A Cube–Packing Problem

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Abstract

In this paper, we discuss a problem on packing a unit cube with smaller cubes, which is a generalization of one of Erdős favorite problems: square–packing problem. We first give the definition of the packing function $f_3(n)$, then give the bounds for $f_3(n)$.

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In 1932, Erdős posed one of his favorite problems on square–packing which was included in [1]: Let S be a unit square. Inscribe n squares with no common interior point. Denote by $e_1, e_2, \cdots, e_n$ the side lengths of these squares. Put $f(n) = max \sum_{i=1}^{n} e_i$. In [2], P. Erdős and Soifer gave some results on $f(n)$.

We [3] generated it to the case using equilateral triangles (isosceles right triangles) to pack a unit equilateral triangle (an isosceles right triangle with legs of length 1). In this paper, we generalize this kind of problem to the case in 3 dimensions, that is, using cubes to pack a unit cube, and obtain corresponding results.

We first give the definition of the packing function:

Definition 1. Let C be a unit cube in 3 dimensions. Inscribe n cubes $C_1, C_2, \cdots, C_n$ with no common interior point in such a way which satisfies: C_i has side of length $c_i (0 < c_i \leq 1)$ and is placed so that its sides are parallel to those of C.

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Define $f_3(n) = max \sum_{i=1}^{n} c_i$.

In the same way, we can define $f_d(n)$ in d dimensions.
We first discuss the upper bounds for $f_3(n)$.

Proposition 2. *The following estimates are true for all positive integers* n*:*
(1) $f_3(n) \leq f_3(n+1)$*;*
(2) $f_3(n) < f_3(n+k)(k = 2, 3, 4, \cdots)$*.*

Proof. It's easy to get the results by replacing a cube C_i with 2 or $k+1$ cubes with sides of length $\frac{c_i}{2}$. □

So $f_3(n)$ is a nondecreasing function. To give an upper bound for $f_3(n)$, we need the following lemma which is a well known result.

Lemma 3. *(Hölder's Inequality) Let* ξ, η *be two random variables,* $1 < p < \infty, 1 < q < \infty$ *and* $1/p + 1/q = 1$*. If* $E|\xi|^p < \infty, E|\eta|^q < \infty$*, then*

$$E|\xi\eta| \leq (E(|\xi|^p))^{1/p} (E(|\xi|^q))^{1/q}, \tag{0.1}$$

where $E(\cdot)$ *denotes mathematical expectation.*

Lemma 4. *Let* $c_1, \cdots, c_n$ *be* n *positive numbers, then*

$$\sum_{i=1}^{n} c_i \leq (\sum_{i=1}^{n} c_i^3)^{1/3} n^{2/3}. \tag{0.2}$$

Proof. Let $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be a sample space. $P(\omega_i) = 1, i = 1, 2, \cdots, n$ is a probability defined on Ω. ξ, η are two random variables with

$$\xi(\omega_i) = c_i, i = 1, \cdots, n$$
$$\eta(\omega_i) = 1, i = 1, 2, \cdots, n.$$

From (0.1), we know

$$\frac{1}{n}\sum_{i=1}^{n} c_i = E\xi = E\xi\eta \leq (E\xi^3)^{1/3} E(\eta^{3/2})^{2/3}$$
$$= (\frac{1}{n}\sum_{i=1}^{n} c_i^3)^{1/3} (\frac{1}{n}\sum_{i=1}^{n} 1^{3/2})^{2/3} = n^{-1/3}(\sum_{i=1}^{n} c_i^3)^{1/3}.$$

Then we get $\sum_{i=1}^{n} c_i \leq (\sum_{i=1}^{n} c_i^3)^{1/3} n^{2/3}$. □

Theorem 5. $f_3(n) \leq n^{\frac{2}{3}}$.

Proof. Let c_i denote the side length of the cube C_i in the packing. Then $\sum_{i=1}^{n} c_i^3 \leq 1$. It follows easily from (0.2) that $\sum_{i=1}^{n} c_i \leq n^{\frac{2}{3}}$, so $f_3(n) \leq n^{\frac{2}{3}}$. □

In a similar way, in the proof of Lemma 4, take $p = d, q = \frac{d}{d-1}$, then we can get the following theorem:

Theorem 6. $f_d(n) \leq n^{\frac{d-1}{d}}$.

Definition 7. For a cube C, dissect each of its sides into n equal parts, then through these dissecting points draw parallel surfaces of the surfaces of C, so we get a packing of C by n^3 cubes with sides of length $\frac{1}{n}$. Such a configuration is called an n^3–grid. When C is a unit cube, the packing is a standard n^3–packing.

See Figure 1 for the case $n = 3$.

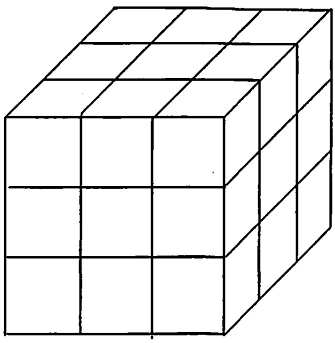

Figure 1: a 3^3-grid

Proposition 8. $f_3(k^3) = k^2$.

Proof. By Definition 6, it's easy to know that for the standard k^3–packing, $n = k^3, c_i = \frac{1}{k}$ and $\sum_{i=1}^{n} c_i = \frac{1}{k} \times k^3 = k^2$. So by the definition of $f_3(n)$, $f_3(k^3) \geq k^2$ which along with Theorem 5 provides the desired equality. □

For $1 < n \leq 7$, we can give the following results.

Theorem 9. $f_3(2) = 1$.

Proof. Let S_1, S_2 be any two small cubes with sides x_1, x_2 which are packed in the unit cube S. Simply observe that the 2 cubes may be moved to rest on a common face of the unit cube. The result then follows from the 2-dimensional consideration. $x_1 + x_2 \leq 1$. It follows that $f_3(2) \leq 1$.

Consider the standard 2^3−packing, that is, dissect the unit cube into 8 congruent small cubes with side length $\frac{1}{2}$. The sum of sides length of any two small cubes is 1.

So $f_3(2) = 1$. □

Theorem 10. *If* $1 < n \leq 7$, *then* $f_3(n) = \frac{n}{2}$;

Proof. We use induction on n.

By Theorem 8, when $n = 2$, $f_3(n) = \frac{n}{2}$. Suppose $f_3(k) = \frac{k}{2}$ holds when $n = k < 7$.

When $n = k+1$, observe that it is impossible for all $k+1$ cubes to have side length larger than $\frac{1}{2}$. Then remove a cube of side length less than or equal to $\frac{1}{2}$. Then we have $f_3(k+1) \leq f_3(k) + \frac{1}{2} = \frac{(k+1)}{2}$.

Dissect the unit cube into 8 congruent small cubes with side length $\frac{1}{2}$, and take $(k+1)$ of them, then the sum of their sides length is $\frac{(k+1)}{2}$.

So $f(k+1) = \frac{(k+1)}{2}$.

By induction, when $n = 2, 3, \cdots, 8$, $f_3(n) = \frac{n}{2}$ holds. □

Now we discuss the lower bounds for $f_3(n)$.

Proposition 11. *For* $k \geq 2$, $f_3(k^3 - 1) \geq k^2 - \frac{1}{k}$.

Proof. Consider the standard k^3-packing with one cube removed. □

Proposition 12. *For any positive integer* n, $f_3(n) \geq (n^{\frac{1}{3}} - 1)^2$.

Proof. By Proposition 7, $f_3(k^3) = k^2$. For any positive integer n, there exists an integer k such that $k^3 \leq n \leq (k+1)^3$. So $f_3(n) \geq f_3(k^3) = k^2 = (k+1-1)^2$. Since the function $f(x) = (x-1)^2$ is increasing in the interval $[1, +\infty)$, $f_3(n) \geq (n^{\frac{1}{3}} - 1)^2$.

□

In the same way, we can generalize the above results to those in d dimensions.

We can easily see that the lower bound for $f_3(n)$ is not good at all, but we can't do better now. We can only give the lower bounds for some exact values.

Proposition 13. $f_3(20) \geq 7$.

Proof. Consider the packing with 20 cubes which can be obtained from a standard 3^3–packing by replacing a 2^3–grid with a single cube whose side is $\frac{2}{3}$. So $\sum_{i=1}^{20} c_i = (27-8) \times \frac{1}{3} + \frac{2}{3} = 7$. □

Proposition 14. *When* $1 \leq k \leq 27$, $f_3(27+7k) \geq 9+k$.

Proof. Begin with the standard 3^3–packing and replace each of k cubes with 8 cubes each of whose side length is $\frac{1}{6}$. $\sum_{i=1}^{27+7k} c_i = 9 + k \times (\frac{1}{6} \times 8 - \frac{1}{3}) = 9 + k$. Then we get the result.

□

Proposition 15. $f_3(38) \geq 10$.

Proof. Consider the packing with 38 cubes which can be obtained from a standard 4^3–packing by replacing a 3^3–grid with a single cube whose side length is $\frac{3}{4}$. So $\sum_{i=1}^{38} c_i = (64-27) \times \frac{1}{4} + \frac{3}{4} = 10$. □

Proposition 16. $f_3(45) \geq \frac{49}{4}$.

Proof. First construct a packing with 38 cubes as in Proposition 14, then replace the largest cube with 8 smaller cubes each of whose side is $\frac{3}{8}$. So $\sum_{i=1}^{45} c_i = (64-27) \times \frac{1}{4} + 8 \times \frac{3}{8} = \frac{49}{4}$. □

Proposition 17. *When* $1 \leq k \leq 8$, $f_3(45+7k) \geq \frac{49}{4} + \frac{9k}{8}$.
When $9 \leq k \leq 45$, $f_3(45+7k) \geq \frac{61}{4} + \frac{3k}{4}$.

Proof. We first construct a packing of 45 cubes as in Proposition 15. When $1 \leq k \leq 8$, we replace each of k cubes with side length $\frac{3}{8}$ by 8 cubes with side length $\frac{3}{16}$. $\sum_{i=1}^{45+7k} c_i = \frac{49}{4} + k \times (\frac{3}{16} \times 8 - \frac{3}{8}) = \frac{49}{4} + \frac{9k}{8}$. When $9 \leq k \leq 45$, we first replace each of 8 cubes with side length $\frac{3}{8}$ by 8 cubes with side length $\frac{3}{16}$, then replace each of $k-8$ cubes with side length $\frac{1}{4}$ by 8 cubes with side length $\frac{1}{8}$. So $\sum_{i=1}^{45+7k} c_i = \frac{49}{4} + \frac{9}{8} \times 8 + (k-8) \times (\frac{1}{8} \times 8 - \frac{1}{4}) = \frac{61}{4} + \frac{3k}{4}$.

□

Proposition 18. $f_3(39) \geq 11$.

Proof. We begin with constructing a packing with 20 cubes as in Proposition 12, then replace the largest cube with 20 smaller cubes constructed as in Proposition 12. So $\sum_{i=1}^{39} c_i = 19 \times \frac{1}{3} + \frac{2}{3} \times \frac{1}{3} \times 19 + \frac{2}{3} \times \frac{2}{3} = 11$. □

Proposition 19. $f_3(46) \geq \frac{37}{3}$.

Proof. First construct a standard 3^3-packing, then replace a 2^3-grid of 8 cubes each of whose side is $\frac{1}{3}$ with a 3^3-grid of 27 cubes each of whose side length is $\frac{2}{9}$. So $\sum_{i=1}^{46} c_i = 19 \times \frac{1}{3} + \frac{2}{3} \times \frac{1}{3} \times 27 = \frac{37}{3}$. □

Proposition 20. *When* $1 \leq k \leq 19$, $f_3(46 + 7k) \geq \frac{37}{3} + k$;
When $20 \leq k \leq 46$, $f_3(46 + 7k) \geq \frac{94}{3} + \frac{(k-19)}{3}$.

Proof. We first construct a packing of 46 cubes as in Proposition 18. When $1 \leq k \leq 19$, we replace each of k cubes with side length $\frac{1}{3}$ by 8 smaller cubes with side length $\frac{1}{6}$. And $\sum_{i=1}^{46+7k} c_i = \frac{37}{3} + k \times (\frac{1}{6} \times 8 - \frac{1}{3}) = \frac{37}{3} + k$. When $19 \leq k \leq 45$, we replace each of 19 cubes with side length $\frac{1}{3}$ by 8 cubes each of with side length $\frac{1}{6}$ and replace each of $(k - 19)$ cubes with side length $\frac{2}{9}$ by 8 cubes each of whose side length is $\frac{1}{9}$. And $\sum_{i=1}^{46+7k} c_i = \frac{37}{3} + 19 \times (\frac{1}{6} \times 8 - \frac{1}{3}) + (k - 19) \times \frac{1}{3} = \frac{94}{3} + \frac{(k-19)}{3}$. Then we can get the results.

□

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