On square-free edge colorings of graphs

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Abstract

An edge coloring of a graph is called square-free, if the sequence of colors on certain walks is not a square, that is not of the form $x_1, \ldots, x_m, x_1, \ldots, x_m$, for any $m \in \mathbb{N}$. Recently, various classes of walks have been suggested to be considered in the above definition. We construct graphs, for which the minimum number of colors needed for a square-free coloring is different if the considered set of walks vary, solving a problem posed by Brešar and Klavžar. We also prove the following: if an edge coloring of G is not square-free (even in the most general sense), then the length of the shortest square walk is at most $8|E(G)|^2$. Hence, the necessary number of colors for a square-free coloring is algorithmically computable.

1 Introduction

A sequence of symbols $a_1, a_2 \ldots$ is called *square-free*, if it does not contain a *square*, that is a subsequence of consecutive terms of the form $XX = x_1, \ldots, x_m, x_1, \ldots, x_m$, for any $m \in \mathbb{N}$. Square-free sequences were first studied by Thue [5]. He proved that there exists an infinite square-free

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sequence consisting of only three symbols. This topic has a vast literature, for more information and references see [4].

Recently, several graph theoretic generalizations of the square-free concept have been suggested in [1] and [2]. We use the following notions: let $W = v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k$ be a walk, where $v_0, v_1, \ldots, v_k$ are vertices and $e_0, e_1, \ldots, e_{k-1}$ are edges of a given graph. For simplicity, we only list the edges sometimes. If $v_0 = v_k$, the walk is *closed*, otherwise *open*. A walk W is called *edge-simple* if $e_i \neq e_j$, when $i \neq j$. A walk is called *admissible*, if its edge sequence is not a square, when we use the same symbol for the identical items in the sequence.

Definition 1.1. Let $\pi_i(G)$ be the minimum number of colors needed to color the edges of a graph G such that the sequence of colors is not a square on

($i = 1$) any path,
($i = 2$) any open edge-simple walk,
($i = 3$) any open walk,
($i = 4$) any admissible walk.

The walks appearing in the various conditions are called *examined walks*. An edge coloring satisfying the ith condition is called a *proper* π_i coloring. A sequence of edges of a walk, whose corresponding color sequence form a square, is called a *square walk*. Observe that even if the coloring is proper, the graph may contain square walks, but these walks should not be examined.

Consider a walk $e_1, e_2 \ldots, e_n, e_1, e_2, \ldots, e_n$. The sequence of colors on the edges of this walk is always a square, thus we must exclude this case. In this sense, π_4 coloring is the most general possible notion.

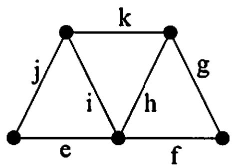

Figure 1: The walk $ejkg$ is examined for $i = 1$, but not $ejkh$; the walk $jeik$ is examined for $i = 2$; the walk $eijefg$ is examined for $i = 3$, but not $ejkh$.

Now Thue's theorem on square-free sequences is equivalent to $\pi_1(P) = 3$, where P is a one-way-infinite ray.

The parameter π_1 was first studied by Alon *et al.* [1], and it was called the Thue number of G. Two adjacent edges form an examined walk of length two, so they must get different colors to avoid squares. Thus π_1

colorings are also classical edge colorings, and $\Delta \leq \pi_1$. On the other hand $\pi_1 \leq O(\Delta^2)$ was proved in [1] using the probabilistic method. The lower bound is valid also for π_2, π_3 and π_4. The proof of the upper bound can only be transferred to π_2, but not to π_3 and π_4. It is not known whether these latter parameters can be bounded by any function of Δ. The parameter π_3 was introduced by Brešar and Klavžar in [2], where it is also mentioned that π_2 had been suggested by Grytczuk and π_4 by Currie.

Coloring the edges of a graph with distinct colors yields a proper π_1, π_2, π_3 and π_4 coloring, so the parameters are well defined. This is not trivial for π_3, see [2] for a proof. Clearly $\pi_1(G) \leq \pi_2(G) \leq \pi_3(G) \leq \pi_4(G)$ for any graph G. The first aim of this paper is to show that the four parameters are different in general, answering a question of Brešar and Klavžar.

2 Distinguishing the parameters

We construct examples to show that the different indices of π denote different parameters. It confirms the interest to study anyone of them on its own.

We first show that there exists a graph G, for which $\pi_1(G) \neq \pi_2(G)$. Let G be the graph in Figure 2. We prove that $\pi_1(G) = 4$, and there is essentially only one way to use four colors. Consider the triangle (e, f, g). These edges must get distinct colors, 1,3,4 say, as in the figure. Then the edges d and h are adjacent to colors 3 and 4, so they receive colors 1 and 2. By symmetry, we may assume that h is colored 1 and d is colored 2. Now the edge i can not get color 3(4) since the sequence $fehi(fghi)$ should not be a square walk. Hence i receives color 2. The edge $b(j)$ is not colored 1, as $dhib(dhij)$ should not be a square walk. Since the edges c, i, b and j get different colors, c receives color 1. Also by symmetry, we may assume that b gets color 4 and j receives color 3. Now $a(k)$ can not get color 2, since the path $fgdcba(fedcjk)$ should not be a square walk. Also a can not receive color 3 and k color 4 simultaneously. So a or k gets color 1. If we set the color of a to be 1, then k receives color 4. The other case is identical. The reader can verify that this is indeed a π_1 coloring.

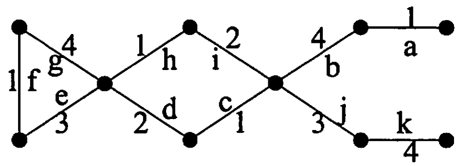

Figure 2: $\pi_1(G) = 4 < \pi_2(G)$

On the other hand, consider the edge-sequence *abcdefghij*. This walk is examined by π_2. But in our essentially unique coloring, it receives the color sequence 1412314123, which is a square. So $\pi_1(G) = 4 < \pi_2(G)$. □

Secondly we separate π_2 and π_3. Let G be the graph in Figure 3. We first show that $\pi_2(G) = 4$, and there is essentially only one way to use four colors. Consider the cut-edge b. We may assume that it has color 4. Both endvertices of b have degree 4. So the three adjacent edges on both sides must get colors 1,2,3 in some order. Since the right-hand side is symmetric, we might pick any arrangement. We take the one in Figure 3. Now the edge d can not get color 4, since then *deba* would be a square walk. So d receives color 3. On the right-hand side, there are six uncolored edges. Two of them are adjacent to the edge a. None of them can get color 4 yielding a 1414 sequence, so they get 2 and 3. Similarly the other two claws receive the colors 1,3 and 1,2. In our figure we depicted the only solution. If we *e.g.* switch the colors 2 and 3, adjacent to a, then we get a sequence 1212. The other cases give similar contradictions. This is now a π_2 coloring of the graph. Since the color 4 appears only once, on the cut-edge, any candidate square walk uses the edges on one side only. Thus $\pi_2(G) = 4$.

On the other hand, consider the edge sequence *abcdebfg*. This walk is examined by π_3. But in our unique coloring it receives the color sequence 14231423, which is a square. So $\pi_2(G) = 4 < \pi_3(G)$.

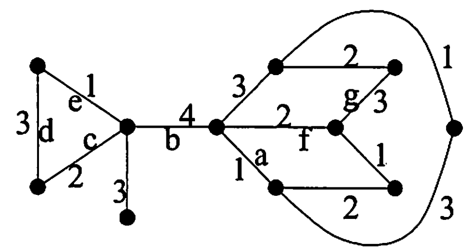

Figure 3: $\pi_2(G) = 4 < \pi_3(G)$

Notice that the separation of π_3 and π_4 is immediate from the definition. Since a cycle is only examined in the second case, *e.g.* $\pi_3(C_4) = 2$ and $\pi_4(C_4) = 3$.

3 Trees

It was shown by Alon *et al.* [1], that for any tree T, $\pi_1(T) \leq 4(\Delta(T) - 1)$. We also know that $\pi_1(G) \geq \Delta$, so this upper bound has the right order of magnitude. We improve the trivial lower bound by a constant factor for certain trees.

Lemma 3.1. *Let T be a rooted tree such that the root u has d neighbours $u_1, \ldots, u_d$, and each u_i has $d-1$ additional neighbors besides u. Then $\pi_1(T) \geq d + \frac{d-1}{2}$.*

Proof. The edges $uu_1, \ldots, uu_d$ are adjacent, thus they get different colors. Set the color of uu_i to be i. Denote by C_i the set of colors used for the $d-1$ edges adjacent to u_i except uu_i. Now any C_i contains $d-1$ different colors, and $i \notin C_i$. Let l_j denote the number of indices $1 \leq i \leq d$ such that $j \in C_i$.

If $j \in C_i$, then $i \notin C_j$, for otherwise we get a path with color sequence $jiji$. Therefore $|C_j \setminus \{1, \ldots, d\}| \geq l_j$. If $l_j \geq \frac{d-1}{2}$, then there are at least $\frac{d-1}{2}$ additional colors in C_j, and we are done.

Assume now that $l_j < \frac{d-1}{2}$ for all j. As $\sum_{i=1}^{d} |\{1, \ldots, d\} \cap C_i| = \sum_{j=1}^{d} l_j < d \cdot \frac{d-1}{2}$, there is a C_i such that $|C_i \cap \{1, \ldots, d\}| < \frac{d-1}{2}$, hence $|C_i \setminus \{1, \ldots, d\}| > \frac{d-1}{2}$, and the proof is complete. □

4 Decidability

It is not *a priori* clear how to decide, whether a given edge coloring of a graph is a proper π_3 (π_4) coloring or not. Proposition 4.2 yields a theoretical algorithm to check this. The following observation will be useful.

Lemma 4.1. *Let an edge coloring of a graph be given such that any adjacent edges receive different colors. Let WW' be a square walk with $|W| = |W'|$. This walk is examined by π_4 if and only if the first vertex of W and the first vertex of W' are different.*

Proof. If WW' is not examined by π_4, then it is not admissible, and by definition $W = W'$. In particular the first vertices are the same.

For the other direction, assume that W and W' start with the same vertex. It is immediate, by induction on k, that the first k vertices of W and W' are the same, since the color of the next edge determines the next vertex. Thus $W = W'$, the square walk is not admissible, thus it is indeed not examined by π_4. □

Proposition 4.2. *If an edge coloring of G is not a proper π_3 (π_4) coloring, then there exists an examined square walk of length at most $8|E(G)|^2$ demonstrating this fact.*

Proof. Assume that W is a square walk of minimal length examined by π_3 (π_4). Assume to the contrary that $|W| > 8|E(G)|^2$. By the Pigeonhole

Principle there is an edge e such that the first half of W goes through e in the same direction more than $2|E(G)|$ times.

Thus $W = AeBeCA'e'B'e'C'$ for some (possibly empty) walks A, B, C, A', B', C' and edge e' such that $|A| = |A'|$, $|B| = |B'|$, $|C| = |C'|$ and W goes through e (respectively e') in the same direction. Consider the walk $W' = AeCA'e'C'$. Clearly W' is a square walk, so for a contradiction we only have to check that W' is examined by π_3 (π_4).

For π_3, notice that the first and the last vertex of W' are the same as of W. So the walk W' is not open, thus it is examined by π_3.

If two adjacent edges have the same color, then there is an examined square walk of length two, which is impossible. For π_4, the first vertex of $AeBeC$ and the first vertex of $A'e'B'e'C'$ are different by Lemma 4.1. Then the first vertices of AeC and $A'e'C'$ are also different, as they are the same as above. Using Lemma 4.1 again, we get that W' is examined by π_4. □

There is a lower bound counterpart of the previous proposition. We show a graph and its coloring, which does not yield a short square walk.

Example 4.3. Consider the graph in Figure 4, and its coloring with colors $\{a_1, b_1, a_2, b_2, a_3, b_3, x, y, z\}$ as follows: let c_k be an arbitrary square-free sequence of the numbers 1, 2 and 3. Then the edge e_{3k-2} is colored by a_{c_k}, e_{3k-1} gets color b_{c_k} and e_{3k} receives color x for $0 < k \leq m$. The color of the other edges can be seen in the figure. There is a square walk of length $6m + 4$ that starts with the edges $f_1, f_2, e_1, e_2, e_3, \ldots, e_{3m}$ and continued by the rightmost nine edges in the required order. In fact, it can be shown that this is the shortest examined square walk. This shows that the upper bound $8|E(G)|^2$ in the previous proposition can not be replaced by $(2 - \varepsilon)|E(G)|$.

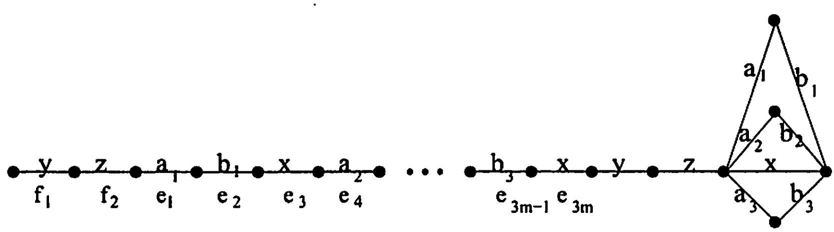

Figure 4: A coloring without short square walks

Open problems

We finish with a collection of open problems. The first one has already been raised in [2].
Problem 1 Can π_3 and π_4 be bounded from above as a function of the maximum degree of the graph?
Problem 2 What is the maximum difference between any of the parameters π_1, π_2 and π_3 for a fixed graph?
Problem 3 Is it possible to improve the quadratic bound in Proposition 4.2 to linear?

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