Critically and Minimally Cochromatic Graphs

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Abstract: The cochromatic number of a graph G, denoted by $z(G)$, is the fewest number of parts we need to partition $V(G)$ so that each part induces in G an empty or a complete graph. A graph G with $z(G) = n$ is called critically n-cochromatic if $z(G - v) = n - 1$ for each vertex v of G, and minimally n-cochromatic if $z(G - e) = n - 1$ for each edge e of G.

We show that for a graph G, $K_1 \cup K_2 \cup \cdots \cup K_{n-1} \cup G$ is a critically n-cochromatic graph if and only if G is $K_n (n \geq 2)$. We consider general minimally cochromatic graphs and obtain a result that a minimally cochromatic graph is either a critically cochromatic graph or a critically cochromatic graph plus some isolated vertices. We also prove that given a graph G, then $K_1 \cup K_2 \cup \cdots \cup K_{n-1} \cup G$ $(n \geq 2)$ is minimally n-cochromatic if and only if G is K_n or $K_n \cup \overline{K_p}$ for $p \geq 1$. We close by giving some properties of minimally n-cochromatic graphs.

Keywords: cochromatic number; critically cochromatic graphs; minimally cochromatic graphs

1. INTRODUCTION

All graphs under consideration are simple, finite and undirected. For undefined terms and concepts the reader is referred to [3]. The cochromatic number of a graph was first proposed by Lesniak and Straight [4]. A *cocoloring* of a graph G is a vertex partition of G in which each part induces a complete or an empty graph. The *cochromatic number* of G, denoted by $z(G)$, is the minimum order of all cocolorings of G.

For any graph G with $z(G) \geq 2$, it is known that $z(G) - 1 \leq z(G - v) \leq z(G)$ for each vertex v of G. A nontrivial graph G is *critically cochromatic* if $z(G - v) = z(G) - 1$ for each vertex v of G, and *critically n-cochromatic* if it is critically cochromatic and $z(G) = n$. Gimbel and Straight [2] showed that the removal of any edge from G alters the cochromatic number by at most one. So we can say that a graph G is *minimally cochromatic* if $z(G-e) = z(G) - 1$ for each edge e of G and *comaximal* if $z(G-e) = z(G) + 1$ for each edge e of G. A graph G is *minimally n-cochromatic* if G is minimally cochromatic and $z(G) = n$. Gimbel and Straight [2] proved that the graph $K_1 \cup K_2 \cup \cdots \cup K_n$ $(n \geq 2)$ is both critically and minimally n-cochromatic. They characterized comaximal graphs and showed that minimally cochromatic graphs without isolated vertices are critically cochromatic. Also, it was shown that if H is an induced subgraph of G, then $z(H) \leq z(G)$. Broere and Burger [1] discussed some properties of critically n-cochromatic graphs and constructed a family of critically n-cochromatic graphs.

In the next section, we show that for a graph G, $K_1 \cup K_2 \cup \cdots \cup K_{n-1} \cup G$, $n \geq 2$, is a critically n-cochromatic graph if and only if G is K_n. In section 3, we consider general minimally cochromatic graphs and show that a minimally cochromatic graph is either a critically cochromatic graph or a critically cochromatic graph plus some isolated vertices. In addition, we prove that for a graph G, $K_1 \cup K_2 \cup \cdots \cup K_{n-1} \cup G$ $(n \geq 2)$ is minimally n-cochromatic if and only if G is K_n or $K_n \cup \overline{K_p}$, $p \geq 1$. We conclude with giving some properties of minimally n-cochromatic graphs.

2. CRITICALLY COCHROMATIC GRAPHS

Many examples of critically cochromatic graphs were given in [1,2]. The path P_3 is a critically cochromatic graph of order 3. There is no critically cochromatic graph of order 4. It is straightforward to verify that the cycle C_5 is both a unique critically cochromatic graph of order 5 and a unique 3-cochromatic graph of order 5. $K_1 \cup K_2 \cup K_3$ is a critically cochromatic graph of order 6. For critically 3-cochromatic graphs with six vertices, we have the following property.

Theorem 2.1. *Let G be a graph of order six with $z(G) = 3$. Then G is a critically 3-cochromatic graph if and only if C_5 is not an induced subgraph of G and $K_1 \cup K_2 \cup K_3$ is a subgraph of G.*

Proof. Suppose that G is a critically 3-cochromatic graph with six vertices. If C_5 is an induced subgraph of G, let $v \in V(G)$ and $v \notin V(C_5)$. Since $G - v = C_5$, $z(C_5) = 3$ contradicts that G is critically 3-cochromatic. Hence G can not contain C_5 as its induced subgraph.

If G has no cycles it is a forest and has chromatic number at most two. This contradicts $z(G) = 3$. Therefore, G has a cycle. We now claim that G contains K_3. Otherwise, the girth of G, that is the minimal length of a cycle in G, is four or six. Since $z(G) = 3$, the girth of G is four. Suppose $v_1v_2v_3v_4v_1$ is a 4-cycle of G and the remaining two vertices of G are v_5

and v_6. Note that since G has no 3-cycle, if v_k, $k = 5, 6$, is adjacent to the vertices of $\{v_1, v_3\}(\{v_2, v_4\})$, then v_k is not adjacent to any vertex of $\{v_2, v_4\}(\{v_1, v_3\})$. If v_5 is not adjacent to any vertex of $\{v_1, v_2, v_3, v_4\}$, then either $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$ or $\{v_1, v_3, v_6\}$, $\{v_2, v_4, v_5\}$ is a 2-cocoloring of G, contradicting that $z(G) = 3$. So, without loss of generality suppose that v_5 is adjacent to the vertices of $\{v_2, v_4\}$. Thus $\{v_1, v_3, v_5\}$ is an independent set in G. If $v_5v_6 \notin E(G)$, then either $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$ or $\{v_1, v_3, v_5, v_6\}$, $\{v_2, v_4\}$ is a 2-cocoloring of G, which is a contradiction. So $v_5v_6 \in E(G)$. Since G contains no 3-cycle, v_i is adjacent to only one vertex of $\{v_5, v_6\}$, $i = 1, 2, 3, 4$. Thus, if $v_5v_2 \in E(G)$, $v_5v_4 \in E(G)$, then $v_2v_6 \notin E(G)$, $v_4v_6 \notin E(G)$. We have $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$ is a 2-cocoloring of G, a contradiction. If v_5 is adjacent to one vertex of $\{v_2, v_4\}$, say v_2, then $v_2v_6 \notin E(G)$. Since C_5 is not an induced subgraph of G, v_4 and v_6 are nonadjacent. It follows that $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$ is a 2-cocoloring of G, a contradiction. Therefore, K_3 is contained in G.

Let $v_1v_2v_3v_1$ be a 3-cycle of G and $H = G - \{v_1, v_2, v_3\}$. If $E(H) = \emptyset$, then $V(H)$, $\{v_1, v_2, v_3\}$ is a 2-cocoloring of G, a contradiction. Hence, there is at least one edge in H and this implies that $K_1 \cup K_2 \cup K_3$ is a subgraph of G.

Conversely, suppose that C_5 is not an induced subgraph of G and $K_1 \cup K_2 \cup K_3 \subseteq G$. We may suppose $\{v_2v_3, v_4v_5, v_5v_6, v_4v_6\} \subseteq E(G)$, where $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Since $\{v_2, v_3\}$, $\{v_4, v_5, v_6\}$ is a 2-cocoloring of $G - v_1$, $z(G - v_1) = 2$. Similarly, $z(G - v_2) = z(G - v_3) = 2$ since $\{v_1, v_3\}$ $(\{v_1, v_2\})$, $\{v_4, v_5, v_6\}$ is a 2-cocoloring of $G - v_2(G - v_3)$. Without loss of generality, we choose any vertex from $\{v_4, v_5, v_6\}$ (say v_4). Then, $2 \leq z(G - v_4) \leq 3$. If $z(G - v_4) = 3$, $G - v_4 \cong C_5$ since C_5 is a unique 3-cochromatic graph with five vertices. This contradicts that C_5 is not an induced subgraph of G. Hence, G is critically 3-cochromatic. ■

It is obvious that if G is a critically cochromatic graph, then $\overline{G}$ is also a critically cochromatic graph. However, if G_1 and G_2 are all critically cochromatic graphs, then $G_1 \cup G_2$ is not necessarily critically cochromatic. Conversely, if $G_1 \cup G_2$ is critically cochromatic, then both G_1 and G_2 are not necessarily critically cochromatic. For example, $K_1 \cup K_2$ is critically cochromatic, but $(K_1 \cup K_2) \cup (K_1 \cup K_2)$ is not; $(K_1 \cup K_2) \cup (K_3 \cup K_4)$ is critically cochromatic, but $K_3 \cup K_4$ is not. Note that the critically n-cochromatic graph $K_1 \cup K_2 \cup \cdots \cup K_n$ $(n \geq 2)$ contains exactly one isolated vertex. In fact, as the next theorem indicates, critically cochromatic graphs have at most one isolated vertex.

Remark 2.1. *If G is a critically cochromatic graph, then G has no more than one isolated vertex.*

Proof. Let G be a critically cochromatic graph with $z(G) = n$. Assume that G has two isolated vertices, say u and v. Since G is critically cochromatic, $z(G-u)=n-1$. Let X_1, X_2, ..., X_{n-1} be an $(n-1)$-cocoloring of $G-u$ such that $v \in X_1$. Since v is an isolated vertex of $G-u$, X_1 is an independent set in $G-u$. Hence, $X_1 \cup \{u\}$, X_2, ..., X_{n-1} is an $(n-1)$-cocoloring of G. This contradicts that $z(G)=n$. ■

Lemma 2.1 (Broere and Burger [1]). *$K_{m_1} \cup K_{m_2} \cup \cdots \cup K_{m_n}$ with $1 \leq m_1 \leq m_2 \leq \ldots \leq m_n$ and $n \geq 2$ is critically n-cochromatic if and only if $m_i = i$ for $i = 1, 2, \ldots, n$.*

Theorem 2.2. *Let G be a graph. Then $K_1 \cup K_2 \cup \cdots \cup K_{n-1} \cup G$ $(n \geq 2)$ is a critically n-cochromatic graph if and only if G is K_n.*

Proof. By Lemma 2.1 we only need to show the necessity. Let $K = K_2 \cup K_3 \cup \cdots \cup K_{n-1}$. Suppose $K_1 \cup K \cup G$ is critically n-cochromatic. Then $z(K \cup G) = n-1$. Let X_1, X_2, ..., X_{n-1} be any $(n-1)$-cocoloring of $K \cup G$. If some X_j, $1 \leq j \leq n-1$, induces an empty graph, then X_1, ..., X_{j-1}, $X_j \cup \{v\}$, X_{j+1}, ..., X_{n-1} would be an $(n-1)$-cocoloring of $K_1 \cup K \cup G$, where $V(K_1) = \{v\}$. This contradicts that $z(K_1 \cup K \cup G) = n$. Therefore, each X_i $(1 \leq i \leq n-1)$ induces a complete graph with at least two vertices. This implies that each X_i induces a complete graph in some K_j $(2 \leq j \leq n-1)$ or G. Hence, each X_i contains only vertices of some K_j or G. Without loss of generality suppose that X_i contains only vertices of K_i, $i = 2, 3, \ldots, n-1$. Thus, X_1 contains only vertices of G and $X_1 = V(G)$. It follows that G is a complete graph K_m, $m \geq 2$. By Lemma 2.1, $m = n$. ■

3. MINIMALLY COCHROMATIC GRAPHS

Lemma 3.1 (Gimbel and Straight [2]). *If G is a minimally cochromatic graph containing no isolated vertices, then G is critically cochromatic.*

For general minimally cochromatic graphs, we have the following result.

Theorem 3.1. *If G is a minimally cochromatic graph, then either G is a critically cochromatic graph or $G = G_1 \cup \overline{K_p}$ $(p \geq 1)$, where G_1 is critically cochromatic.*

Proof. Let G be a minimally n-cochromatic graph and p the number of isolated vertices in G. Then $G = \overline{K_p} \cup G_1$, where G_1 contains no isolated vertices. We examine the following three cases.

Case 1. $p = 0$.

By Lemma 3.1 we see that G is critically cochromatic.

Case 2. $p = 1$.

Case 2.1. $z(G_1) = n$.

For each edge e of G_1, since $G_1 - e$ is an induced subgraph of $(G_1 - e) \cup K_1$, $z(G_1 - e) \leq z((G_1 - e) \cup K_1) = z(G - e) = n - 1$. It follows that G_1 is minimally cochromatic. Thus, G_1 is critically cochromatic since G_1 contains no isolated vertices.

Case 2.2. $z(G_1) = n - 1$.

Since G_1 contains no isolated vertices, the degree of any vertex of G_1 is at least one. Select a vertex v from G_1 and suppose that $vw \in E(G_1)$. Note that $G_1 - v$ is an induced subgraph of $G_1 - vw$. Hence, $z(G - v) = z((G_1 - v) \cup K_1) \leq z((G_1 - vw) \cup K_1) = z(G - vw) = n - 1$. This implies that G is critically cochromatic.

Case 3. $p \geq 2$.

Case 3.1. $z(G_1) = n$.

Similar to the proof of Case 2.1, we have G_1 is critically cochromatic.

Case 3.2. $z(G_1) = n - 1$.

Let $X_1, X_2, \ldots, X_{n-1}$ be any $(n-1)$-cocoloring of G_1. Then each X_i $(1 \leq i \leq n-1)$ induces a complete graph with at least two vertices in G_1. Otherwise, if some X_j $(1 \leq j \leq n-1)$ induces an empty graph in G_1, then $X_1, \ldots, X_{j-1}, X_j \cup V(\overline{K_p}), X_{j+1}, \ldots, X_{n-1}$ would be an $(n-1)$-cocoloring of G, a contradiction. Note that $G = \overline{K_p} \cup G_1 = \overline{K_{p-1}} \cup K_1 \cup G_1$. Let $V(K_1) = \{v\}$. We assert that $z(K_1 \cup G_1) = n$. Clearly, $z(K_1 \cup G_1) \leq n$. Suppose that $z(K_1 \cup G_1) \leq n - 1$. Let $Y_1, Y_2, \ldots, Y_{n-1}$ be an $(n-1)$-cocoloring of $K_1 \cup G_1$. Without loss of generality say that $v \in Y_1$. Thus Y_1 induces an empty graph in $K_1 \cup G_1$. If Y_1 contains the vertices of G_1, then $Y_1 \backslash \{v\}, Y_2, \ldots, Y_{n-1}$ is an $(n-1)$-cocoloring of G_1, where $Y_1 \backslash \{v\}$ induces an empty graph, a contradiction. Hence, Y_1 does not contain any vertex of G_1. Thus, $z(G_1) \leq n - 2$. However, this is impossible since $z(G_1) = n - 1$. So we have $z(K_1 \cup G_1) = n$. For each edge e of G_1, $z(K_1 \cup G_1 - e) \leq z(\overline{K_p} \cup G_1 - e) = n - 1$. This implies that $K_1 \cup G_1$ is minimally cochromatic. From Case 2.2 we conclude that $K_1 \cup G_1$ is critically cochromatic. Thus $G = \overline{K_{p'}} \cup G_1'$, where $p' = p - 1 \geq 1$ and $G_1' = K_1 \cup G_1$ which is critically cochromatic. ■

A graph was given in [2] which is critically cochromatic but not minimally cochromatic. The following Lemma is similar to proposition 3 in [2].

Lemma 3.2. *Let G be a minimally n-cochromatic graph and uv any edge of G. Then there is an n-cocoloring of G which contains $\{u, v\}$ as a cocolor class.*

Proof. Since G is minimally n-cochromatic, $z(G - uv) = n - 1$. Let $V_2, V_3, \ldots, V_n$ be an $(n-1)$-cocoloring of $G - uv$. We assert that $u, v \in V_i$ for some i, $2 \leq i \leq n$. Otherwise, let $u \in V_i$ and $v \in V_j$, $j \neq i$, then $V_2, V_3, \ldots, V_n$ is also an $(n-1)$-cocoloring of G, a contradiction. Let $V_1 = \{u, v\}$. Then $V_1, \ldots, V_{i-1}, V_i - \{u, v\}, V_{i+1}, \ldots, V_n$ is an n-cocoloring of G. ■

From the proof of Lemma 3.2 we see that V_i is an independent set in $G-uv$. Let $H = G \cup \overline{K_p}$, $p \geq 1$. Then $V_2, \ldots, V_{i-1}, V_i \cup V(\overline{K_p}), V_{i+1}, \ldots, V_n$ is an $(n-1)$-cocoloring of $(G-uv) \cup \overline{K_p}$. Thus $z((G-uv) \cup \overline{K_p}) = z(H-uv) \leq n-1$. Since G is an induced subgraph of H, $n = z(G) \leq z(H)$. This implies that H is also minimally n-cochromatic. Therefore, we conclude that if G is minimally cochromatic, then $G \cup \overline{K_p}$ ($p \geq 1$) is also minimally cochromatic. Thus, for any (fixed) integer p, $p \geq 3$, there is a minimally cochromatic graph of order p. Next, we will show that $K_1 \cup K_2 \cup \cdots \cup K_{n-1} \cup G$ is minimally n-cochromatic if and only if G is K_n or $K_n \cup \overline{K_p}$ for $p \geq 1$.

Lemma 3.3. *If G is a complete or an empty graph, then $z(K_2 \cup K_3 \cup \cdots \cup K_{n-1} \cup G) = n-1$.*

Proof. Let $K = K_2 \cup K_3 \cup \cdots \cup K_{n-1}$. It is obvious that $V(K_2), V(K_3), \ldots, V(K_{n-1}), V(G)$ is an $(n-1)$-cocoloring of $K \cup G$. Then $z(K \cup G) \leq n-1$. On the other hand, $n-1 = z(K_1 \cup K) \leq z(K \cup G)$ since $K_1 \cup K$ is an induced subgraph of $K \cup G$. Hence $z(K \cup G) = n-1$. ■

Theorem 3.2. *Let G be a graph. Then $K_1 \cup K_2 \cup \cdots \cup K_{n-1} \cup G$ ($n \geq 2$) is minimally n-cochromatic if and only if G is K_n or $K_n \cup \overline{K_p}$ for $p \geq 1$.*

Proof. The sufficiency is obvious. So we prove only the necessity.

Suppose that $K_1 \cup K_2 \cup \cdots \cup K_{n-1} \cup G$ is minimally n-cochromatic. For $n = 2$, the proof is easy, so we suppose that $n \geq 3$ and let $K = K_2 \cup K_3 \cup \cdots \cup K_{n-1}$. We distinguish the following two cases.

Case 1. $z(K \cup G) = n-1$.

Similar to the proof of Theorem 2.2, we can see that G is a complete graph K_t. Since $n = z(K_1 \cup K \cup G) \leq \chi(K_1 \cup K \cup G) = \chi(G) = t$, we have $t \geq n$. Suppose that $t \geq n+1$. Note that for each edge uv of K_t, $K_1 \cup K \cup (K_t - u)$ is an induced subgraph of $K_1 \cup K \cup (K_t - uv)$. Hence, $z(K_1 \cup K \cup K_t - uv) \geq z(K_1 \cup K \cup (K_t - u)) = z(K_1 \cup K \cup K_{t-1}) \geq z(K_1 \cup K \cup K_n) = n$. This contradicts that $K_1 \cup K \cup K_t$ is minimally n-cochromatic. Thus $G = K_n$.

Case 2. $z(K \cup G) = n$.

Since $K_1 \cup K \cup G$ is minimally n-cochromatic, $z(K_1 \cup K \cup G - e) = n-1$ for each edge e of $K \cup G$. Notice, $K_1 \cup (K \cup G - e)$ contains $K \cup G - e$ as an induced subgraph. Hence, $z(K \cup G - e) \leq z(K_1 \cup K \cup G - e) = n-1$. It follows that $K \cup G$ is a minimally n-cochromatic graph. By Lemma 3.2, there is an n-cocoloring of $K \cup G$ which contains $V(K_2)$ as a cocolor class. Let $X_1, X_2, \ldots, X_n$ be such an n-cocoloring of $K \cup G$, where $X_n = V(K_2) = \{u, v\}$. Thus, $X_1, X_2, \ldots, X_{n-1}$ is a cocoloring of $K_3 \cup \cdots \cup K_{n-1} \cup G$. Without loss of generality suppose that $X_1, X_2, \ldots, X_l$ contain the vertices of G, $l \leq n-1$. If each class X_i, $1 \leq i \leq l$, induces an empty graph, then $\chi(G) \leq l \leq n-1$. This contradicts that $n = z(K_1 \cup K \cup G) \leq \chi(K_1 \cup K \cup G) = \chi(G)$. Without loss of generality suppose that X_1 induces a complete graph with at least two vertices in G, and hence contains no vertices of K_j, $j = 3, 4, \ldots, n-1$. By the pigeonhole principle, there exists some X_j $(2 \leq j \leq n-1)$, say X_{n-1}, which contains at least two vertices of K_{n-1}. So X_{n-1} induces a complete graph in K_{n-1}, and hence contains no vertices of K_3, K_4, ..., K_{n-2} and G. Similarly, we may suppose that X_i induces a complete graph in K_i, $i = n-1, n-2, \ldots, 3$, and hence contains no vertices of K_j and G, $j = 3, 4, \ldots, n-1$, $j \neq i$. It follows that $l \leq 2$. Note that X_1, $X_2 \cap V(G)$ is also a cocoloring of G, so $z(G) \leq 2$.

Claim 1. $z(G) = 2$.
Proof. If $z(G) = 1$, then G is either empty or complete. By Lemma 3.3, $z(K \cup G) = n - 1$, a contradiction. ■

Claim 2. There is a 2-cocoloring of G such that one cocolor class induces a complete graph and the other induces an empty graph in G.
Proof. Let H_1, H_2 be any 2-cocoloring of G. Since $3 \leq n \leq \chi(G)$, either H_1 or H_2 induces a complete graph with at least two vertices in G. Without loss of generality suppose that H_1 induces a complete graph with at least two vertices in G. If H_2 also induces a complete graph with at least two vertices in G, then G contains no isolated vertices, and hence $K \cup G$ contains no isolated vertices. By Lemma 3.1, $K \cup G$ is critically n-cochromatic. Let $V(K_2) = \{u, v\}$. Hence, for $u \in V(K_2)$, $z(K \cup G - u) = z(K_1 \cup K_3 \cup \cdots \cup K_{n-1} \cup G) = n - 1$. Let X_1, X_2, ..., X_{n-1} be an $(n-1)$-cocoloring of $K \cup G - u$. Since $n \leq \chi(G)$, there exists some X_i $(1 \leq i \leq n-1)$, say X_{n-1}, which induces a complete graph with at least two vertices in G, and hence contains no vertices of K_1, K_3, ..., K_{n-1}. By the pigeonhole principle, there exists some X_j $(1 \leq j \leq n-2)$, say X_{n-2}, containing two or more vertices of K_{n-1}. Therefore, X_{n-2} induces a complete graph with at least two vertices in K_{n-1}, and hence contains no vertices of K_1, K_3, ..., K_{n-2} and G. By the similar manner, we may suppose that X_i induces a complete graph with at least two vertices in K_{i+1}, $i = 2, 3, \ldots, n-2$, and hence contains no vertices of K_j and G, $j = 1, 3, \ldots, n-1$, $j \neq i+1$. This implies that $v \in X_1$. Note that since $z(G) = 2$, $X_1 \cap V(G)$, X_{n-1} is a 2-cocoloring of G, where $X_1 \cap V(G) \neq \emptyset$ induces an empty graph in G. ■

By Claim 2 we choose a 2-cocoloring H_1^*, H_2^* of G, where H_1^* induces a complete graph K_m and H_2^* induces an empty graph $\overline{K_p}$ $(p \geq 1)$ in G, such that H_1^* has as many vertices as possible. We denote by $[K_m, \overline{K_p}]$ the set of edges with one end in K_m and the other in $\overline{K_p}$. Then $E(G) = E(K_m) \cup [K_m, \overline{K_p}]$.

Claim 3. $\chi(G) = m = n$.

Proof. Let $V(K_m) = \{u_1, u_2, \ldots, u_m\}$, $V(\overline{K_p}) = \{w_1, w_2, \ldots, w_p\}$. Color K_m with m colors. For each w_i, $1 \le i \le p$, there is a vertex u_j, $1 \le j \le m$, such that $w_i u_j \notin E(G)$. For otherwise, if $w_i u_j \in E(G)$ for any vertex u_j, $j = 1, 2, \ldots, m$, then let $H_1' = V(K_m) \cup \{w_i\}$, $H_2' = V(\overline{K_p}) \setminus \{w_i\}$. Thus H_1', H_2' is also a 2-cocoloring of G, and H_1' has more vertices than H_1^*. This contradicts our choice of H_1^*, H_2^*. Therefore, color w_i with the same color as u_j. Thus G is m-colorable. Since $K_m \subseteq G$, $\chi(G) = m$.

Now we show that $m = n$. Since $\chi(G) \ge n$, $m \ge n$. Suppose that $m \ge n+1$. For each edge e of G, $K_1 \cup K \cup (G-e)$ contains $K_1 \cup K \cup K_n$ as an induced subgraph. Hence, $n = z(K_1 \cup K \cup K_n) \le z(K_1 \cup K \cup (G-e)) = z(K_1 \cup K \cup G - e)$. This contradicts that $K_1 \cup K \cup G$ is minimally n-cochromatic. Thus, $m = n$. ■

Claim 4. $[K_n, \overline{K_p}] = \emptyset$.

Proof. Suppose that $e \in [K_n, \overline{K_p}]$. Note that $K_1 \cup K \cup K_n$ is an induced subgraph of $K_1 \cup K \cup (G-e)$. Hence, $n = z(K_1 \cup K \cup K_n) \le z(K_1 \cup K \cup (G-e))$, a contradiction. ■

By Claim 3 and 4, we have $G = K_n \cup \overline{K_p}$, $p \ge 1$. ■

In [1], some properties of critically n-cochromatic graphs were discussed. We next obtain the analogous properties of minimally n-cochromatic graphs.

Lemma 3.4 (Broere and Burger [1]). *Let G be a graph with $z(G) = n$ and $z(G \cup K_n) = n$. Then $\chi(G \cup K_n) = \chi(G) = n$.*

Theorem 3.3. *Let G be a minimally n-cochromatic graph. Then $G \cup K_{n+1}$ is a minimally $(n+1)$-cochromatic graph if and only if $\chi(G) = n$.*

Proof. Suppose that $G \cup K_{n+1}$ is minimally $(n+1)$-cochromatic. Hence, $z((G \cup K_{n+1}) - uv) = z(G \cup (K_{n+1} - uv)) = n$ for each edge uv of K_{n+1}. Since $G \cup (K_{n+1} - u)$ is an induced subgraph of $G \cup (K_{n+1} - uv)$, $z(G \cup (K_{n+1} - u)) = z(G \cup K_n) \le z((G \cup K_{n+1}) - uv) = n$. This implies that $z(G \cup K_n) = n$. By Lemma 3.4, $\chi(G) = n$.

Now, we show that the converse statement holds. Since any n-cocoloring of G can be extended to an $(n+1)$-cocoloring of $G \cup K_{n+1}$, $z(G \cup K_{n+1}) \le n+1$. If $z(G \cup K_{n+1}) \le n$, then let Y_1, Y_2, $\ldots$, Y_n be an n-cocoloring of $G \cup K_{n+1}$. By the pigeonhole principle, there exists some Y_j, $1 \le j \le n$, which contains at least two vertices of K_{n+1}, and hence contains no vertices of G. Therefore $z(G) \le n-1$, a contradiction. Thus $z(G \cup K_{n+1}) = n+1$. We consider two cases to show that $G \cup K_{n+1}$ is minimally cochromatic.

If $e \in E(G)$, then $z((G \cup K_{n+1}) - e) = z((G-e) \cup K_{n+1}) \le z(G-e) + 1 = n$. If $e \in E(K_{n+1})$, then $z((G \cup K_{n+1}) - e) = z(G \cup (K_{n+1} - e)) \le \chi(G \cup (K_{n+1} - e)) = \max\{\chi(G), \chi(K_{n+1} - e)\} = n$. Thus, $G \cup K_{n+1}$ is minimally $(n+1)$-cochromatic. ■

Theorem 3.4. *If $G \cup K_{n+1}$ is minimally $(n+1)$-cochromatic, then G is minimally n-cochromatic.*

Proof. First, we prove that $z(G) = n$. If $z(G) = k < n$, then $z(G \cup K_{n+1}) \leq k+1 < n+1$, a contradiction. Therefore $z(G) \geq n$. If $z(G) \geq n+1$, then for each edge e of K_{n+1}, $z(G \cup K_{n+1} - e) = z(G \cup (K_{n+1} - e)) \geq z(G) \geq n+1$ since G is an induced subgraph of $G \cup K_{n+1} - e$. This contradicts that $G \cup K_{n+1}$ is minimally cochromatic. Thus, $z(G) = n$.

We now prove that G is minimally cochromatic. Since $G \cup K_{n+1}$ is minimally $(n+1)$-cochromatic, $z((G-e) \cup K_{n+1}) = z(G \cup K_{n+1} - e) = n$ for each edge e of G. Let $X_1, X_2, \ldots, X_n$ be an n-cocoloring of $(G-e) \cup K_{n+1}$. By the pigeonhole principle, there exists some X_i, $1 \leq i \leq n$, which contains at least two vertices of K_{n+1}, and hence contains no vertices of $G-e$. Hence, $z(G-e) \leq n-1$. This implies that G is minimally n-cochromatic. ■

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