A New σ_3 Type Condition for Heavy Cycles in Weighted Graphs

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Abstract. A weighted graph is one in which every edge e is assigned a nonnegative number $w(e)$, called the weight of e. The weight of a cycle is defined as the sum of the weights of its edges. The weighted degree of a vertex is the sum of the weights of the edges incident with it. In this paper, motivated by a recent result of Fujisawa, we prove that a 2-connected weighted graph G contains either a Hamilton cycle or a cycle of weight at least $2m/3$ if it satisfies the following conditions: (1) The weighted degree sum of every three pairwise nonadjacent vertices is at least m; (2) In each induced claw and each induced modified claw of G, all edges have the same weight. This extends a theorem of Zhang, Broersma and Li.

1 Terminology and notation

We use Bondy and Murty [3] for terminology and notation not defined here and consider finite simple graphs only.

Let $G = (V, E)$ be a simple graph. G is called a *weighted graph* if each edge e is assigned a nonnegative number $w(e)$, called the *weight* of e. For a subgraph H of G, $V(H)$ and $E(H)$ denote the sets of vertices and edges of H, respectively. The *weight* of H is defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

For a vertex $v \in V$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of vertices in H that are adjacent to v. We define the *weighted degree* of v in

H by

$$d_H^w(v) = \sum_{h \in N_H(v)} w(vh).$$

When no confusion occurs, we will denote $N_G(v)$, $d_G(v)$ and $d_G^w(v)$ by $N(v)$, $d(v)$ and $d^w(v)$, respectively.

An unweighted graph can be regarded as a weighted graph in which each edge e is assigned weight $w(e) = 1$. Thus, in an unweighted graph, $d^w(v) = d(v)$ for every vertex v, and the weight of a subgraph is simply the number of edges of it.

An (x, y)*-path* is a path connecting the two vertices x and y. The *distance* between two vertices x and y, denoted by $d(x, y)$, is the length of a shortest (x, y)-path. If u and v are two vertices on a path P, $P[u, v]$ denotes the segment of P from u to v.

The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$. If G is noncomplete, then for a positive integer $k \leq \alpha(G)$ we denote by $\sigma_k(G)$ the minimum value of the degree sum of any k pairwise nonadjacent vertices, and by $\sigma_k^w(G)$ the minimum value of the weighted degree sum of any k pairwise nonadjacent vertices. If G is complete, then both $\sigma_k(G)$ and $\sigma_k^w(G)$ are defined as ∞.

We call the graph $K_{1,3}$ a *claw*, and the graph obtained by joining a pendant edge to some vertex of a triangle a *modified claw*.

2 Results

There have been many results on the existence of long cycles in graphs. The following theorem is well-known.

Theorem A (Pósa [7]). *Let G be a 2-connected graph such that $\sigma_2(G) \geq s$. Then G contains either a Hamilton cycle or a cycle of length at least s.*

This result was generalized by the following two theorems along different lines.

Theorem B (Fan [4]). *Let G be a 2-connected graph such that* $\max\{d(x), d(y) | d(x, y) = 2\} \geq c/2$. *Then G contains either a Hamilton cycle or a cycle of length at least c.*

Theorem C (Fournier & Fraisse [5]). *Let G be a k-connected graph where $2 \leq k < \alpha(G)$, such that $\sigma_{k+1}(G) \geq m$. Then G contains either a Hamilton cycle or a cycle of length at least $2m/(k + 1)$.*

Bondy *et al.* [2] gave a weighted generalization of Theorem A as follows.

Theorem 1 (Bondy *et al.* [2]). *Let G be 2-connected weighted graph such that $\sigma_2^w(G) \geq s$. Then G contains either a Hamilton cycle or a cycle of weight at least s.*

In [9], it was showed that if one wants to give a generalization of Theorem B to weighted graphs some extra conditions cannot be avoided. By adding two extra conditions, the authors gave a weighted generalization of Theorem B.

Theorem 2 (Zhang *et al.* [9]). *Let G be a 2-connected weighted graph which satisfies the following conditions:*
(1) $max\{d^w(x), d^w(y) | d(x,y) = 2\} \geq s/2$;
(2) $w(xz) = w(yz)$ *for every vertex* $z \in N(x) \cap N(y)$ *with* $d(x,y) = 2$;
(3) *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*
Then G contains either a Hamilton cycle or a cycle of weight at least s.

Motivated by this result, Zhang *et al.* [8] gave a weighted generalization of Theorem C in the case $k = 2$.

Theorem 3 (Zhang *et al.* [8]). *Let G be a 2-connected weighted graph which satisfies the following conditions:*
(1) $\sigma_3^w(G) \geq m$;
(2) $w(xz) = w(yz)$ *for every vertex* $z \in N(x) \cap N(y)$ *with* $d(x,y) = 2$;
(3) *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*
Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$.

Theorem B was further extended by the following result.

Theorem D (Bedrossian *et al.* [1]). *Let G be a 2-connected graph. If* $\max\{d(x), d(y)\} \geq c/2$ *for each pair of nonadjacent vertices x and y, which are vertices of an induced claw or an induced modified claw of G, then G contains either a Hamilton cycle or a cycle of length at least c.*

Fujisawa [6] gave a weighted generalization of Theorem D. The result also generalizes Theorem 2.

Theorem 4 (Fujisawa [6]). *Let G be a 2-connected weighted graph which satisfies the following conditions:*
(1) *For each induced claw and each induced modified claw of G, all its nonadjacent pair of vertices x and y satisfy* $max\{d^w(x), d^w(y)\} \geq s/2$;
(2) *For each induced claw and each induced modified claw of G, all of its edges have the same weight.*
Then G contains either a Hamilton cycle or a cycle of weight at least s.

It is clear that Condition (2) of Theorem 4 is weaker than Conditions (2) and (3) of Theorem 3. Our main result in this paper is a further weighted generalization of Theorem C in the case $k = 2$. It turns out that Conditions (2) and (3) of Theorem 3 can be replaced by Condition (2) of Theorem 4.

Theorem 5. *Let G be a 2-connected weighted graph which satisfies the following conditions:*
(1) $\sigma_3^w(G) \geq m$;
(2) For each induced claw and each induced modified claw of G, all of its edges have the same weight.
Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$.

We postpone the proof of Theorem 5 to the next section.

3 Proof of Theorem 5

In the proof of Theorem 5, we call a path P a *heaviest longest path* if P has the following properties

- P is a longest path of G, and
- $w(P)$ is maximum among all longest paths in G.

To prove Theorem 5, we need the following lemmas. The proof of Lemma 1 is implicit in [2].

Lemma 1 (Bondy *et al.* [2]). *Let G be a non-hamiltonian 2-connected weighted graph and $P = v_1v_2\cdots v_p$ be a heaviest longest path in G. Then there is a cycle C in G with $w(C) \geq d^w(v_1) + d^w(v_p)$.*

Lemma 2 (Fujisawa [6]). *Let G be a weighted graph satisfying Condition (2) of Theorem 5. If x_1yx_2 is an induced path with $w(x_1y) \neq w(x_2y)$ in G, then each vertex $x \in N(y)\backslash\{x_1, x_2\}$ is adjacent to both x_1 and x_2.*

Lemma 3 (Fujisawa [6]). *Let G be a weighted graph satisfying Condition (2) of Theorem 5. Suppose x_1yx_2 is an induced path such that $w_1 = w(x_1y)$ and $w_2 = w(x_2y)$ with $w_1 \neq w_2$, and yz_1z_2 is a path such that $\{z_1, z_2\} \cap \{x_1, x_2\} = \emptyset$ and $x_2z_2 \notin E(G)$. Then*
(i) $\{z_1x_1, z_1x_2, z_2x_1\} \subseteq E(G)$, and $yz_2 \notin E(G)$. Moreover, all edges in the subgraph induced by $\{x_1, y, x_2, z_1, z_2\}$, other than x_1y, have the same weight w_2.
(ii) Let Y be the component of $G - \{x_2, z_1, z_2\}$ with $y \in V(Y)$. For each vertex $v \in V(Y)\backslash\{x_1, y\}$, v is adjacent to all of x_1, x_2, y and z_2. Furthermore, $w(vx_1) = w(vx_2) = w(vy) = w(vz_2) = w_2$.

Proof of Theorem 5. Let G be a 2-connected weighted graph satisfying the conditions of Theorem 5. Suppose that G does not contain a Hamilton cycle. Then it suffices to prove that G contains a cycle of weight at least $2m/3$.

Choose a path $P = v_1v_2 \cdots v_p$ in G such that
(a) P is a heaviest longest path in G;
(b) $d^w(v_1) + d^w(v_p)$ is as large as possible, subject to (a).
From the choice of P, we can immediately see that $N(v_1) \cup N(v_p) \subseteq V(P)$. And it is not difficult to prove that there exists no cycle of length p.

It follows from Lemma 1 that there exists a cycle C of weight $w(C) \geq d^w(v_1) + d^w(v_p)$. Without loss of generality, we assume $d^w(v_1) \leq w(C)/2$.

Claim 0. *Let P_1 and P_2 be two heaviest longest paths such that P_1 has v' and v_p as its end-vertices, and P_2 has v'' and v_p as its end-vertices. If $v'v'' \notin E(G)$, then $w(C) \geq 2m/3$.*

Proof. Since P_1 and P_2 are heaviest longest paths, $v'v_p \notin E(G)$ and $v''v_p \notin E(G)$. Then v', v'' and v_p are pairwise nonadjacent. By the choice of the path P in (b), $d^w(v') \leq d^w(v_1) \leq w(C)/2$ and $d^w(v'') + d^w(v_p) \leq w(C)$. So we have $d^w(v') + d^w(v'') + d^w(v_p) \leq 3w(C)/2$. It follows from Condition (1) of the theorem that $w(C) \geq 2m/3$. □

Since G is 2-connected, v_1 is adjacent to at least one vertex on P other than v_2. Choose $v_k \in N(v_1)$ such that k is as large as possible. It is clear that $3 \leq k \leq p-1$.

Since $G - v_k$ is connected, there must be a path Q such that
- Q has end-vertices v_r and v_s, such that $r < k < s$, and
- $V(Q) \cap V(P) = \{v_r, v_s\}$.

We assume that such a path Q was chosen so that
(i) s is as large as possible;
(ii) r is as large as possible, subject to (i).

Case 1. $v_1v_i \in E(G)$ for every i with $r \leq i \leq k$.

Claim 1.1. $v_rv_s \in E(G)$.

Proof. Since $r < k$, we have $v_1v_{r+1} \in E(G)$. If there exists a vertex $u \notin \{v_r, v_s\}$ on Q, then the path $Q[u, v_r]v_rv_{r-1} \cdots v_1v_{r+1}v_{r+2} \cdots v_p$ is longer than P, a contradiction. □

Case 1.1. $s > k+1$.

Claim 1.2. $w(v_1v_{r+1}) = w(v_rv_{r+1})$.

Proof. First, we consider the case $r < k-1$. By the choices of v_k and v_r, $v_1v_s \notin E(G)$ and $v_{r+1}v_s \notin E(G)$. So $\{v_r, v_{r+1}, v_1, v_s\}$ induces a modified claw. Then we get $w(v_1v_{r+1}) = w(v_rv_{r+1})$.

Now consider the case $r = k-1$. We need prove $w(v_1v_k) = w(v_{k-1}v_k)$.

By the choices of v_k and v_s, $v_1v_{s+1} \notin E(G)$ and $v_{k-1}v_{s+1} \notin E(G)$. So, if $v_kv_{s+1} \in E(G)$, then $\{v_k, v_{k-1}, v_1, v_{s+1}\}$ induces a modified claw. Then we get $w(v_1v_k) = w(v_{k-1}v_k)$.

By the choice of v_k, $v_1v_s \notin E(G)$. So, if $v_kv_s \notin E(G)$, then $\{v_{k-1}, v_k, v_1, v_s\}$ induces a modified claw. Then we get $w(v_1v_k) = w(v_{k-1}v_k)$.

Clearly we need only consider the case $v_kv_{s+1} \notin E(G)$ and $v_kv_s \in E(G)$. By the choice of v_k, $v_1v_{k+1} \notin E(G)$ and $v_1v_s \notin E(G)$. So $\{v_k, v_{k+1}, v_1, v_s\}$ induces a claw or a modified claw, which implies that $w(v_1v_k) = w(v_kv_s)$; On the other hand, by the choice of v_s, $v_{k-1}v_{s+1} \notin E(G)$. So $\{v_s, v_{k-1}, v_k, v_{s+1}\}$ induces a modified claw, which implies that $w(v_kv_s) = w(v_{k-1}v_k)$. Thus we have $w(v_1v_k) = w(v_{k-1}v_k)$. □

Claim 1.3. $w(v_{s-1}v_s) = w(v_rv_s)$.

Proof. By the choice of v_k, $v_1v_{s-1} \notin E(G)$ and $v_1v_s \notin E(G)$. So, if $v_{s-1}v_r \in E(G)$, then $\{v_r, v_{s-1}, v_s, v_1\}$ induces a modified claw, which implies that $w(v_{s-1}v_s) = w(v_rv_s)$. By the choice of v_s, $v_rv_{s+1} \notin E(G)$. So, if $v_{s-1}v_r \notin E(G)$, then $\{v_s, v_{s-1}, v_{s+1}, v_r\}$ induces a claw or a modified claw. Thus we have $w(v_{s-1}v_s) = w(v_rv_s)$. □

It follows from Claims 1.2 and 1.3 that $v_{s-1}v_{s-2}\cdots v_{r+1}v_1v_2\cdots v_rv_s v_{s+1}\cdots v_p$ is a heaviest longest path different from P and with v_p as one of its end-vertices. At the same time, by the choice of v_k, $v_1v_{s-1} \notin E(G)$. From Claim 0 we have $w(C) \geq 2m/3$.

Case 1.2. $s = k + 1$.

From the choice of the path Q and the connectedness of $G - v_{k+1}$, there exists a path R such that

- R has end-vertices v_k and v_t with $k + 2 \leq t < p$, and
- $V(R) \cap V(P) = \{v_k, v_t\}$.

Choose R such that t is as large as possible. Similar to the proof of Claim 1.1, we have the following claim:

Claim 1.4. $v_kv_t \in E(G)$. □

Claim 1.5. $w(v_1v_{r+1}) = w(v_rv_{r+1})$.

Proof. Suppose $r < k - 1$. By the same proof as for Claim 1.2, we get $w(v_1v_{r+1}) = w(v_rv_{r+1})$.

Suppose $r = k - 1$. By the choices of v_k and v_s, $v_1v_t \notin E(G)$ and $v_{k-1}v_t \notin E(G)$. So $\{v_k, v_{k-1}, v_1, v_t\}$ induces a modified claw, so we get $w(v_1v_k) = w(v_{k-1}v_k)$. □

Claim 1.6. $w(v_kv_t) = w(v_{t-1}v_t)$.

Proof. Suppose $v_k v_{t-1} \notin E(G)$. By the choice of v_t, $v_k v_{t+1} \notin E(G)$, then $\{v_t, v_{t-1}, v_{t+1}, v_k\}$ induces a claw or a modified claw. So we get $w(v_k v_t) = w(v_{t-1} v_t)$.

Suppose $v_k v_{t-1} \in E(G)$. By the choice of v_k, $v_1 v_{t-1} \notin E(G)$ and $v_1 v_t \notin E(G)$, then $\{v_k, v_{t-1}, v_t, v_1\}$ induces a modified claw. So we get $w(v_k v_t) = w(v_{t-1} v_t)$. □

Claim 1.7. $w(v_r v_{k+1}) = w(v_k v_{k+1})$.

Proof. Suppose $v_r v_k \notin E(G)$. By the choice of v_s, $v_r v_{k+2} \notin E(G)$. So $\{v_{k+1}, v_k, v_{k+2}, v_r\}$ induces a claw or a modified claw. Thus $w(v_r v_{k+1}) = w(v_k v_{k+1})$.

Suppose $v_r v_k \in E(G)$. By the choices of v_k and v_t, $v_1 v_{k+1} \notin E(G)$, $v_1 v_t \notin E(G)$ and $v_r v_t \notin E(G)$. So $\{v_k, v_{k+1}, v_t, v_1\}$ induces a claw or a modified claw and $\{v_k, v_1, v_r, v_t\}$ induces a modified claw. Thus $w(v_1 v_k) = w(v_k v_{k+1})$ and $w(v_1 v_k) = w(v_1 v_r)$. So $w(v_1 v_r) = w(v_1 v_k) = w(v_k v_{k+1})$. We conclude that $w(v_r v_{k+1}) = w(v_1 v_r)$. Otherwise, apply Lemma 3 (*ii*) to the subgraph induced by $\{v_1, v_r, v_{k+1}, v_k, v_t\}$. Since there must be a vertex $v_{t'} \in V(P[v_{k+2}, v_p]) \backslash \{v_t\}$ such that $v_{k+1} v_{t'} \in E(G)$, we have $v_{t'}$ is in the component of $G - \{v_1, v_k, v_t\}$. Thus we have $v_{t'} v_1 \in E(G)$, contradicting the choice of v_k. So we get $w(v_r v_{k+1}) = w(v_k v_{k+1})$. □

It follows from Claims 1.5, 1.6 and 1.7 that $v_{t-1} v_{t-2} \cdots v_{k+1} v_r v_{r-1} \cdots v_1 v_{r+1} \cdots v_k v_t \, v_{t+1} \cdots v_p$ is a heaviest longest path different from P and with v_p as one of its end-vertices. By the choice of v_k, $v_1 v_{t-1} \notin E(G)$. Then from Claim 0, we have $w(C) \geq 2m/3$. This completes the proof of Case 1.

Case 2. *$v_1 v_i \notin E(G)$ for some i with $r \leq i < k$.*

Choose $v_l \notin N(v_1)$ with $r \leq l < k$ such that l is as large as possible. It is clear that $3 \leq l < k$ and $v_1 v_i \in E(G)$ for every i with $l < i \leq k$. Let j be the smallest index such that $j > l$ and $v_j \notin N(v_1) \cap N(v_l)$. Since $v_{l+1} \in N(v_1) \cap N(v_l)$, we have $j \geq l + 2$. Also, it is obvious that $j \leq k + 1$.

If $w(v_1 v_{l+1}) = w(v_l v_{l+1})$, then $v_l v_{l-1} \cdots v_1 v_{l+1} v_{l+2} \cdots v_p$ is a heaviest longest path different from P and with v_p as one of its end-vertices. By the choice of v_l, $v_1 v_l \notin E(G)$. Then from Claim 0, we have $w(C) \geq 2m/3$. From now on, we have the following assumption.

Assumption 1. $w(v_1 v_{l+1}) \neq w(v_l v_{l+1})$.

Claim 2.1. $v_{l+1} v_j \notin E(G)$.

Proof. If $v_{l+1} v_j \in E(G)$, then $v_j \in N(v_1) \cap N(v_l)$ by Lemma 2, contradicting the choice of v_j. □

Claim 2.2. $v_{l+1} v_{j-1} \notin E(G)$ and $v_{l+2} v_j \notin E(G)$.

Proof. Suppose $v_{l+1}v_{j-1} \in E(G)$. If $j \leq k$, then by the choice of v_j, we have $v_lv_j \notin E(G)$. Apply Lemma 3 (*ii*) to the subgraph induced by $\{v_1, v_{l+1}, v_l, v_{j-1}, v_j\}$. Since v_s is in the component of $G - \{v_l, v_{j-1}, v_j\}$, we have $v_sv_1 \in E(G)$, contradicting the choice of v_k. So $v_{l+1}v_{j-1} \notin E(G)$. The case $j = k+1$ can be proved similarly.

Similar to the proof of above, we can prove that $v_{l+2}v_j \notin E(G)$. □

From Claim 2.2, we know that $l + 4 \leq j \leq k + 1$.

Case 2.1. $j \leq k$.

Claim 2.3. $w(v_{l+1}v_{l+2}) = w(v_1v_{l+2}) = w(v_{j-1}v_j) = w(v_1v_j)$.

Proof. By Claims 2.1 and 2.2, we know that both $\{v_1, v_{l+1}, v_{l+2}, v_j\}$ and $\{v_1, v_{j-1}, v_j, v_{l+1}\}$ induce modified claws. Thus $w(v_{l+1}v_{l+2}) = w(v_1v_{l+2}) = w(v_{j-1}v_j) = w(v_1v_j)$. □

It follows from Claim 2.3 that each of $v_{l+1}v_l \cdots v_1v_{l+2}v_{l+3} \cdots v_p$ and $v_{j-1}v_{j-2} \cdots v_1v_j\ v_{j+1} \cdots v_p$ is a heaviest longest path with v_p as one of its end-vertices. By Claim 2.2 and Claim 0, we have $w(C) \geq 2m/3$.

Case 2.2. $j = k + 1$.

Claim 2.4. $w(v_1v_k) = w(v_lv_k) = w(v_kv_{k+1})$.

Proof. By the choices of v_k and v_l, $v_1v_{k+1} \notin E(G)$ and $v_1v_l \notin E(G)$. So $\{v_k, v_{k+1}, v_l, v_1\}$ induces a claw or a modified claw, thus $w(v_1v_k) = w(v_lv_k) = w(v_kv_{k+1})$. □

Claim 2.5. *For any vertex* $v \in N(v_1) \cap N(v_l) \backslash \{v_{l+1}, v_k\}$, *we have* $vv_k \in E(G), vv_{l+1} \in E(G)$ *and* $vv_{k+1} \notin E(G)$.

Proof. From Assumption 1 and Claim 2.4, we know that $w(v_lv_{l+1}) = w(v_lv_k)$ and $w(v_1v_{l+1}) = w(v_1v_k)$ cannot hold at the same time. Suppose $w(v_lv_{l+1}) \neq w(v_lv_k)$. Then applying Lemma 2 to the induced path $v_kv_lv_{l+1}$and $v \in N(v_l)\backslash\{v_k, v_{l+1}\}$, we get $vv_k \in E(G)$ and $vv_{l+1} \in E(G)$. Now if $vv_{k+1} \in E(G)$, then apply Lemma 3 (*ii*) to the subgraph induced by $\{v_k, v_l, v_{l+1}, v, v_{k+1}\}$. Since v_1 is in the component of $G - \{v_{l+1}, v, v_{k+1}\}$, we have $v_1v_l \in E(G)$, contradicting the choice of v_l. So $vv_{k+1} \notin E(G)$. The case $w(v_1v_{l+1}) \neq w(v_1v_k)$ can be proved similarly. □

Claim 2.6. $w(v_1v_{k-1}) = w(v_{k-1}v_k)$.

Proof. By Claim 2.5, $v_{k-1}v_{k+1} \notin E(G)$. So $\{v_k, v_{k-1}, v_1, v_{k+1}\}$ induces a modified claw, thus $w(v_1v_{k-1}) = w(v_{k-1}v_k)$. □

Suppose $v_1v_{l-1} \in E(G)$. Then by Claim 2.5, we have $v_{l-1}v_k \in E(G)$, $v_{l-1}v_{l+1} \in E(G)$ and $v_{l-1}v_{k+1} \notin E(G)$. So $\{v_k, v_1, v_{l-1}, v_{k+1}\}$ induces a modified claw, thus $w(v_1v_{l-1}) = w(v_{l-1}v_k)$. We claim that $w(v_1v_{l-1}) = w(v_{l-1}v_l)$. Otherwise, apply Lemma 3 (*ii*) to the subgraph induced by $\{v_l, v_{l-1}, v_1, v_k, v_{k+1}\}$. Since v_{l+1} is in the component of $G - \{v_1, v_k, v_{k+1}\}$, we have $v_{l+1}v_{k+1} \in E(G)$, contradicting Claim 2.1. So we get $w(v_{l-1}v_l) = w(v_{l-1}v_k)$. Therefore, from Claim 2.6 we know that $v_lv_{l+1}\cdots v_{k-1}v_1v_2\cdots v_{l-1}v_kv_{k+1}\cdots v_p$ is a heaviest longest path different from P and with v_p as one of its end-vertices. By the choice of v_l, $v_1v_l \notin E(G)$. Then from Claim 0, we have $w(C) \geq 2m/3$.

Suppose $v_1v_{l-1} \notin E(G)$, we have $v_{l-1}v_{l+1} \notin E(G)$. Otherwise, $\{v_{l+1}, v_l, v_{l-1}, v_1\}$ induces a modified claw, contradicting the assumption $w(v_1v_{l+1}) \neq w(v_lv_{l+1})$. From Claim 2.2, we know that $\{v_l, v_{l-1}, v_k, v_{l+1}\}$ induces a claw or a modified claw. So $w(v_{l-1}v_l) = w(v_lv_k)$. Therefore, from Claim 2.6, we know that $v_{l-1}v_{l-2}\cdots v_1v_{k-1}\ v_{k-2}\cdots v_lv_kv_{k+1}\cdots v_p$ is a heaviest longest path different from P and with v_p as one of its end-vertices. Then from Claim 0, we have $w(C) \geq 2m/3$.

The proof is complete.

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