How close to regular must a multipartite tournament be to secure a given path covering number?

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Abstract

If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and the indegree of x, respectively. The global irregularity of a digraph D is defined by $i_g(D) = \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}$ over all vertices x and y of D (including $x = y$).

A c-partite tournament is an orientation of a complete c-partite graph. Recently, Volkmann and Winzen [9] proved that c-partite tournaments with $i_g(D) = 1$ and $c \geq 3$ or $i_g(D) = 2$ and $c \geq 5$ contain a Hamiltonian path. Furthermore, they showed that these bounds are best possible.

Now, it is a natural question to generalize this problem by asking for the minimal value $g(i, k)$ with $i, k \geq 1$ arbitrary such that all c-partite tournaments D with $i_g(D) \leq i$ and $c \geq g(i, k)$ have a path covering number $pc(D) \leq k$. In this paper, we will prove that $4i - 4k \leq g(i, k) \leq 4i - 3k - 1$, when $i \geq k + 2$. Especially in the case $k = 1$, this yields that $g(i, 1) = 4i - 4$, which means that all c-partite tournaments D with the global irregularity $i_g(D) = i$ and $c \geq 4i - 4$ contain a Hamiltonian path.

Keywords: Multipartite tournaments; Path covering number; Hamiltonian path

1 Terminology and introduction

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y, and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the fact that there is no arc leading from Y to X. By $d(X,Y)$ we denote the number of arcs from the set X to the set Y, i.e.,

$$d(X,Y) = |\{xy \in E(D) : x \in X, y \in Y\}|.$$

If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if there is the arc $xy \in E(D)$, then y is an *outer neighbor* of x and x is an *inner neighbor* of y. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and *indegree* of x, respectively. For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. A *cycle* or *path* here is always a directed cycle or directed path, and a cycle of length n is called an *n-cycle.* A cycle or path of a digraph D is *Hamiltonian,* if it includes all the vertices of D. The *path covering number,* $pc(D)$, of a digraph D is the minimum number of paths in D that are pairwise vertex disjoint and cover the vertices of D. A *factor* is a spanning subgraph of a digraph. A factor is a *k-path-cycle,* if it consists of a set of vertex disjoint paths and cycles, where k stands for the number of paths in the set.

There are several measures of how much a digraph differs from being regular. In [12], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x \in V(D)}\{d^+(x), d^-(x)\} - \min_{y \in V(D)}\{d^+(y), d^-(y)\}$$

and the *local irregularity* as $i_l(D) = \max|d^+(x) - d^-(x)|$ over all vertices x of D. Clearly, $i_l(D) \le i_g(D)$. If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \le 1$, then D is called *almost regular.*

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. A *semicomplete multipartite digraph* is obtained by replacing each edge of a complete multipartite graph by an arc or by a pair of two mutually opposite arcs. If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament or semicomplete c-partite digraph D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$. If D is a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_i| = n_i$ for $i = 1, 2, \ldots, c$, then we speak of the *partition-sequence* $(n_i) = n_1, n_2, \ldots, n_c$.

In 1934, Rédei [5] showed that every tournament contains a Hamiltonian path. Clearly, for multipartite tournaments, this result becomes false. Hence, an interesting question is to find sufficient conditions for a multipartite tournament to contain a Hamiltonian path. Noting that $pc(D) \leq 1$ means that D contains a Hamiltonian path, we recognize that in 1988 Gutin, respectively Gutin and Yeo in 2000, gave a characterization of semicomplete multipartite digraphs having this property.

Theorem 1.1 (Gutin [2], Gutin, Yeo [4]) *If D be a semicomplete multipartite digraph, then $pc(D) \leq k$ $(k \geq 1)$ holds if and only if D contains a k-path-cycle factor.*

In this paper, we will study the existence of a k-path-cycle factor depending on the global irregularity of a multipartite tournament D. A first result in this direction was made by Zhang [13] in 1989.

Theorem 1.2 (Zhang [13]) *Every regular c-partite tournament D with $c \geq 2$ contains a Hamiltonian path.*

This result was improved by Yeo [11].

Theorem 1.3 (Yeo [11]) *Every regular semicomplete c-partite digraph D with $c \geq 2$ contains a Hamiltonian cycle.*

Recently, Volkmann and Winzen [9] examined the cases $i_g(D) = 1$ and $i_g(D) = 2$.

Theorem 1.4 (Volkmann, Winzen [9]) *Let D be a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$.*

i) If $i_g(D) = 1$, then D contains a Hamiltonian path if and only if $c \geq 3$ or $c = 2$ and $|V_2| \leq |V_1| + 1$.

ii) If $i_g(D) = 2$ and $c \geq 5$, then D contains a Hamiltonian path.

Furthermore, they showed that the bound $c \geq 5$ in Theorem 1.4 ii) is best possible. The aim is now to extend such kind of results to multipartite tournaments of arbitrary irregularity, which means to solve the following problem.

Problem 1.5 *For all i and k, find the smallest value $g(i,k)$, such that all c-partite tournaments with $i_g \leq i$ and $c \geq g(i,k)$ have a k-path-cycle factor.*

Theorems 1.3 and 1.4 lead to $g(0,1) = 2$, $g(1,1) = 3$ and $g(2,1) = 5$. Using the methods as in the article [9], it is a simple matter to obtain the inequality $g(i,k) \leq 4i - 3k + 1$ for $i \geq k \geq 1$. In this article, we suppose that $i \geq k + 2 \geq 3$. This condition is essential to improve the bound by $g(i, k) \leq 4i - 3k - 1$. A class of examples will demonstrate that $g(i, k) \geq 4i - 4k$ such that especially in the case $k = 1$ we arrive at $g(i, 1) = 4i - 4$, when $i \geq 3$.

Further new results about Hamiltonian paths in multipartite tournaments can be found in [8, 10]. For more informations about multipartite tournaments, we refer to [1, 3, 7].

2 Preliminary results

The following results play an important role in our investigations.

Lemma 2.1 (Tewes, Volkmann, Yeo [6]) *If D is a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$, then $|V_c| \leq |V_1| + 2i_g(D)$.*

The following two results can be found in [12]. The cases of equality can implicitly be found in the proofs of the lemmas.

Lemma 2.2 (Yeo [12]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a semicomplete multipartite digraph D. Let $X \subset Y \subseteq V(D)$ and let $y_i = |Y \cap V_i|$ and $x_i = |X \cap V_i|$ for all $i = 1, 2, \ldots, c$. Then*

$$\begin{aligned} &\frac{d(X, Y - X) + d(Y - X, X)}{|X|} + \frac{d(X, Y - X) + d(Y - X, X)}{|Y - X|} \\ \geq\ & |Y| - \max\{y_i | i = 1, 2, \ldots, c\}. \end{aligned}$$

Furthermore, if equality holds above, then $y_i - 2x_i = y_j - 2x_j$ and $y_j - x_j = y_i - x_i$ and thus $x_i = x_j$ and $y_i = y_j$ for all $1 \leq i, j \leq c$.

Lemma 2.3 (Yeo [12]) *If D is a semicomplete c-partite digraph, then the following holds.*

$$i_l(D) \geq \max_{\emptyset \neq X \subseteq V(D)} \left\{ \frac{|d(X, V(D) - X) - d(V(D) - X, X)|}{|X|} \right\}$$

In the case of equality, we observe that $d^+(x) = d^-(x) + i_l(D)$ for all $x \in X$, if $d(X, V(D) - X) \geq d(V(D) - X, X)$ and $d^-(x) = d^+(x) + i_l(D)$ for all $x \in X$, if $d(V(D) - X, X) \geq d(X, V(D) - X)$.

If we interpret the inequality $pc(D) > 0$ in such a way that the digraph D does not contain a cycle-factor, then we can combine two results of Yeo and Gutin and Yeo, respectively.

Theorem 2.4 (Yeo [12], Gutin, Yeo [4]) *Let D be a semicomplete c-partite digraph. Then $pc(D) > k \geq 0$ if and only if $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 such that*

$$R_1 \leadsto Y,\ (R_1 \cup Y) \leadsto R_2,\ Y \text{ is an independent set} \qquad (1)$$

and $|Y| > |Z| + k$.

The following theorem is almost identical with Theorem 3.1 in [9]. The proof is similar to the proof of Lemma 4.3 in [12] and Theorem 3.2 in [4]. Since we will use the proof of this theorem in the following section, we will echo it here.

Theorem 2.5 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of the semicomplete multipartite digraph D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. Assume that $pc(D) > k$ for an integer $k \geq 0$. According to Theorem 2.4, $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 satisfying (1) such that $|Z| + k + 1 \leq |Y| \leq |V_c| - t$ with an integer $t \geq 0$. Let V_i be the partite set with the property that $Y \subseteq V_i$. Let $Q = V(D) - Z - V_i$, $Q_1 = Q \cap R_1$, $Q_2 = Q \cap R_2$, $Y_1 = R_1 \cap V_i$ and $Y_2 = R_2 \cap V_i$. Then*

$$i_g(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + |Y_2|}{2},$$

if $Q_1 = \emptyset$,

$$i_g(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + |Y_1|}{2},$$

if $Q_2 = \emptyset$, and

$$i_g(D) \geq i_l(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + t}{2},$$

if $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$.

Proof. Let $V(D)$ be partitioned into the subsets Y, Z, R_1, R_2 satisfying (1) such that $|Z| + k + 1 \leq |Y| \leq |V_c| - t$ for integers $k \geq 0$ and $t \geq 0$. If Q_1, Q_2, Y_1 and Y_2 are defined as above, then we observe that $|Z| \leq |Y| - 1 - k \leq |V_c| - 1 - k - t$, $Q_1 \to Y \to Q_2$, $(Q_1 \cup Y_1) \leadsto (Q_2 \cup Y_2)$ and $Y_1 \cup Y_2 \cup Y \subseteq V_i$. If $i = c$, then let $j = c - 1$ and if $i < c$, then let $j = c$. We now consider the following three cases.

Case 1. Let $Q_1 = \emptyset$. Then it follows that $Q_2 = Q$. Let $\delta^* = \min\{d^-(w) | w \in V_i\}$. Since $Y \subseteq V_i$ and thus $d^-(y) \leq |Z|$ for all $y \in Y$ we observe that $\delta^* \leq |Z| \leq |Y| - k - 1 \leq |V_i| - |Y_2| - 1 - k$. Let $\Delta^* = \max\{d^+(w), d^-(w) | w \in V(D) - V_i\}$ and note that $d^+(w) + d^-(w) \geq$

$|V(D)| - |V_j|$ for all $w \in V(D) - V_i$. The fact that $\sum_{x \in Q_2}(d^-(x) - d^+(x)) \geq |Q_2|(|Y| - |Z| - |Y_2|) \geq |Q_2|(1 + k - |Y_2|)$ implies that there is a vertex $q \in Q_2$ such that $d^-(q) \geq d^+(q) + k - |Y_2| + 1$. This leads to $2d^-(q) - k + |Y_2| - 1 \geq d^+(q) + d^-(q) \geq |V(D)| - |V_j|$, and thus we conclude that $\Delta^* \geq \frac{|V(D)|-|V_j|+k-|Y_2|+1}{2}$. This implies

$$\begin{aligned} i_g(D) &\geq \Delta^* - \delta^* \geq \frac{|V(D)| - |V_j| + k - |Y_2| + 1}{2} - |V_i| + |Y_2| + k + 1 \\ &= \frac{|V(D)| - |V_j| - 2|V_i| + 3k + 3 + |Y_2|}{2} \\ &\geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + |Y_2|}{2}, \end{aligned}$$

and the first part is proved.

Case 2. Let $Q_2 = \emptyset$. This is analogous to Case 1 (change the orientation of all arcs in D).

Case 3. Let $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. Since $|V_i| + |V_j| \leq |V_{c-1}| + |V_c|$, we deduce that $|Q| - |V_j| \geq |V(D)| - |V_i| - |Z| - |V_j| \geq |V(D)| - |V_{c-1}| - |V_c| - (|V_c| - 1 - k - t)$. By Lemma 2.2 with $X = Q_1$ and $Y = Q_1 \cup Q_2 = Q$, and because $Q \cap V_i = \emptyset$, it follows that

$$\begin{aligned} & \frac{d(Q_1, Q_2) + d(Q_2, Q_1)}{|Q_1|} + \frac{d(Q_1, Q_2) + d(Q_2, Q_1)}{|Q_2|} \\ = \; & \frac{d(Q_1, Q_2)}{|Q_1|} + \frac{d(Q_1, Q_2)}{|Q_2|} \\ \geq \; & |Q| - |V_j| \geq |V(D)| - |V_{c-1}| - 2|V_c| + 1 + k + t. \end{aligned}$$

Consequently

$$\begin{aligned} \text{(i)} \quad \frac{d(Q_1, Q_2)}{|Q_1|} &\geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 1 + k + t}{2} - |Y_2| + |Y_1| \text{ or} \\ \text{(ii)} \quad \frac{d(Q_1, Q_2)}{|Q_2|} &\geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 1 + k + t}{2} + |Y_2| - |Y_1|. \end{aligned}$$

Assume that (i) holds as the case when (ii) holds can be treated similarly. Because of $R_1 = Q_1 \cup Y_1$ and $R_2 = Q_2 \cup Y_2$, Lemma 2.3 yields

$$\begin{aligned} i_g(D) \geq i_l(D) &\geq \frac{d(Q_1, V(D) - Q_1) - d(V(D) - Q_1, Q_1)}{|Q_1|} \\ &= \frac{d(Q_1, Q_2)}{|Q_1|} + \frac{d(Q_1, Y \cup Y_2) - d(Y \cup Y_2, Q_1)}{|Q_1|} \\ &+ \frac{d(Q_1, Z \cup Y_1) - d(Z \cup Y_1, Q_1)}{|Q_1|} - \frac{d(Q_2, Q_1)}{|Q_1|} \end{aligned}$$

$$
\begin{aligned}
&\geq \left(\frac{|V(D)| - |V_{c-1}| - 2|V_c| + 1 + k + t}{2} - |Y_2| + |Y_1|\right) \\
&+ \ (|Y| + |Y_2|) - (|Z| + |Y_1|) \\
&= \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 1 + k + t}{2} + |Y| - |Z| \\
&\geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3 + 3k + t}{2}.
\end{aligned}
$$

This completes the proof of the theorem. □

An analysis of the proof of the last theorem yields the following result.

Corollary 2.6 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of the semicomplete multipartite digraph D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. Assume that $pc(D) > k$ for an integer $k \geq 1$. Let $Y, Z, R_1, R_2, Q, Q_1, Q_2, V_i, Y_1$ and Y_2 be defined as in Theorem 2.5.*

If $Q_1 = \emptyset$ and $i_g(D) = \frac{|V(D)|-|V_{c-1}|-2|V_c|+3k+3+|Y_2|}{2}$, then the following holds.

i) $\min\{d^-(w)|w \in V_i\} = |Z| = |Y| - k - 1$.

ii) $|Y| = |V_i| - |Y_2|$, *which means that* $|Y_1| = 0$ *and* $|V_i \cap Z| = 0$.

iii) $Y \to Q_2 \to (Y_2 \cup Z)$.

iv) $d^-(q_2) = d^+(q_2) + k - |Y_2| + 1$ *for all* $q_2 \in Q_2$.

v) $\max\{d^+(w), d^-(w)|w \in V(D) - V_i\} = d^-(q)$ *for a vertex* $q \in Q_2$ *such that* $|V(q)| = |V_{c-1}|$

vi) $i_g(D) = \max\{d^-(q)|q \in Q_2\} - \min\{d^-(w)|w \in V_i\}$.

vii) $|V_i| = |V_c|$.

viii) $|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + |Y_2|$ *is even.*

Let $j = c - 1$, if $i = c$ and $j = c$, if $i < c$. If $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$ and $i_g(D) = \frac{|V(D)|-|V_{c-1}|-2|V_c|+3k+3+t}{2}$, then we conclude that

a) $i_g(D) = i_l(D)$.

b) $\{|V_i|, |V_j|\} = \{|V_c|, |V_{c-1}|\}$.

c) $V_i \cap Z = \emptyset$, $|Z| = |Y| - 1 - k$, $|Y| = |V_c| - t$.

d) there is equality in Lemma 2.2 with $X = Q_1$ and $Y = Q = Q_1 \cup Q_2$, which means that $|V_m \cap Q_1| = |V_l \cap Q_1|$ and $|V_m \cap Q| = |V_l \cap Q|$ for all $1 \leq l, m \leq c$ such that $V_m \cap Q \neq \emptyset$ and $V_l \cap Q \neq \emptyset$.

e) $V_j \subseteq Q$.

f) $\frac{d(Q_1,Q_2)}{|Q_1|} = \frac{|V(D)|-|V_{c-1}|-2|V_c|+1+k+t}{2} - |Y_2| + |Y_1|$ *and* $\frac{d(Q_1,Q_2)}{|Q_2|} = \frac{|V(D)|-|V_{c-1}|-2|V_c|+1+k+t}{2} + |Y_2| - |Y_1|$.

g) $d^+(q_1) = d^-(q_1) + i_g(D)$ *for all* $q_1 \in Q_1$ *and* $d^-(q_2) = d^+(q_2) + i_g(D)$ *for all* $q_2 \in Q_2$.

h) $Q_2 \rightarrow (Z \cup Y_2)$, $(Z \cup Y_1) \rightarrow Q_1$.

j) $|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + t$ *is even.*

Corollary 2.7 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a semicomplete multipartite digraph D such that $|V_1| \le |V_2| \le \ldots \le |V_c|$. If there exists a positive integer k such that $i_g(D) \le \frac{|V(D)|-|V_{c-1}|-2|V_c|+3k+2}{2}$, then $pc(D) \le k$.*

The proof of the following theorem is also important in the following section and can be found in a weaker form in [9].

Theorem 2.8 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of the semicomplete c-partite digraph D such that $1 \le r = |V_1| \le |V_2| \le \ldots \le |V_c| \le r + p$ for an integer $p \ge 0$. If $c \ge \max\{2, 3 + \frac{2i_g(D)-3k-2+p}{r}\}$ for an integer $k \ge 0$, then it follows that $pc(D) \le k$.*

Proof. According to Corollary 2.7, it is sufficient to show that

$$i_g(D) \le \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 2}{2}.$$

Because of $c \ge 3 + \frac{2i_g(D)-3k-2+p}{r}$, we conclude that $i_g(D) \le \frac{(c-3)r+3k+2-p}{2}$, and together with $|V_1|, |V_2|, \ldots, |V_{c-2}| \ge r$, $|V_c| \le r + p$ and $c \ge 2$ this implies

$$\begin{aligned} & \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 2}{2} \\ = & \frac{|V_1| + |V_2| + \ldots + |V_{c-2}| - |V_c| + 3k + 2}{2} \\ \ge & \frac{(c-3)r - p + 3k + 2}{2} \ge i_g(D), \end{aligned}$$

the desired result. □

If D is a multipartite tournament, then, according to Theorem 2.1, we can choose $p = 2i_g(D)$ in the previous theorem.

Corollary 2.9 (Volkmann, Winzen [9]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $1 \le r = |V_1| \le |V_2| \le \ldots \le |V_c|$. If $c \ge \max\{2, \frac{4i_g(D)-3k-2}{r} + 3\}$ for an integer $k \ge 0$, then it follows that $pc(D) \le k$.*

3 Main results

If $i_g(D) \geq k+2$, then Corollary 2.9 is not best possible as we will see in Theorem 3.2. But at first, we will prove the following helpful lemma.

Lemma 3.1 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a multipartite tournament D with $i_g(D) \geq k+2$ for an integer $k \geq 1$ such that $r = |V_1| \leq \ldots \leq |V_c| \leq r + 2i_g(D) - p_1$ and $|V_1| + |V_2| + \ldots + |V_{c-2}| = (c-2)r + p_2$ for integers $0 \leq p_1 \leq 2i_g(D)$ and $0 \leq p_2$. Furthermore let D have the property*

$$(|d^+(x) - d^-(x)| > l \quad \textit{for a vertex} \quad x \in V(D)) \Rightarrow x \in V_{c-1} \qquad (2)$$

with $l \in \{0,1\}$. Let $pc(D) > k$ and let the sets Y, Y_1, Y_2, Q, Q_1, Q_2 and Z be defined as in Theorem 2.5 such that $Y \subseteq V_c$. If g) of Corollary 2.6 holds for the case that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$, and if iv) of Corollary 2.6 with $k \geq |Y_2| + l$ holds for the case that $Q_1 = \emptyset$, and if

iv) $d^+(q_1) = d^-(q_1) + k - |Y_1| + 1$ for all $q_1 \in Q_1$*

with $k \geq |Y_1| + l$ holds for the case that $Q_2 = \emptyset$, then it follows that

$$c \leq 3 + \frac{4i_g(D) - 3k - 5 - p_1 - p_2}{r}.$$

Proof. Let $pc(D) > k$ and let the sets Y, Y_1, Y_2, Q, Q_1, Q_2 and Z be defined as in Theorem 2.5. First we will show that the assumptions of this lemma imply that $Q \subseteq V_{c-1}$.

If $Q_1 = \emptyset$ and iv) of Corollary 2.6 with $k \geq |Y_2| + l$ holds, then we arrive at $|d^+(q_2) - d^-(q_2)| > l$ for all $q_2 \in Q_2$ and (2) implies that $Q_2 = Q \subseteq V_{c-1}$.

If $Q_2 = \emptyset$ and iv)* with $k \geq |Y_1| + l$ holds, then we arrive at $|d^+(q_1) - d^-(q_1)| > l$ and (2) implies that $Q_1 = Q \subseteq V_{c-1}$.

If $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$ and g) of Corollary 2.6 holds, then it follows that $|d^+(q_1) - d^-(q_1)| \geq i_g(D) > 1$ for all $q_1 \in Q_1$ and $|d^+(q_2) - d^-(q_2)| \geq i_g(D) > 1$ for all $q_2 \in Q_2$. Hence, (2) implies that $Q \subseteq V_{c-1}$.

So, in all cases we have shown that $Q \subseteq V_{c-1}$. Since $Y \subseteq V_c$ this yields $V_1 \cup V_2 \cup \ldots \cup V_{c-2} \subseteq Z$. Suppose that $c \geq \frac{4i_g(D)-3k-4-p_1-p_2}{r} + 3$. This implies

$$\begin{aligned} r + 2i_g(D) - p_1 &\geq |V_c| \geq |Y| \geq |Z| + k + 1 \geq (c-2)r + p_2 + k + 1 \\ &\geq \left(\frac{4i_g(D) - 3k - 4 - p_1 - p_2}{r} + 1\right) r + k + 1 + p_2 \\ &= 4i_g(D) - 3k - 4 - p_1 + r + k + 1 \end{aligned}$$

$$\Rightarrow 2i_g(D) \leq 2k + 3 \Rightarrow i_g(D) \leq k + \frac{3}{2},$$

a contradiction. □

Theorem 3.2 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D with $i_g(D) \geq k+2$ for an integer $k \geq 1$ such that $1 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If $c \geq \frac{4i_g(D)-3k-3}{r} + 3$, then $pc(D) \leq k$ holds.*

Proof. Suppose that $pc(D) > k$. According to Corollary 2.9, this leads to $c = \frac{4i_g(D)-3k-3}{r} + 3 > 2$. Regarding Theorem 2.8, we observe that $p = 2i_g(D)$ and D has the partition-sequence $r, r, \ldots, r, |V_{c-1}|, r + 2i_g(D)$. To get no contradiction, it follows that $d^+(x) = d^-(x) = \frac{(c-2)r+|V_{c-1}|}{2}$ for all $x \in V_c$ and $d^+(y) = d^-(y) = \frac{(c-2)r+|V_{c-1}|}{2} + i_g(D)$ for all $y \in V_1 \cup V_2 \cup \ldots \cup V_{c-2}$. In other words,

$$\big(|d^+(x) - d^-(x)| > 0 \quad \text{for a vertex} \quad x \in V(D)\big) \Rightarrow x \in V_{c-1}.$$

Note that because of $c = \frac{4i_g(D)-3k-3}{r} + 3$ in the following we may consider the case that $i_g(D) = \frac{|V(D)|-|V_{c-1}|-2|V_c|+3k+3}{2}$.

Let the sets $Y, R_1, R_2, Z, Q, Q_1, Q_2, V_i, Y_1, Y_2$ and t be defined as in Theorem 2.5.

Case 1. Let $Q_1 = \emptyset$, and thus $Q = Q_2$. This yields that i)-viii) of Corollary 2.6 with $|Y_2| = 0$, $|Y| = |V_c|$ and $k > 0$ are valid. Now Lemma 3.1 yields $c \leq 3 + \frac{4i_g(D)-3k-5}{r}$, a contradiction.

Case 2. Assume that $Q_2 = \emptyset$. By symmetry, we arrive at a contradiction similarly to Case 1.

Case 3. Suppose that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. Hence a)-j) of Corollary 2.6 with $t = 0$ hold. With c) we see that $|Y| = |V_c|$ and thus $|Y_1| = |Y_2| = 0$. Again, using Lemma 3.1, we arrive at a contradiction. This completes the proof of this theorem. □

Since $\max\left\{\frac{4i_g-3k-3}{r} + 3 \mid r \in \mathbb{N}\right\} = 4i_g(D) - 3k$, we see that the following result is an improvement of Theorem 3.2.

Theorem 3.3 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament such that $1 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If $c = 4i_g(D) - 3k - 1$ and $i_g(D) \geq k+2$ for an integer $k \geq 1$, then $pc(D) \leq k$.*

Proof. If $r \geq 2$, then Theorem 3.2 and the fact that $i_g(D) \geq k+2$ yield the desired result. Let $r = 1$, $c = 4i_g(D) - 3k - 1$ and suppose that $pc(D) > k$. If $|V_c| \leq 2i_g(D) - 1$, then $\frac{|V(D)|-|V_{c-1}|-2|V_c|+3k+2}{2} \geq \frac{c+3k+1-2i_g(D)}{2} = i_g(D)$, a contradiction to Corollary 2.7. If $|V_c| = 2i_g(D)$ and $|V_1| + |V_2| + \ldots + |V_{c-2}| \geq c - 1$, then, similarly as in the proof of Theorem 2.8 with $p = 2i_g(D) - 1$, we see that D contains a Hamiltonian path, a contradiction. Analogously, if $|V_c| = 2i_g(D) + 1$ and $|V_1| + |V_2| +$

$\ldots + |V_{c-2}| \geq c$, then we arrive at a contradiction. The following three cases remain to be considered.

Case 1. Assume that $|V_c| = 2i_g(D)$. As seen above this implies that D has the partition-sequence $1, 1, \ldots, 1, |V_{c-1}|, 2i_g(D)$.

If $|V_{c-1}| - 3k = 2p + 1$ with $p \in \mathbb{Z}$, we conclude that $\{d^+(x), d^-(x)\} = \{3i_g(D) + p - 2, 3i_g(D) + p - 1\}$, if $x \in V_1 \cup V_2 \cup \ldots \cup V_{c-2}$ and $d^+(y) = d^-(y) = 2i_g(D) + p - 1$ for all $y \in V_c$.

If $|V_{c-1}| - 3k = 2m$ with $m \in \mathbb{Z}$, it follows that $d^+(x) = d^-(x) = 3i_g(D) + m - 2$ for all $x \in V_1 \cup V_2 \cup \ldots \cup V_{c-2}$ and $\{d^+(y), d^-(y)\} = \{2i_g(D) - 1 + m, 2i_g(D) - 2 + m\}$ for all $y \in V_c$.

In both cases we deduce that

$$(|d^+(x) - d^-(x)| > 1 \quad \text{for a vertex} \quad x \in V(D)) \Rightarrow x \in V_{c-1}.$$

Furthermore, we observe that $i_g(D) = \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3}{2}$.

Let the sets $Y, R_1, R_2, Z, Q, Q_1, Q_2, V_i, t, Y_1$ and Y_2 be defined as in Theorem 2.5. Now Corollary 2.6 g) and iv) and condition iv)* of Lemma 3.1 hold, and this lemma with $p_1 = 1$ yields

$$c \leq 3 + 4i_g(D) - 3k - 6 = 4i_g(D) - 3k - 3,$$

a contradiction.

Case 2. Let $1, 1, \ldots, 1, 2, |V_{c-1}|, 2i_g(D) + 1$ be the partition-sequence of D. If $|V_{c-1}| - 3k = 2m + 1$ for an $m \in \mathbb{Z}$, then there is a vertex $x \in V_1$ such that $d^+(x) \geq 3i_g(D) + m$ or $d^-(x) \geq 3i_g(D) + m$ and a vertex $y \in V_c$ such that $d^+(y) \leq 2i_g(D) + m - 1$ or $d^-(y) \leq 2i_g(D) + m - 1$, a contradiction to the definition of $i_g(D)$. Hence, in the following we can assume that $|V_{c-1}| - 3k = 2p$ for an $p \in \mathbb{Z}$. This implies that $d^+(x) = d^-(x) = 3i_g(D) - 1 + p$ for all $x \in V_1 \cup V_2 \cup \ldots \cup V_{c-3}$, $\{d^+(y), d^-(y)\} = \{3i_g(D) + p - 1, 3i_g(D) + p - 2\}$ for all $y \in V_{c-2}$ and $d^+(z) = d^-(z) = 2i_g(D) + p - 1$ for all $z \in V_c$. In other words this means that

$$(|d^+(x) - d^-(x)| > 1 \quad \text{for a vertex} \quad x \in V(D)) \Rightarrow x \in V_{c-1}.$$

The partition-sequence yields $i_g(D) = \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3}{2}$.

Let the sets $Y, R_1, R_2, Z, Q, Q_1, Q_2, V_i, Y_1, Y_2$ and t be defined as in Theorem 2.5. Now Corollary 2.6 g) and iv) and condition iv)* of Lemma 3.1 hold, and this lemma with $p_2 = 1$ yields

$$c \leq 3 + 4i_g(D) - 3k - 6 = 4i_g(D) - 3k - 3,$$

a contradiction.

Case 3. It remains to treat the case that D has the partition-sequence $1, 1, \ldots, 1, |V_{c-1}|, 2i_g(D) + 1$. If $|V_{c-1}| - 3k = 2m$ for an $m \in \mathbb{Z}$, then we deduce that there are vertices $x \in V_1$ and $y \in V_c$ such that $d^+(x) \geq 3i_g(D) + m - 1$ or $d^-(x) \geq 3i_g(D) + m - 1$ and $d^+(y) \leq 2i_g(D) + m - 2$ or $d^-(y) \leq 2i_g(D) + m - 2$, a contradiction. Hence, we may assume that $|V_{c-1}| - 3k = 2p + 1$ with $p \in \mathbb{Z}$, $d^+(x) = d^-(x) = 3i_g(D) - 1 + p$ for all $x \in V_1 \cup V_2 \cup \ldots \cup V_{c-2}$ and $d^+(y) = d^-(y) = 2i_g(D) + p - 1$ for all $y \in V_c$. This leads to

$$\left(|d^+(x) - d^-(x)| > 0 \quad \text{for a vertex} \quad x \in V(D)\right) \Rightarrow x \in V_{c-1}. \tag{3}$$

The given partition-sequence implies that $i_g(D) = \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 4}{2}$. Let the sets $Y, Z, R_1, R_2, Q, Q_1, Q_2, V_i, V_j, Y_1, Y_2$ and t be defined as in Theorem 2.5.

Subcase 3.1. Let $t \geq 1$.

Subcase 3.1.1. Suppose that $Q_1 = \emptyset$.

First, let $i \neq c$, and thus $j = c$. Since $|Y| \geq |Z| + k + 1 \geq |Z| + 2$, we conclude that $i = c - 1$. Note that $|V_c|$ is odd. If $|V_{c-1}|$ is even, and thus k is odd, then this implies that $|V_j| = |V_c| = |V_{c-1}| + s = |V_i| + s$ for an integer $s \geq 1$. As in Case 1 of the proof of Theorem 2.5, we see that

$$\begin{aligned} i_g(D) &\geq \frac{|V(D)| - |V_j| - 2|V_i| + 3k + 3}{2} \\ &= \frac{|V(D)| - |V_{c-1}| - 2|V_c| + s + 3k + 3}{2}. \end{aligned} \tag{4}$$

To present no contradiction it follows that $s = 1$, and thus $|V_i| = |V_{c-1}| = 2i_g(D)$. Furthermore, equality holds in the inequality (4). This is possible, only if $|Y_2| = 0$ and ii) and iv) of Corollary 2.6 hold. Using ii), we see that $Y = V_{c-1}$, and iv) means that $d^-(q_2) = d^+(q_2) + k + 1$ for all $q_2 \in Q_2$. According to (3), this yields $Q \subseteq V_{c-1}$, a contradiction to $Y = V_{c-1}$.

If $|V_{c-1}|$ is odd, and thus k is even, then $|V_c| = |V_{c-1}| + s$ for an integer $s \geq 2$ would lead to a contradiction as above. Hence, we conclude that $|V_{c-1}| = |V_c| = 2i_g(D) + 1$, and without loss of generality, we may suppose that $i = c$. If $|Y_2| \geq 2$, then Theorem 2.5 yields a contradiction. If $|Y_2| = 1$, then Theorem 2.5 and Corollary 2.6 iv) imply that $d^-(q_2) = d^+(q_2) + k$ for all $q_2 \in Q_2$, and thus Lemma 3.1 leads to the contradiction $c \leq 4i_g(D) - 3k - 2$. If $|Y_2| = 0$, then, from the fact that $t \geq 1$, we conclude that $|Z| \leq |Y| - k - 1 \leq |V_i| - k - 2$. If δ^* and Δ^* are defined as in Case 1 of the proof of Theorem 2.5, then, as there, we observe that

$$\begin{aligned} i_g(D) &\geq \Delta^* - \delta^* \geq \frac{|V(D)| - |V_j| + k + 1}{2} - |V_i| + k + 2 \\ &\geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 5}{2}, \end{aligned}$$

a contradiction.

Subcase 3.1.2. Let $Q_2 = \emptyset$. If we reverse each arc of D, we arrive at a contradiction by using Subcase 3.1.1.

Subcase 3.1.3. Assume that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. To get no contradiction it follows that $t = 1$ and a)-g) of Corollary 2.6 holds, and thus Lemma 3.1 yields the contradiction $c \leq 4i_g(D) - 3k - 2$.

Subcase 3.2. Let $t = 0$ and thus $|Y| = |V_c|$ and $|Y_1| = |Y_2| = 0$.

Subcase 3.2.1. Suppose that $Q_1 = \emptyset$. If $d^-(q_2) \geq d^+(q_2) + k + 1$ for all $q_2 \in Q_2$, then with Lemma 3.1 we arrive at the contradiction $c \leq 4i_g(D) - 3k - 2$. Hence, observing Case 1 of the proof of Theorem 2.5, we conclude that there is a vertex $q \in Q \cap V_{c-1}$ such that $d^-(q) \geq d^+(q) + k + 2$. Let δ^* and Δ^* be defined as in the proof of Theorem 2.5. Similarly as there, we deduce that $\Delta^* \geq \frac{|V(D)|-|V_j|+k+2}{2}$ and thus $i_g(D) \geq \frac{|V(D)|-|V_{c-1}|-2|V_c|+3k+4}{2}$. To present no contradiction, it must be the case that $|Z| = |Y| - k - 1 = 2i_g(D) - k$ and

$$\begin{aligned} |Z| = 2i_g(D) - k &= \delta^* = \delta(G) = \min\{d^+(x), d^-(x) \mid x \in V(D)\} \\ &= 2i_g(D) + p - 1, \end{aligned}$$

and thus $p = 1-k$ and $|V_{c-1}| = k+3$. Hence, D has the partition-sequence $1, 1, \ldots, 1, k+3, 2i_g(D)+1$ and thus $d^+(x) = d^-(x) = 3i_g(D) - k$ for all $x \in V_1 \cup V_2 \cup \ldots \cup V_{c-2}$ and $d^+(y) = d^-(y) = 2i_g(D) - k$ for all $y \in V_c$. Since $|Y| = 2i_g(D) + 1 = |Z| + k + 1$, it follows that

$$\begin{aligned} |Q_2| &= |V(D)| - |Y| - |Z| \\ &= c - 2 + k + 3 + 2i_g(D) + 1 - (2i_g(D) + 1) - (2i_g(D) - k) \\ &= 2i_g(D) - k. \end{aligned}$$

The facts that $Y \to Q_2$ and $d^+(y) = 2i_g(D) - k$ for all $y \in Y = V_c$ yield that $Z \to Y$.

Suppose now, that $|V_{c-1} \cap Q_2| \leq \frac{k+3}{2}$. If there is a vertex $q_2 \in Q_2$ such that $d^-_{D[Q_2]}(q_2) \geq i_g(D) - k$, then this leads to the contradiction

$$\begin{aligned} d^-(q_2) &\geq |Y| + d^-_{D[Q_2]}(q_2) \geq 2i_g(D) + 1 + i_g(D) - k \\ &= 3i_g(D) + 1 - k. \end{aligned}$$

Hence, we have $d^-_{D[Q_2]}(q_2) \leq i_g(D) - k - 1$ for all $q_2 \in Q_2$, and thus we arrive at

$$\begin{aligned} |E(Q_2)| &= \sum_{q_2 \in Q_2} d^-_{D[Q_2]}(q_2) \leq (2i_g(D) - k)(i_g(D) - k - 1) \\ &= 2i_g^2(D) - (3k+2)i_g(D) + k^2 + k. \end{aligned}$$

On the other hand we observe that

$$\begin{aligned}|E(Q_2)| &\geq \frac{(2i_g(D)-k-\frac{k+3}{2})(2i_g(D)-k-1)+\frac{k+3}{2}(2i_g(D)-k-\frac{k+3}{2})}{2}\\ &= 2i_g^2(D)-(2k+1)i_g(D)+\frac{3k^2-3}{8}.\end{aligned}$$

Combining these two results we conclude

$$\begin{aligned}& 2i_g(D)^2-(2k+1)i_g(D)+\frac{3k^2-3}{8}\\ \leq\; & 2i_g(D)^2-(3k+2)i_g(D)+k^2+k\\ \Leftrightarrow\; & i_g(D)\leq\frac{5k^2+8k+3}{8k+8},\end{aligned}$$

a contradiction to $i_g(D)\geq k+2$ and $k\geq 1$.

If $|V_{c-1}\cap Z|\leq\frac{k+3}{2}$, then analogously we arrive at a contradictiion (replacing Q_2 by Z and $d^-_{D[Q_2]}(q_2)$ by $d^+_{D[Z]}(z)$ for a vertex $z\in Z$).

Subcase 3.2.2. Assume that $Q_2=\emptyset$. Caused by symmetry this leads to a contradiction analogously as in Subcase 3.2.1.

Subcase 3.2.3. Finally, let $Q_1\neq\emptyset$ and $Q_2\neq\emptyset$. If $i_g(D)\geq i_l(D)+1$, then Theorem 2.5 yields that $i_g(D)\geq\frac{|V(D)|-|V_{c-1}|-2|V_c|+3k+5}{2}$, a contradiction. Hence let $i_g(D)=i_l(D)$. It follows that there is a vertex $x\in V_{c-1}$ such that $\{d^+(x),d^-(x)\}=\{\frac{7i_g(D)-3k-2}{2},\frac{5i_g(D)-3k-2}{2}\}$ and thus

$$\frac{7i_g(D)-3k-2}{2}=3i_g(D)-1+p\Rightarrow|V_{c-1}|-3k=2p+1=i_g(D)+1-3k.$$

Hence, we may assume that D has the partition-sequence $1,1,\ldots,1,i_g(D)+1,2i_g(D)+1$. Observing the proof of Theorem 2.5, we recognize that the case $|Y|>|Z|+k+1$ also yields a contradiction. Consequently it remains to consider the case that $|Z|=|Y|-k-1=2i_g(D)-k$. This implies that

$$|Q|=|V(D)|-|Y|-|Z|=c-2+i_g(D)+1-(2i_g(D)-k)=3i_g(D)-2k-2.$$

Without loss of generality, we assume that $|Q_1|\geq|Q_2|$ and thus $|Q_1|\geq\frac{3i_g(D)-2k-2}{2}$. If there is a vertex $x\in Q_2-V_{c-1}$, we arrive at the contradiction

$$d^-(x)\geq|Q_1|+|Y|\geq 3i_g(D)+\frac{i_g(D)-3k}{2}+\frac{k}{2}=3i_g(D)+p+\frac{k}{2}.$$

Hence, let $Q_2\subseteq V_{c-1}$. Now we conclude for an arbitrary vertex $x\in Q_2$ that

$$d^-(x)\geq|Y|+|Q_1-V_{c-1}|\geq|Y|+|Q_1|-|V_{c-1}|+|Q_2|=4i_g(D)-2k-2.$$

Since $4i_g(D) - 2k - 2 > 3i_g(D) - 1 + p = 3i_g(D) - 1 + \frac{i_g(D)-3k}{2} = d^+(y) + i_g(D)$ with $y \in Y$ arbitrary if and only if $i_g(D) > k+2$, it remains to treat the case that $i_g(D) = k+2$, D has the partition-sequence $1, 1, 1, 1, 1, 1, k+3, 2k+5$, $|Y| = 2k+5 = |Z| + k + 1$, $|Q| = k+4$, $d^+(x) = d^-(x) = 2k+6$ for all $x \in V_1 \cup V_2 \cup \ldots \cup V_{k+5}$ and $d^+(z) = d^-(z) = k+4$ for all $z \in V_{k+7}$. Because of $|Q| = k+4$, there is a vertex $v \in Q \cap (V_1 \cup V_2 \cup \ldots \cup V_{k+5})$. If $v \in Q_2$ and $|Q_1| \geq 2$, then it follows that $d^-(v) \geq |Y| + |Q_1| \geq 2k+7$, a contradiction. If $|Q_1| = 1$, then we arrive at a contradiction to our assumption $|Q_1| \geq |Q_2|$. Hence, let $v \in Q_1$, $|Q_1| = k+3$ and $|Q_2| = 1$. It follows that $Q_2 \cup (Q_1 - \{v\}) = V_{k+6}$, $(Q_1 - \{v\}) \to v$, $Q_2 \to Z \to Q_1$ and $D[Z]$ is a tournament. Since

$$\begin{aligned} 2k^2 + 14k + 24 &= (k+4)(2k+6) = \sum_{z \in Z} d^+(z) \\ &= |Z||Q_1| + d(Z,Y) + \sum_{z \in Z} d^+_{D[Z]}(z) \\ &= \frac{3k^2 + 21k + 36}{2} + d(Z,Y), \end{aligned}$$

we deduce that $d(Z,Y) = \frac{k^2+7k+12}{2}$. On the other hand, we observe that

$$\begin{aligned} (2k+5)(k+4) &= \sum_{y \in Y} d^-(y) = |Y||Q_1| + d(Z,Y) \\ &= (2k+5)(k+4) - (2k+5) + d(Z,Y), \end{aligned}$$

and thus $d(Z,Y) = 2k+5$, a contradiction to $k \geq 1$.

This completes the proof of this theorem. □

Combining the Theorems 3.2 and 3.3, we observe that $g(i,k) \leq 4i - 3k - 1$, when $i \geq k+2$. The following example yields the estimation $g(i,k) \geq 4i - 4k$, when $i \geq k+2$.

Example 3.4 *If $k \geq 1$ and $i \geq k+2$ are integers and $c = 4i - 4k - 1$, we define the c-partite tournament $G_{i,k}$ with the partite sets $V_j = \{v_j\}$ for $1 \leq j \leq 4i - 4k - 2$ and $V_{4i-4k-1} = \{y_1, y_2, \ldots, y_{2i-k+2}\}$ as follows.*

The partite sets $V_1, V_2, \ldots, V_{2i-2k-1}$ induce an $(i-k-1)$-regular tournament A and the partite sets $V_{2i-2k}, V_{2i-2k+1}, \ldots, V_{4i-4k-2}$ induce an $(i-k-1)$-regular tournament B. In addition, let $A \to B \to (V_{4i-4k-1} - \{y_{2i-k+1}, y_{2i-k+2}\}) \to A \to \{y_{2i-k+1}, y_{2i-k+2}\} \to B$. It is straightforward to verify that $G_{i,k}$ is a $(4i-4k-1)$-partite tournament with $i_g(G_{i,k}) = i$ and $pc(G_{i,k}) > k$. If we remove the vertices y_{2i-k+1} and y_{2i-k+2}, then for each integer $i \geq k+2$ we get a $(4i-4k-1)$-partite tournament $D_{i,k}$ with $i_g(D_{i,k}) = i$ and $pc(D_{i,k}) > k$ (see Figure 1 for $D_{3,1}$).

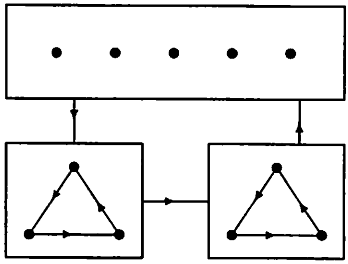

Figure 1: The 7-partite tournament $D_{3,1}$ without any Hamiltonian path

Note that for all multipartite tournaments $G_{i,1}$ with $i \geq 3$ it follows that $i_l(G_{i,1}) = 0$. This demonstrates that the existence of a Hamiltonian path does not depend on the local irregularity. There are local regular c-partite tournaments with c arbitrary large that do not contain any Hamiltonian path.

Combining Example 3.4 together with the Theorems 3.2 and 3.3, we arrive at the following main result of this paper.

Theorem 3.5 *Let $k \geq 1$ and $i \geq k+2$ be integers. If $g(i,k)$ is the minimal value such that all c-partite tournaments D with $i_g(D) \leq i$ and $c \geq g(i,k)$ have a path covering number $pc(D) \leq k$, then it follows that*

$$4i - 4k \leq g(i,k) \leq 4i - 3k - 1.$$

In the case that $k = 1$, this yields $g(i,1) = 4i - 4$, when $i \geq 3$.

Hence, for the case $k = 1$, Problem 1.5 is completely solved. If $k > 1$ and $i < k+2$, then it is still an open problem to find the values or bounds for $g(i,k)$.

Observing the global irregularity of a multipartite tournament the upper bound for $g(i,k)$ directly implies the following corollary.

Corollary 3.6 *If D is a c-partite tournament such that*

$$k + 2 \leq i_g(D) \leq \frac{c + 3k + 1}{4},$$

then $pc(D) \leq k$. Moreover, for $k = 1$ and $i_g(D) \geq 3$ the upper bound of $i_g(D)$ is optimal.

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