Restrained Domination Excellent Trees

[1]Johannes H. Hattingh and [2]Michael A. Henning*

[1]Department of Mathematics and Statistics
Georgia State University
Atlanta, Georgia 30303, USA

[2]School of Mathematical Sciences
University of KwaZulu-Natal
Pietermaritzburg, 3209 South Africa

Abstract

A set S of vertices in a graph $G = (V, E)$ is a restrained dominating set of G if every vertex not in S is adjacent to a vertex in S and to a vertex in $V \setminus S$. The graph G is called restrained domination excellent if every vertex belongs to some minimum restrained dominating set of G. We provide a characterization of restrained domination excellent trees.

Keywords: restrained domination, excellent trees
AMS subject classification: 05C69

1 Introduction

For a graph $G = (V, E)$, a set S is a *dominating set* if every vertex in $V \setminus S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. We call a dominating set of cardinality $\gamma(G)$ a $\gamma(G)$-*set* and use similar notation for other parameters. Domination and its many variations have been surveyed in [7, 8].

*Research supported in part by the South African National Research Foundation and the University of KwaZulu-Natal.

In this paper we study a variation on the domination theme called restrained domination, introduced by Telle and Proskurowski [13], albeit indirectly, as a vertex partitioning problem and further studied in [1, 2, 3, 5, 6]. A set $S \subseteq V$ is a *restrained dominating set* (RDS) if every vertex in $V \setminus S$ is adjacent both to a vertex in S (i.e., S is a dominating set of G) and to a vertex in $V \setminus S$. Every graph has a RDS, since $S = V$ is such a set. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a RDS of G. Clearly, $\gamma(G) \leq \gamma_r(G)$. A RDS of G of cardinality $\gamma_r(G)$ we call a $\gamma_r(G)$-set.

Fricke, Haynes, Hedetniemi, Hedetniemi, and Laskar [4] defined a graph G to be *γ-excellent* if every vertex of G belongs to some $\gamma(G)$-set. They showed that the family of γ-excellent trees (trees where every vertex is in some minimum dominating set) is properly contained in the set of i-excellent trees (trees where every vertex is in some minimum independent dominating set). The γ-excellent trees have been characterized by Sumner [12], while the i-excellent trees have been characterized in [9] where it is shown that any such tree of order at least three can be constructed using a double-star as a base tree and recursively applying one of two operations. A constructive characterization of γ_t-excellent trees (trees where every vertex is in some minimum total dominating set) is given in [11]. In this paper, we present a characterization of γ_r-excellent trees (trees where every vertex is in some minimum RDS).

For notation and graph theory terminology we in general follow [7]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. The *open neighborhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and its closed neighborhood is the set $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$.

For ease of presentation, we mostly consider *rooted trees*. For a vertex v in a (rooted) tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *leaf* of T is a vertex of degree 1, while a *support vertex* of T is a vertex adjacent to a leaf. A *strong support vertex* is adjacent to at least two leaves. A *double star* is a tree with exactly two vertices that are not leaves.

For $k \geq 1$ an integer, a *k-branch* of a tree T is a path P in T of length k (and order $k+1$) that contains a leaf of T and such that every internal vertex of P has degree 2 in T. A *branch* of T is a k-branch of T for some $k \geq 1$. If P is a k-branch of T and P is a u–v path where u is a leaf of T, then we call v a *k-branch vertex* of T. In particular, a 1-branch vertex of T is a support vertex of T. By *attaching a k-branch* to a vertex v of a tree T, we mean adding to T a path of order k and joining a leaf of this path to v.

Our aim in this paper is present a characterization of γ_r-excellent trees. For this purpose, we introduce some additional notation. Let $G = (V, E)$ be a graph, and let $u \in V$.

Definition 1. We define the *restrained domination number of G relative to u*, denoted $\gamma_r^u(G)$, as the minimum cardinality of a RDS in G that contains u. A RDS of cardinality $\gamma_r^u(G)$ containing u we call a $\gamma_r^u(G)$-set. Hence, the graph G is γ_r*-excellent* if $\gamma_r^u(G) = \gamma_r(G)$ for every vertex $u \in V$.

Definition 2. We define a *type-I almost restrained dominating set* (type-I ARDS) of G *relative to* u as a set $S \subseteq V \setminus \{u\}$ such that S dominates $V \setminus \{u\}$ (possibly, S also dominates u) and $G[V \setminus S]$ has no isolated vertex.

Definition 3. We define a *type-II almost restrained dominating set* (type-II ARDS) of G *relative to* u as a set $S \subseteq V \setminus \{u\}$ such that S dominates V and $G[V \setminus S]$ has no isolated vertex, except possibly for the vertex u.

Definition 4. We define an *almost restrained dominating set* (ARDS) of G *relative to* u as a type-I ARDS or a type-II ARDS of G *relative to* u.

Definition 5. The *almost restrained domination number* of G *relative* to u, denoted $\gamma_r(G; u)$, is the minimum cardinality of an ARDS of G relative to u. An ARDS of G relative to u of cardinality $\gamma_r(G; u)$ we call a $\gamma_r(G; u)$-set. A type-I (respectively, type-II) ARDS of G relative to u of cardinality $\gamma_r(G; u)$ we call a type-I (respectively, type-II) $\gamma_r(G; u)$-set.

Definition 6. We say that G is *γ_r-excellent relative to u* if

- $\gamma_r^u(G) = \gamma_r(G; u) + 1$,
- there exists both a type-I and a type-II $\gamma_r(G; u)$-set, and
- every vertex of G is in some $\gamma_r^u(G)$-set or in some $\gamma_r(G; u)$-set.

2 The Family $\mathcal{F}$

We define the family $\mathcal{F}$ to consist of all trees that can be obtained from a γ_r-excellent tree T' relative to a vertex u of T' by adding a path P_6 to T' and joining u to a central vertex of this path. As an example, the tree T shown in Figure 1 and the tree T shown in Figure 2 both belong to the family $\mathcal{F}$. (Notice that in both examples, the component T' of $T - uv$ that contains the vertex u is a γ_r-excellent tree relative to a vertex u.)

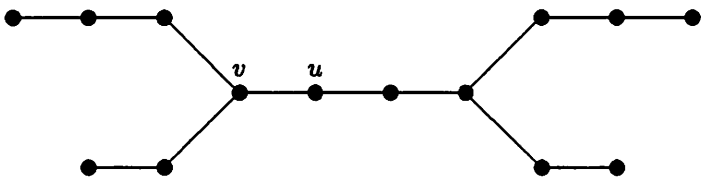

Figure 1: A γ_r-excellent tree T in $\mathcal{F}$.

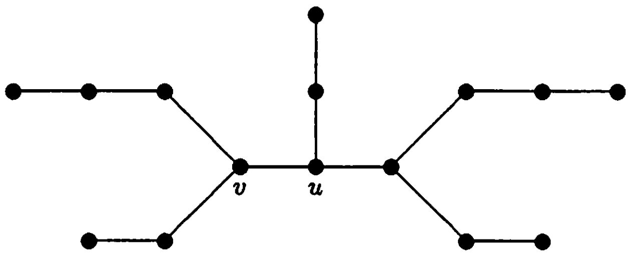

Figure 2: A γ_r-excellent tree T in $\mathcal{F}$.

3 The Family $\mathcal{T}$

In this section we construct a family of γ_r-excellent trees. Let $\mathcal{T}$ be the family of trees that contain $\{P_2\} \cup \mathcal{F}$ and is closed under the four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ listed below, which extend a tree T' by attaching one or more branches to a vertex of T', called its *attacher*. We say that a leaf v of a tree T is a *good leaf* if T has order at least 3 and there exists a $\gamma_r(T)$-set that contains $N[w]$ where w is the neighbor of v.

- **Operation $\mathcal{O}_1$.** Attach to a support vertex of T' a 1-branch.
- **Operation $\mathcal{O}_2$.** Attach to a good leaf of T' a 3-branch.
- **Operation $\mathcal{O}_3$.** Attach to a 2-branch vertex of degree 2 in T' a 2-branch and a 3-branch.
- **Operation $\mathcal{O}_4$.** Attach to a 3-branch vertex of degree at least 2 in T' both a 2-branch and a 3-branch.

If $T \in \mathcal{T}$, and T is obtained from a sequence $T_1, \ldots, T_m$ of trees where $T_1 = P_2$ and $T = T_m$, and, if $m \geq 2$, T_{i+1} can be obtained from T_i by operation $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ or $\mathcal{O}_4$ for $i = 1, \ldots, m-1$, then we say that T has *length* m in $\mathcal{T}$.

4 Preliminary Results

Lemma 1 *Let T be a tree that contains a strong support vertex. Let T' be obtained from T by deleting a leaf-neighbor of a strong support vertex. Then,* (i) $\gamma_r(T) = \gamma_r(T') + 1$, *and* (ii) *$T$ is γ_r-excellent if and only if T' is γ_r-excellent.*

Proof. Let w be a strong support vertex, and let $T' = T - v$ where v is a leaf-neighbor of w. Removing v from a $\gamma_r(T)$-set produces a RDS of T', and so $\gamma_r(T') \leq \gamma_r(T) - 1$. On the other hand, every RDS of T' can be extended to a RDS of T by adding to it the vertex v, and so $\gamma_r(T) \leq \gamma_r(T') + 1$. This establishes Statement (i). Statement (ii) follows readily from the observation that removing v from a $\gamma_r(T)$-set produces a $\gamma_r(T')$-set, while adding v to a $\gamma_r(T')$-set produces a $\gamma_r(T)$-set. □

As an immediate consequence of Lemma 1, we have the following result.

Corollary 1 *Let T' be a nontrivial tree and let T be obtained from T' by operation $\mathcal{O}_1$. Then,* (i) $\gamma_r(T) = \gamma_r(T') + 1$, *and* (ii) *$T$ is γ_r-excellent if and only if T' is γ_r-excellent.*

Lemma 2 *Let T' be a tree of order at least 3. Let v be a leaf of T' and let w be the neighbor of v. Let T be the tree obtained from T' by adding to it the path x, y, z and the edge vx. Then,* (i) $\gamma_r(T) = \gamma_r(T') + 1$, *and* (ii) *if T is γ_r-excellent, then T' is γ_r-excellent and there exists a $\gamma_r(T')$-set containing $N[w]$.*

Proof. We begin the proof by proving two claims.

Claim 1 $\gamma_r(T) \leq \gamma_r(T') + 1$.

Proof. Let S' be a $\gamma_r(T')$-set. Then, $v \in S'$ and $S' \cup \{z\}$ is a RDS of T, and so $\gamma_r(T) \leq |S' \cup \{z\}| = |S| + 1 = \gamma_r(T') + 1$. □

Claim 2 $\gamma_r(T') \leq \gamma_r(T) - 1$.

Proof. Let S be a $\gamma_r(T)$-set, and let $S' = S \cap V(T')$. If $v \in S$, then S' is a RDS of T' and $S \cap \{x, y, z\} = \{z\}$, and so $\gamma_r(T') \leq |S'| = |S| - 1 = \gamma_r(T) - 1$. If $v \notin S$ and $w \in S$, then $S \cap \{x, y, z\} = \{y, z\}$ and $S' \cup \{v\}$ is a RDS of T', and so $\gamma_r(T') \leq |S' \cup \{v\}| = |S'| + 1 = |S| - 1 = \gamma_r(T) - 1$. Hence we may assume that $v \notin S$ and $w \notin S$, for otherwise the desired result follows. Then, $\{x, y, z\} \subset S$. If $S' \cup \{v\}$ is a RDS of T', then, by Claim 1, $\gamma_r(T) - 1 \leq \gamma_r(T') \leq |S' \cup \{v\}| = |S'| + 1 = |S| - 2 = \gamma_r(T) - 2$, which is impossible. Hence, $S' \cup \{v\}$ is not a RDS of T'. However, $S' \cup \{v\}$ is a dominating set of T'. Thus, w must be isolated in $T'[S' \cup \{v\}]$. Hence, $S' \cup \{v, w\}$ is a RDS of T', and so $\gamma_r(T') \leq |S' \cup \{v, w\}| = |S'| + 2 = |S| - 1 = \gamma_r(T) - 1$. □

We now return to the proof of Lemma 2. Statement (i) of Lemma 2 is a consequence of Claims 1 and 2. To prove Statement (ii), suppose that T is γ_r-excellent. Let $u \in V(T')$ and let S be a $\gamma_r(T)$-set containing u. Let $S' = S \cap V(T')$. Proceeding exactly as in the proof of Claim 2, we can show that there is a $\gamma_r(T')$-set containing the set S' (either S' or $S' \cup \{v\}$ or $S' \cup \{v, w\}$ is a $\gamma_r(T')$-set). Hence there is a $\gamma_r(T')$-set containing the vertex u. Therefore, T' is γ_r-excellent.

Let D be a $\gamma_r(T)$-set containing the vertex x, and let $D' = D \cap V(T')$. Then, $\{x, y, z\} \subset D$ and $v \notin D$ (for otherwise, $D \setminus \{x, y\}$ is a RDS, contradicting the minimality of D), and so $w \notin D$. Thus as in the proof of Claim 2, the set $D' \cup \{v, w\}$ is a $\gamma_r(T')$-set containing $N[w]$. □

As a consequence of Lemma 2 we have the following result.

Corollary 2 *Let T' be a tree of order at least* 3 *and let T be obtained from T' by operation $\mathcal{O}_2$. Then,* (i) $\gamma_r(T) = \gamma_r(T') + 1$ *and* (ii) *T is γ_r-excellent if and only if T' is γ_r-excellent.*

Proof. Let v be the attacher of T' (and so, v is a leaf of T') and let w be the neighbor of v in T'. Statement (i) follows by Lemma 2(i). If T is γ_r-excellent, then by Lemma 2(ii), T' is γ_r-excellent. Suppose that T' is γ_r-excellent. Let $u \in V(T')$ and let S' be a $\gamma_r(T')$-set containing u. Then, $v \in S'$ and $S' \cup \{z\}$ is a $\gamma_r(T)$-set containing u. If $w \in S'$, then $(S' \setminus \{v\}) \cup \{y, z\}$ is a $\gamma_r(T)$-set containing y. It remains to show that there is a $\gamma_r(T)$-set containing x. Since v is a good leaf in T', there is a $\gamma_r(T)$-set S^* that contains $N[w]$. Thus the set $(S^* \setminus \{v, w\}) \cup \{x, y, z\}$ is a $\gamma_r(T)$-set containing the vertex x. Therefore, T is γ_r-excellent. This establishes Statement (ii). □

Lemma 3 *Let T be obtained from a tree T' by operation $\mathcal{O}_3$. Then,* (i)

$\gamma_r(T) = \gamma_r(T')+3$ *and* (ii) T *is* γ_r*-excellent if and only if* T' *is* γ_r*-excellent.*

Proof. Let v be the attacher of T' (and so, v is a 2-branch vertex of degree 2 in T'), and let v, x, y be a 2-branch of T'. Further, let $N(v) = \{u, x\}$. Let v, x_1', y_1' and v, x_1, y_1, z_1 be the 2-branch and 3-branch attached to v when applying operation $\mathcal{O}_3$ to obtain the tree T. We begin the proof by proving two claims.

Claim 3 $\gamma_r(T) \leq \gamma_r(T') + 3$.

Proof. Let S' be a $\gamma_r(T')$-set. If $v \in S'$, let $S = S' \cup \{x_1', y_1', z_1\}$, while if $v \notin S$, let $S = S' \cup \{y_1, y_1', z_1\}$. In both cases, S is a RDS of T and $|S| = |S'| + 3$, whence $\gamma_r(T) \leq |S| = \gamma_r(T') + 3$. □

Claim 4 $\gamma_r(T) \geq \gamma_r(T') + 3$.

Proof. Let S be a $\gamma_r(T)$-set, and let $S' = S \cap V(T')$. If $v \in S$, then S' is a RDS of T' and $S \setminus S' = \{x_1', y_1', z_1\}$, and so $\gamma_r(T') \leq |S'| = |S| - 3 = \gamma_r(T)-3$. Hence we may assume that $v \notin S$, for otherwise the desired result follows. Then, $\{y, y_1, y_1', z_1\} \subset S$. If $|S \cap \{x, x_1, x_1'\}| \geq 2$, then removing all but one vertex from this intersection produces a RDS of T, contradicting the minimality of S. Hence, $|S \cap \{x, x_1, x_1'\}| \leq 1$. If $S \cap \{x, x_1, x_1'\} = \emptyset$, then $|S'| = |S| - 3$ and S' is a RDS of T'. On the other hand, suppose $|S \cap \{x, x_1, x_1'\}| = 1$. Without loss of generality, we may assume $x \in S$. Thus, $|S'| = |S| - 3$. If $u \in S$, then $S \setminus \{x\}$ is a RDS of T, a contradiction. Hence, $S \cap N[v] = \{x\}$, and so S' is a RDS of T'. Thus, in both cases S' is a RDS of T' and $|S'| = |S| - 3$, whence $\gamma_r(T') \leq |S'| = \gamma_r(T) - 3$. □

Statement (i) of Lemma 3 is an immediate consequence of Claims 3 and 4.

Claim 5 *If* T' *is* γ_r*-excellent, then* T *is* γ_r*-excellent.*

Proof. Let $w \in V(T')$ and let S' be a $\gamma_r(T')$-set containing w. If $v \notin S'$, then $S' \cup \{y_1, y_1', z_1\}$ is a $\gamma_r(T)$-set containing w. If $v \in S'$, then $S' \cup \{x_1', y_1', z_1\}$ is a $\gamma_r(T)$-set containing w. Observe that if $w = u$, then $v \notin S'$, while if $w = v$, then $v \in S'$. Hence it remains to show that there is a $\gamma_r(T)$-set containing the vertex x_1. Let S^* be a $\gamma_r(T')$-set containing the vertex x. Suppose $v \in S^*$. If $u \in S^*$, then $S^* \setminus \{v, x\}$ is a RDS of T', contradicting the minimality of S^*. Hence, $u \notin S^*$. The set $(S^* \setminus \{v, x\}) \cup \{u\}$ is not a RDS of T' but is a dominating set of T'. Hence there must be a neighbor w of u such that $w \notin S^*$ and w is isolated in $T'[S^* \cup \{u\}]$. But then $(S^* \setminus \{v, x\}) \cup \{w\}$ is a RDS of T', contradicting the minimality of S^*. Hence, $v \notin S^*$. Thus, $(S^* \setminus \{x\}) \cup \{x_1, y_1, y_1', z_1\}$ is a RDS of T of cardinality $|S^*| + 3 = \gamma_r(T') + 3 = \gamma_r(T)$ and is therefore a $\gamma_r(T)$-set containing the vertex x_1. Therefore, T is γ_r-excellent. □

Claim 6 *If T is γ_r-excellent, then T' is γ_r-excellent.*

Proof. Let $w \in V(T')$ and let S be a $\gamma_r(T)$-set containing w. Let $S' = S \cap V(T')$. If $v \in S$, then $S \setminus S' = \{x_1', y_1', z_1\}$ and S' is a RDS of T' of cardinality $|S| - 3 = \gamma_r(T) - 3 = \gamma_r(T')$ and is therefore a $\gamma_r(T')$-set containing the vertex w. If $v \notin S$, then proceeding as in the proof of Claim 4, we can choose such a set S so that S' is a RDS of T' and $|S'| = |S| - 3$. Thus, S' is a $\gamma_r(T')$-set containing the vertex w. Hence, T' is γ_r-excellent. □

Statement (ii) of Lemma 3 is an immediate consequence of Claims 5 and 6. This completes the proof of Lemma 3. □

Lemma 4 *Let T be obtained from a tree T' by operation $\mathcal{O}_4$. Then,* (i) $\gamma_r(T) = \gamma_r(T') + 3$ *and* (ii) *T is γ_r-excellent if and only if T' is γ_r-excellent.*

Proof. Let v be the attacher of T' (and so, v is a 3-branch vertex of degree at least 2 in T'), and let v, x, y, z be a 3-branch of T'. Further, let $U = N(v) \setminus \{x\}$ in T'. Let v, x_1', y_1' and v, x_1, y_1, z_1 be the 2-branch and 3-branch, respectively, attached to v when applying operation $\mathcal{O}_4$ to obtain the tree T. We begin the proof by proving two claims.

Claim 7 $\gamma_r(T) \leq \gamma_r(T') + 3$.

Proof. Let S' be a $\gamma_r(T')$-set. If $v \in S'$, let $S = S' \cup \{x_1', y_1', z_1\}$, while if $v \notin S$, let $S = S' \cup \{y_1, y_1', z_1\}$. In both cases, S is a RDS of T, and so $\gamma_r(T) \leq |S| = |S'| + 3 = \gamma_r(T') + 3$. □

Claim 8 $\gamma_r(T) \geq \gamma_r(T') + 3$.

Proof. Let S be a $\gamma_r(T)$-set, and let $S' = S \cap V(T')$. If $v \in S$, then S' is a RDS of T' and $S \setminus S' = \{x_1', y_1', z_1\}$, and so $\gamma_r(T') \leq |S'| = |S| -$

$3 = \gamma_r(T) - 3$. Hence we may assume that $v \notin S$, for otherwise the desired result follows. Then, $\{y, y_1, y'_1, z, z_1\} \subset S$. If $|S \cap \{x, x_1, x'_1\}| \geq 2$, then removing all but one vertex from this intersection produces a RDS of T, contradicting the minimality of S. Hence, $|S \cap \{x, x_1, x'_1\}| \leq 1$. If $S \cap \{x, x_1, x'_1\} = \emptyset$, then $|S'| = |S| - 3$ and S' is a RDS of T'. On the other hand, suppose $|S \cap \{x, x_1, x'_1\}| = 1$. Without loss of generality, we may assume $x \in S$. Thus, $|S'| = |S| - 3$. If $|S \cap U| \geq 1$, then $S \setminus \{x\}$ is a RDS of T, a contradiction. Hence, $S \cap N[v] = \{x\}$, and so S' is a RDS of T'. Thus, in both cases S' is a RDS of T' and $|S'| = |S| - 3$, whence $\gamma_r(T') \leq |S'| = |S'| - 3 = \gamma_r(T) - 3$. □

Statement (i) of Lemma 4 is an immediate consequence of Claims 7 and 8.

Claim 9 *If T' is γ_r-excellent, then T is γ_r-excellent.*

Proof. Let $w \in V(T')$ and let S' be a $\gamma_r(T')$-set containing w. If $v \notin S'$, then $S' \cup \{y_1, y'_1, z_1\}$ is a $\gamma_r(T)$-set containing w. If $v \in S'$, then $S' \cup \{x'_1, y'_1, z_1\}$ is a $\gamma_r(T)$-set containing w. Observe that if $w = x$, then $v \notin S'$, while if $w = v$, then $v \in S'$. Hence it remains to show that there is a $\gamma_r(T)$-set containing the vertex x_1. Let S^* be a $\gamma_r(T')$-set containing the vertex x. Then, $\{y, z\} \subset S^*$. If $v \in S^*$, then $S^* \setminus \{x, y\}$ is a RDS of T', contradicting the minimality of S^*. Hence, $v \notin S^*$. Thus, $(S^* \setminus \{x\}) \cup \{x_1, y_1, y'_1, z_1\}$ is a RDS of T of cardinality $|S^*| + 3 = \gamma_r(T') + 3 = \gamma_r(T)$ and is therefore a $\gamma_r(T)$-set containing the vertex x_1. Hence, T is γ_r-excellent. □

Claim 10 *If T is γ_r-excellent, then T' is γ_r-excellent.*

Proof. Let $w \in V(T')$ and let S be a $\gamma_r(T)$-set containing w. Let $S' = S \cap V(T')$. If $v \in S$, then $S \setminus S' = \{x'_1, y'_1, z_1\}$ and S' is a RDS of T' of cardinality $|S| - 3 = \gamma_r(T) - 3 = \gamma_r(T')$ and is therefore a $\gamma_r(T')$-set containing the vertex w. If $v \notin S$, then proceeding as in the proof of Claim 8, we can choose such a set S so that S' is a RDS of T' and $|S'| = |S| - 3$. Thus, S' is a $\gamma_r(T')$-set containing the vertex w. Hence, T' is γ_r-excellent. □

Statement (ii) of Lemma 4 is an immediate consequence of Claims 9 and 10. This completes the proof of Lemma 4. □

Lemma 5 *If $T \in \mathcal{F}$, then T is a γ_r-excellent tree.*

Proof. The tree T can be obtained from a γ_r-excellent tree T' relative to a vertex u of T' by adding a path $P_6\colon a, b, c, d, e, f$ and the edge ud to T'.

We show first that $\gamma_r^u(T') = \gamma_r(T) - 3$. Every $\gamma_r^u(T')$-set can be extended to a RDS of T by adding to it the vertices in the set $\{a, b, f\}$, and so $\gamma_r(T) \leq \gamma_r^u(T') + 3$. Conversely, let S be a $\gamma_r(T)$-set and let S' be the restriction of S to the tree T', i.e., $S' = S \cap V(T')$. If $u \in S$, then S' is a RDS of T' containing u and $S \setminus S' = \{a, b, f\}$. Thus, $\gamma_r^u(T') \leq |S'| = |S| - 3 = \gamma_r(T) - 3$. If $u \notin S'$, then S' is an ARDS of T' relative to u and $|S \setminus S'| = 4$, and so $\gamma_r^u(T') - 1 = \gamma_r(T'; u) \leq |S'| = |S| - 4 = \gamma_r(T) - 4$, whence $\gamma_r^u(T') \leq \gamma_r(T) - 3$. Consequently, $\gamma_r^u(T') = \gamma_r(T) - 3$.

Let $v \in V(T')$. Since T' is a γ_r-excellent tree relative to the vertex u, there is a set S_v containing the vertex v that is a $\gamma_r^u(T')$-set or a $\gamma_r(T'; u)$-set. If S_v is a $\gamma_r^u(T')$-set, then the set $S_v \cup \{a, b, f\}$ is a $\gamma_r(T)$-set containing v. If S_v is a $\gamma_r(T'; u)$-set, then the set $S_v \cup \{a, d, e, f\}$ or the set $S_v \cup \{a, b, c, f\}$ is a $\gamma_r(T)$-set containing v (observe that $\gamma_r(T'; u) + 4 = \gamma_r^u(T') + 3 = \gamma_r(T)$). Hence every vertex in $V(T')$ is in some $\gamma_r(T)$-set.

It remains for us to establish that every vertex in $V(T) \setminus V(T')$ is in some $\gamma_r(T)$-set. Every $\gamma_r^u(T')$-set can be extended to a $\gamma_r(T)$-set by adding to it the vertices in the set $\{a, b, f\}$. Since T' is a γ_r-excellent tree relative to the vertex u, there exists both a type-I and a type-II $\gamma_r(T'; u)$-set. Let D_1 be a type-I $\gamma_r(T'; u)$-set and let D_2 be a type-II $\gamma_r(T'; u)$-set. Then, D_1 can be extended to a $\gamma_r(T)$-set by adding to it the vertices in the set $\{a, d, e, f\}$, while D_2 can be extended to a $\gamma_r(T)$-set by adding to it the vertices in the set $\{a, b, c, f\}$. Hence, every vertex in $V(T) \setminus V(T')$ is in some $\gamma_r(T)$-set. □

Lemma 6 *If $T \in \mathcal{T}$, then T is a γ_r-excellent tree.*

Proof. We proceed by induction on the length m of the sequence of trees needed to construct the tree $T \in \mathcal{T}$. If $m = 1$, then $T = P_2$, which is a γ_r-excellent tree, or $T \in \mathcal{F}$ and T is a γ_r-excellent tree by Lemma 6. This establishes the base case. Assume, then, that the result holds for all trees in $\mathcal{T}$ of length less than m, where $m \geq 2$. Let T be a tree of length m in $\mathcal{T}$. Thus, $T \in \mathcal{T}$ can be obtained from a sequence $T_1, T_2, \ldots, T_m$ of m trees where $T_1 = P_2$ and $T = T_m$, and for $i = 1, \ldots, m-1$, the tree T_{i+1} can be obtained from T_i by operation $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ or $\mathcal{O}_4$. Applying the inductive hypothesis to the tree $T' = T_{m-1} \in \mathcal{T}$, we have that T' is a γ_r-excellent tree. Since the tree T is obtained from T' by operation $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ or $\mathcal{O}_4$, the desired result follows from Corollary 1, Corollary 2, Lemma 3 and Lemma 4. □

5 Main Result

Theorem 7 *A nontrivial tree T is a γ_r-excellent tree if and only if $T \in \mathcal{T}$.*

Proof. The sufficiency follows from Lemma 6. To prove the necessity, we proceed by induction on the order $n \geq 2$ of a γ_r-excellent tree T. If $n = 2$, then $T = P_2 \in \mathcal{T}$. This establishes the base case. Assume then that $n \geq 3$ and that if T' is a γ_r-excellent tree of order at least 2 but less than n, then $T' \in \mathcal{T}$. Let T be a γ_r-excellent tree of order n. If T is a star, then T can be obtained from P_2 by repeated applications of operation $\mathcal{O}_1$, and so $T \in \mathcal{T}$. Hence we may assume that $\operatorname{diam} T \geq 3$.

If $\operatorname{diam} T = 3$, then T is a double star and the set of leaves of T is a unique $\gamma_r(T)$-set. But then T is not γ_r-excellent, a contradiction. Hence, $\operatorname{diam} T \geq 4$ (and so, $n \geq 5$).

Suppose that T has a strong support vertex w with v as one of its leaf-neighbors. Let $T' = T - v$. By Lemma 1, T' is γ_r-excellent. Applying the inductive hypothesis to T', we have $T' \in \mathcal{T}$. We can now restore the tree T from T' by applying operation $\mathcal{O}_1$ with w as the attacher, and so $T \in \mathcal{T}$. Thus we may assume that T has no strong support vertex, for otherwise $T \in \mathcal{T}$ as required.

Let T be rooted at a leaf r of a longest path P. Let P be a r–z path, and let y be the neighbor of z. Then, z is a leaf and y is a support vertex. By assumption, $\deg_T y = 2$. Let x denote the parent of y on this path, v denote the parent of x, and u the parent of v. We show that $\deg_T x = 2$.

Claim 11 $\deg_T x = 2$.

Proof. Suppose that $\deg_T x \geq 3$ and consider a $\gamma_r(T)$-set S that contains the vertex x. Then, every descendant of x is in S. In particular, $\{y, z\} \subset S$. If $v \in S$, then $S \setminus \{x, y\}$ is a RDS of T, contradicting the minimality of S. Hence, $v \notin S$. Let $S' = (S \setminus \{x, y\}) \cup \{v\}$. By the minimality of S, the set S' is not a RDS. However, S' is a dominating set. This implies that there exists a vertex $w \notin S'$ such that $N(w) \subset S'$. Necessarily, $w \in N(v) \setminus \{x\}$. It now follows that $(S \setminus \{x, y\}) \cup \{w\}$ is a RDS of T, contradicting the minimality of S. Hence, $\deg_T x = 2$. □

By Claim 11, every vertex at distance $\operatorname{diam}(T) - 2$ from r on a longest path emanating from r has degree 2.

Suppose that $\deg_T v = 2$. Let $T' = T - \{x, y, z\}$. Then, v is a leaf of T'. By Lemma 2(ii), the tree T' is γ_r-excellent. Applying the inductive hypothesis to T', we have $T' \in \mathcal{T}$. By Lemma 2(ii), there exists a $\gamma_r(T')$-set containing $N[u]$. Hence the vertex v is a good leaf in the tree T'. Thus we can restore the tree T from T' by applying operation $\mathcal{O}_2$ with v as the attached vertex, and so $T \in \mathcal{T}$. Thus we may assume that $\deg_T v \geq 3$, for otherwise $T \in \mathcal{T}$ as required.

Claim 12 *The vertex v is not a support vertex.*

Proof. Suppose that v is a support vertex of T. Let S be a $\gamma_r(T)$-set that contains x. Then, $\{x, y, z\} \subset S$. Since S contains the leaf-neighbor of v, the set $S \setminus \{x\}$ is a RDS of T, contradicting the minimality of S. □

By Claim 12 and our earlier assumption that there is no strong support vertex, the maximal subtree T_v of v can be obtained from a star $K_{1,k+\ell}$, where $k + \ell \geq 2$ and $k \geq 1$, by subdividing k edges exactly twice and subdividing ℓ edges exactly once. For $i = 1, \ldots, k$, let v, x_i, y_i, z_i denote the 3-branches attached to v in T_v (where $x_1 = x$, $y_1 = y$ and $z_1 = z$), and if $\ell \geq 1$, then for $i = 1, \ldots, \ell$, let v, x'_i, y'_i denote the 2-branches attached to v in T_v. Let X, Y and Z denote the set of vertices at distance 1, 2 and 3, respectively, from v in T_v. If $\ell \geq 1$, let $X' = \{x'_1, \ldots, x'_\ell\}$ and $Y' = \{y'_1, \ldots, y'_\ell\}$.

Claim 13 $\ell \leq k + 1$.

Proof. Suppose that $\ell \geq k + 2$. Let S be a $\gamma_r(T)$-set that contains the vertex v. Then, $S \cap V(T_v) = X' \cup Y' \cup Z$. Let $S' = (S \setminus (X' \cup \{v\}) \cup (Y \setminus Y') \cup \{u, x'_\ell\}$. Then, S' is a dominating set of T with $|S'| \leq |S| - 1 = \gamma_r(T) - 1$. By the minimality of the set S, the set S' is not a RDS of T. This implies that there exists a vertex $w \notin S'$ such that $N(w) \subset S'$. Since $N(w) \not\subset S$, it follows that $u \notin S$ and $w \in N(u) \setminus \{v\}$. This implies that $(S' \setminus \{u\}) \cup \{w\}$ is a RDS of T of cardinality $\gamma_r(T) - 1$ (observe that the vertex v is dominated by the vertex $x'_\ell \in S'$ in this RDS), contradicting the minimality of S. Hence, $\ell \leq k + 1$. □

Claim 14 $k - 1 \leq \ell$.

Proof. Suppose that $\ell \leq k - 2$. Let S be a $\gamma_r(T)$-set that contains the vertex x_1. If $v \in S$, then $S \setminus \{x_1, y_1\}$ is a RDS of T, contradicting the minimality of S. Hence, $v \notin S$. If $u \in S$, then $S \setminus \{x_1\}$ is a RDS of T, a contradiction. Hence, $u \notin S$. Further by the minimality of S, $S \cap V(T_v) = \{x_1\} \cup Y \cup Z$. Let $S' = (S \setminus \{x_1, y_1, \ldots, y_k\}) \cup X' \cup \{v\}$. Then, S' is a dominating set of T with $|S'| = |S| - (k+1) + (\ell+1) = \gamma_r(T) - k + \ell \leq \gamma_r(T) - 2$. By the minimality of the set S, the set S' is not a RDS of T. This implies that there exists a vertex $w \notin S'$ such that $N(w) \subset S'$. Since $N(w) \not\subset S$ and $u \notin S$, it follows that $u = w$. But then $S' \cup \{u\}$ is a RDS of T of cardinality $\gamma_r(T) - 1$, contradicting the minimality of S. Hence, $k - 1 \leq \ell$. □

By Claim 13 and Claim 14, $k - 1 \leq \ell \leq k + 1$.

If $\ell = k - 1$, then let T' be obtained from T by deleting all vertices in T_v except for the path v, x, y, z. Hence, T can be obtained from the tree T' by repeatedly applying operation $\mathcal{O}_4$. By Lemma 4, the tree T' is γ_r-excellent. Applying the inductive hypothesis to T', we have $T' \in \mathcal{T}$. Thus we can restore the tree T from T' by repeatedly applying operation $\mathcal{O}_4$ with v as the attached vertex, and so $T \in \mathcal{T}$.

If $\ell = k + 1$, then let T' be obtained from T by deleting all vertices in T_v except for the path v, x'_1, y'_1. Hence T can be obtained from the tree T' by first applying operation $\mathcal{O}_3$ with v as the attached vertex and then, if $k \geq 2$, by repeatedly applying operation $\mathcal{O}_4$ with v as the attached vertex. By Lemma 3 and Lemma 4, the tree T' is γ_r-excellent. Applying the inductive hypothesis to T', we have $T' \in \mathcal{T}$. Thus we can restore the tree T from T' by applying operation $\mathcal{O}_3$ and repeatedly applying operation $\mathcal{O}_4$ with v as the attached vertex, and so $T \in \mathcal{T}$.

If $\ell = k \geq 2$, then let T' be obtained from T by deleting all vertices in T_v except for the path v, x, y, z and the path v, x'_1, y'_1. Hence, T can be obtained from the tree T' by repeatedly applying operation $\mathcal{O}_4$. By Lemma 4, the tree T' is γ_r-excellent. Applying the inductive hypothesis to T', we have $T' \in \mathcal{T}$, whence $T \in \mathcal{T}$.

Hence we may assume that $k = \ell = 1$, for otherwise $T \in \mathcal{T}$ as desired. Thus the maximal subtree T_v rooted at v is the path z, y, x, v, x'_1, y'_1. Let $T' = T - V(T_v)$. We proceed further with the following two claims.

Claim 15 $\gamma_r^u(T') = \gamma_r(T'; u) + 1 = \gamma_r(T) - 3$.

Proof. Let S_u be a $\gamma_r(T)$-set containing the vertex u, and let $S'_u = S_u \cap V(T')$. Then, S'_u is a RDS of T' containing u and $S_u \setminus S'_u = \{y, y'_1, z\}$. Thus, $\gamma_r^u(T') \leq |S'_u| = |S_u| - 3 = \gamma_r(T) - 3$. On the other hand, every

$\gamma_r^u(T')$-set can be extended to a RDS of T by adding to it the vertices in the set $\{y, y_1', z\}$, and so $\gamma_r(T) \le \gamma_r^u(T')+3$. Consequently, $\gamma_r^u(T') = \gamma_r(T)-3$.

Let S_x be a $\gamma_r(T)$-set containing the vertex x, and let $S_x' = S_x \cap V(T')$. Then, $u \notin S_x$ and $S_x \backslash S_x' = \{x, y, y_1', z\}$. Thus, $\gamma_r(T'; u) \le |S_x'| = |S_x|-4 = \gamma_r(T) - 4$. On the other hand, let D be a $\gamma_r(T'; u)$-set. Then, $u \notin D$, and D dominates $V(T')$ or $T'[V \setminus D]$ has no isolated vertex. If D dominates $V(T')$, let $D^* = D \cup \{x, y, y_1', z\}$. If $T'[V \setminus D]$ has no isolated vertex, let $D^* = D \cup \{v, x_1', y_1', z\}$. In both cases, D^* is a RDS of T, and so $\gamma_r(T) \le |D^*| = \gamma_r(T'; u) + 4$. Consequently, $\gamma_r(T'; u) = \gamma_r(T) - 4$. □

Claim 16 *The tree T' is γ_r-excellent relative to u.*

Proof. By Claim 15, $\gamma_r^u(T') = \gamma_r(T'; u) + 1$. Let S_v be a $\gamma_r(T)$-set containing the vertex v, and let $S_v' = S_v \cap V(T')$. Then, S_v' is a type-I $\gamma_r(G; u)$-set. Let S_x be a $\gamma_r(T)$-set containing the vertex x, and let $S_x' = S_x \cap V(T')$. Then, S_x' is a type-II $\gamma_r(G; u)$-set. It remains for us to show that every vertex of T' is in some $\gamma_r^u(T')$-set or in some $\gamma_r(T'; u)$-set. Let $w \in V(T')$. Let S_w be a $\gamma_r(T)$-set containing w and let $S_w' = S_w \cap V(T')$. If $u \in S_w$, then $S_w \setminus S_w' = \{y, y_1', z\}$. Thus, S_w' is a RDS of T' containing u with $|S_w'| = \gamma_r(T) - 3$. Hence, by Claim 15, S_w' is a $\gamma_r^u(T')$-set. If $u \notin S_w$, then $|S_w \setminus S_w'| = 4$ and S_w' is a ARDS of T' relative to u with $|S_w'| = \gamma_r(T) - 4$. Hence, by Claim 15, S_w' is a $\gamma_r(T'; u)$-set. □

By Claim 16, the tree T' is γ_r-excellent relative to u. Since the tree T can be obtained from T' by adding the path z, y, x, v, x_1', y_1' and the edge uv, we have that $T \in \mathcal{F} \subset \mathcal{T}$. □

References

[1] G. S. Domke, J. H. Hattingh, S. T. Hedetniemi, R. C. Laskar, and L. R. Markus, Restrained domination in graphs. *Discrete Math.* **203** (1999), 61–69.

[2] G. S. Domke, J. H. Hattingh, M. A. Henning, and L. R. Markus, Restrained domination in graphs with minimum degree two. *J. Combin. Math. Combin. Comput.* **35** (2000), 239–254.

[3] G. S. Domke, J. H. Hattingh, M. A. Henning, and L. R. Markus, Restrained domination in trees. *Discrete Math.* **211** (2000), 1–9.

[4] G. H. Fricke, T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, and R. C. Laskar, Excellent trees. *Bulletin of ICA* **34** (2002), 27–38.

[5] J. H. Hattingh and M. A. Henning, Characterisations of trees with equal domination parameters. *J. Graph Theory* **34** (2000), 142–153.

[6] M. A. Henning, Graphs with large restrained domination number. 16th British Combinatorial Conference (London, 1997). *Discrete Math.* **197/198** (1999), 415–429.

[7] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[8] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[9] T. W. Haynes and M. A. Henning, A characterization of i-excellent trees. *Discrete Mathematics* **248** (2002), 69–77.

[10] M.A. Henning, Graphs with large total domination number. *J. Graph Theory* **35(1)** (2000), 21–45.

[11] M.A. Henning, Total domination excellent trees. *Discrete Math.* **263** (2003), 93–104.

[12] D. Sumner, talk presented at the *Sixteenth Cumberland Conference on Graph Theory, Combinatorics, and Computing* held at Georgia State University, Atlanta, USA, May 2003.

[13] J. A. Telle and A. Proskurowski, Algorithms for vertex partitioning problems on partial k-trees. *SIAM J. Discrete Math.* **10** (1997), 529–550.