - I_n + I_n)^l e \\ &= \sum_{l=0}^{n-1} e_m^T \sum_{p=0}^{l} \binom{l}{p} (\mathcal{P}_n - I_n)^p e = \sum_{l=0}^{n-1} \sum_{p=0}^{n} \binom{l}{p} e_m^T (\mathcal{P}_n - I_n)^p e \\ &= \sum_{p=0}^{n} \left[\sum_{l=0}^{n-1} \binom{l}{p} \right] e_m^T (\mathcal{P}_n - I_n)^p e = \sum_{p=0}^{n} \binom{n}{p+1} e_m^T (\mathcal{P}_n - I_n)^p e \\ &= \sum_{p=0}^{m} \binom{n}{p+1} \sum_{l=0}^{p} (-1)^{p-l} \binom{p}{l} S_{l+1}^{(m)}. \qquad \square \end{aligned}$$

Remark 5.8. The identity (5.8) enabled us to reduce the summation to m summands, the values of which depend only on n for fixed m. The first two instances of identities (5.8) are as follows.

$$(a)\quad \sum_{k=1}^{n} S_k^{(1)} = \left[\binom{n}{1} - \binom{n}{2}\right] S_1^{(1)} + \binom{n}{2} S_2^{(1)},$$

$$(b)\quad \sum_{k=1}^{n} S_k^{(2)} = \left[\binom{n}{1} - \binom{n}{2} + \binom{n}{3}\right] S_1^{(2)} + \left[\binom{n}{2} - 2\binom{n}{3}\right] S_2^{(2)} + \binom{n}{3} S_3^{(2)}.$$

6. The exponential form for q-Pascal functional matrix

In this section we shall use two well-known formulas ([10, 12]):

(i) $$\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ r \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix}\begin{bmatrix} n-r \\ k-r \end{bmatrix}. \tag{6.1}$$

(ii) Euler's formula: For any positive integer n,

$$(x\dot{-}y)^n = \prod_{k=0}^{n-1}(x - q^k y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} x^{n-k} y^k, \tag{6.2}$$

$(x\dot{-}y)^0 = 1$.

Theorem 6.1. *Let x and y be any two nonzero real numbers. Then we have*

$$\mathcal{P}_n[x]\widetilde{\mathcal{P}}_n[y] = \widetilde{\mathcal{P}}_n[x\dot{-}y]. \tag{6.3}$$

Proof: Clearly, both $\mathcal{P}_n[x]\widetilde{\mathcal{P}}_n[y]$ and $\widetilde{\mathcal{P}}_n[x\dot{-}y]$ are lower triangular, and have all main diagonal entries equal to 1. So we assume $i > j$ and write $i = j + l$ with $l > 0$. Then

$$\begin{aligned}
(\mathcal{P}_n[x]\widetilde{\mathcal{P}}_n[y])_{i,j} &= \sum_{k=0}^{l} (\mathcal{P}_n[x])_{j+l,j+k} (\widetilde{\mathcal{P}}_n[y])_{j+k,j} \\
&= \sum_{k=0}^{l} \begin{bmatrix} j+l \\ j+k \end{bmatrix}\begin{bmatrix} j+k \\ j \end{bmatrix} q^{\binom{k}{2}} x^{l-k} (-y)^k \text{ (by(6.1))} \\
&= \sum_{k=0}^{l} \begin{bmatrix} j+l \\ j \end{bmatrix}\begin{bmatrix} l \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} x^{l-k} y^k \text{ (by(6.2))} \\
&= \begin{bmatrix} i \\ j \end{bmatrix} (x\dot{-}y)^l. \quad \square
\end{aligned}$$

Call and Velleman [5] found the exponential form of the Pascal matrix $P[x]$. Now, using the q-exponential function [10] $\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}$, we also can conclude that the q-Pascal functional matrix $\mathcal{P}_n[x]$ has an exponential form.

Theorem 6.2. *For any real number x, we have*

$$\mathcal{P}_n[x] = \exp_q(x\mathcal{L}_n), \tag{6.4}$$

where $\mathcal{L}_n$ is the $(n+1) \times (n+1)$ matrix with entries

$$\mathcal{L}_n(i,j) = \begin{cases} [j] & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in particular, if $q = 1$ in Theorem 6.2, then we can get the following results of Call and Velleman [5]: $P_n[x] = \exp(xL)$. If taking $x = 1$ in Theorem 6.2, we have $\mathcal{P}_n = \exp_q(\mathcal{L}_n)$. To prove Theorem 6.2, we will need the following Lemma.

Lemma 6.3. *For every positive integer k, the entries of $\mathcal{L}_n^k$ are given by the formula*

$$\mathcal{L}_n^k(i,j) = \begin{cases} [i]!/[j]! & \text{if } i = j+k, \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

Proof: By induction on k, it is easy to complete the proof. Note that for $k \geq n+1$ we have $\mathcal{L}_n^k = 0$. □

Proof of Theorem 6.2: Since $\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}$, $\mathcal{L}_n^k = 0$ for $k \geq n+1$, then we have

$$\exp_q(x\mathcal{L}_n) = \sum_{k=0}^{n} \frac{x^k}{[k]!}\mathcal{L}_n^k.$$

Applying Lemma 6.3, $\exp_q(x\mathcal{L}_n)$ is a lower triangular matrix, and the diagonal entries are all 1. Now suppose $i > j$, and let $i - j = k$. Then the only matrix in the sum above which has a nonzero (i,j) entry is $(x^k/[k]!)\mathcal{L}_n^k$, so

$$(\exp_q(x\mathcal{L}_n))_{i,j} = \frac{x^k}{[k]!}\mathcal{L}_n^k(i,j) = \frac{x^k}{[k]!}\frac{[i]!}{[j]!} = \begin{bmatrix} i \\ j \end{bmatrix} x^{i-j} = \mathcal{P}_n(x;i,j). \quad \square$$

Similarly to $\mathcal{P}_n[x]$, $\widetilde{\mathcal{P}}_n[x]$ also has an exponential form.

Theorem 6.4. *For any real number x, there holds*

$$\widetilde{\mathcal{P}}_n[x] = \exp_{1/q}(-x\mathcal{L}_n), \tag{6.6}$$

where $\exp_{1/q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} q^{\binom{n}{2}}$.

Proof: The proof is similar to the proof in Theorem 6.2, hence we omit it here.

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