Graph designs for eight graphs with six vertices and eight edges (index >1) *

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Abstract

Let K_v be a *complete graph* with v vertices, and $G=(V(G),E(G))$ be a finite simple graph. A G-design G-$GD_\lambda(v)$ is a pair $(X,\mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are jointed in exactly λ blocks of $\mathcal{B}$. In this paper, the existence of graph designs G-$GD_\lambda(v)$, $\lambda > 1$, for eight graphs G with six vertices and eight edges is completely solved.

Key words: graph design, holey graph design, quasi-group.

1 Introduction

A *group-divisible design* $GDD(2,K,v;r_1\{m_1\},\cdots,r_s\{m_s\})$, where $K\subseteq N$, $\sum_{i=1}^{s} r_i m_i = v$, and for any $k\in K, k\geq 2$, is a triple $(X,\mathcal{G},\mathcal{B})$ such that

1) X is a set of v points,

2) $\mathcal{G}$ is a partition of X into r_i sets of m_i points (called *groups*), $i = 1,2,\cdots,s$,

3) $\mathcal{B}$ is a collection of subsets of X (called *blocks*), where $|B|\in K$,

4) Every 2-set of X is contained in exactly one member of $\mathcal{G}\bigcup\mathcal{B}$.

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*Research supported by NSFC Grant 10671055 and NSFHB Grant A2007000230.
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Usually, we write K-$GDD(m_1^{r_1}\cdots m_s^{r_s})$ and k-$GDD(m_1^{r_1}\cdots m_s^{r_s})$ instead of $GDD(2,K,v;r_1\{m_1\},\cdots,r_s\{m_s\})$ and $\{k\}$-$GDD(m_1^{r_1}\cdots m_s^{r_s})$. A k-$GDD(m^k)$ is called a *transversal design* and denoted by $TD(k,m)$.

A $GDD(2,K,v;v\{1\})=(X,\mathcal{G},\mathcal{B})$ is often called *pairwise balanced design* and denoted by $B[K,1;v]=(X,\mathcal{B})$.

Let K_v be a *complete graph* with v vertices, and G=$(V(G),E(G))$ be a finite simple graph. A G-design G-$GD_\lambda(v)$ is a pair $(X,\mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are jointed in exactly λ blocks of $\mathcal{B}$. Obviously, the necessary conditions for the existence of a G-$GD_\lambda(v)$ are

$$v\geq|V(G)|,\ \lambda v(v-1)\equiv 0 \mod 2|E(G)|,\ \lambda(v-1)\equiv 0 \mod d, \qquad (*)$$

where d is the greatest common divisor of the degrees of the vertices in $V(G)$.

Let $K_{n_1,n_2,\cdots,n_t}$ be a *complete multipartite graph* with vertex set $\bigcup_{i=1}^{t} X_i$, where these X_i are disjoint and $|X_i|=n_i$, $1\leq i\leq t$. For a given graph G, a *holey G-design*, denoted by G-$HD_\lambda(n_1^1n_2^1\cdots n_t^1)$, is a partition $\mathcal{A}$ of edges of $\lambda K_{n_1,n_2,\cdots,n_t}$, such that each member of $\mathcal{A}$ is isomorphic to G. If $n_1=\cdots=n_t=n$, then the holey G-design may be denoted by G-$HD_\lambda(n^t)$. For $\lambda=1$, the index 1 is often omitted. A G-$HD_\lambda(1^vw^1)$ is called an *incomplete G-design*, denoted by G-$ID_\lambda(v+w,w)$. Obviously, a G-$GD_\lambda(v)$ can be regarded as a G-$HD_\lambda(1^v)$, a G-$ID_\lambda(v+0,0)$ or a G-$ID_\lambda((v-1)+1,1)$.

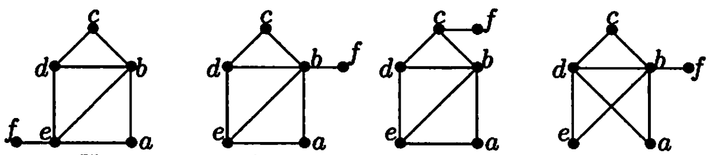
From [1], there are 22 graphs with six vertices and eight edges without isolated vertex, which are shown in [3]. For $\lambda=1$, the existence of graph designs for these graphs has been solved by us:

Lemma 1.1[3]

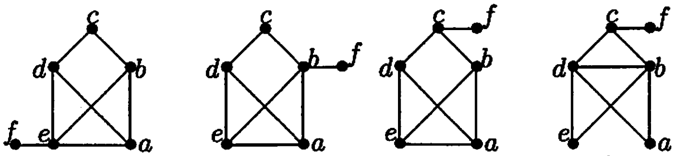
(1) *For graph* $G\in\{H_i,D_i,R_j,Q_i,M_j,C_k,W_3: 1\leq i\leq 3, 1\leq j\leq 2, 1\leq k\leq 6\}$, *there exists a* G-$GD(v)$ *if and only if* $v\equiv 0,1 \pmod{16}$ *and* $v\geq 16$ *with possible exception* $v=32$ *for graphs* M_1 *and* M_2.

(2) *For graphs* $G=W_1$ *and* W_2, *there exists a* G-$GD(v)$ *if and only if* $v\equiv 1 \pmod{16}$ *and* $v\geq 17$.

In this paper, we shall focus on graph designs of the following eight graphs for $\lambda>1$.



H_1 H_2 H_3 R_1



D_1 D_2 D_3 R_2

For convenience, all graphs above are denoted by (a, b, c, d, e, f). Our main conclusions will be:

Theorem 1.2 *For graph $G \in \{H_i, D_i, R_j : 1 \le i \le 3, 1 \le j \le 2\}$, the necessary conditions for the existence of G-$GD_\lambda(v)$ with $\lambda > 1$ are also sufficient with the exceptions $(G, v, \lambda) \in \{(D_3, 8, 2), (D_3, 6, 16t+8) : t \ge 0\}$.*

By $(*)$, we need discuss the following v and λ:

$\lambda = 2,\ v \equiv 0, 1 \pmod 8$; $\lambda = 4,\ v \equiv 0, 1 \pmod 4$; $\lambda = 8,\ v \ge 6$. $(**)$

The following lemmas are important for our constructing methods in this paper.

Lemma 1.3 *Let G be a simple graph, K be a set of positive integers, and m, u, v, λ, μ be positive integers.*

(1) *If there exist a K-$GDD(a^u b^v)$ and a G-$HD_\lambda(m^k)$ for any $k \in K$, then there exists a G-$HD_\lambda((ma)^u (mb)^v)$.*

(2) *If there exists a G-$HD_\lambda(m^h)$, then there exists a G-$HD_{\lambda\mu}(m^h)$.*

Proof. Obviously, the conclusions hold.

Lemma 1.4[4] *Let G be a simple graph, and h, m, n, λ be positive integers, $w \ge 0$.*

(1) *If there exist a G-$HD_\lambda(m^h)$, a G-$ID_\lambda(m+w, w)$ and a G-$GD_\lambda(m+w)$ (or G-$GD_\lambda(w)$), then there exists a G-$GD_\lambda(mh + w)$.*

(2) *If there exist a G-$HD_\lambda(m^h n^1)$, a G-$ID_\lambda(m+w, w)$ and a G-$GD_\lambda(n+w)$, then there exists a G-$GD_\lambda(mh + n + w)$.*

2 Main structures

The following lemma is the modified version of Theorem 2.2.7 in [3], where G is a graph with six vertices and eight edges.

Lemma 2.1 *Let m be a positive integer, $q = 3, 4, 5$, $w = 0, 1$ and $i = 1, 2$. If there exist a G-$HD_2(m^q)$ and a G-$GD_2(im + w)$, then there exists a G-$GD_2(v)$ for $v \equiv 0, 1 \pmod m$ and $v \ge m$.*

Lemma 2.2 *Let $q \in \{3, 4, 5\}$, $m \in \{1, 2, 5\}$, $w = 2, 3, 6, 7$. If there exist a G-$HD_\lambda(8^q)$, a G-$ID_\lambda(8+w, w)$, a G-$ID_\lambda(16+w, w)$ and a G-$GD_\lambda(8m+w)$, then there exists a G-$GD_\lambda(v)$ for $v \equiv 2, 3, 6, 7 \pmod 8$ and $v \ge 10$.*

Proof. Let $v = 8n + w$, $w = 2, 3, 6, 7$.

(1) For $n \equiv 1, 3 \bmod 6$, there exists a $B[3, 1; n]$ by [2], which implies the existence of a 3-$GDD(1^n)$. And, by Lemma 1.3(1) and Lemma 1.4(1),

the existence of $G\text{-}HD_\lambda(8^3)$, $G\text{-}ID_\lambda(8+w,w)$ and $G\text{-}GD_\lambda(8+w)$ implies the existence of $G\text{-}GD_\lambda(v)$.

(2) For $n \equiv 0,2 \bmod 6$, there exists a $B[3,1;n+1]$ by [2], which implies the existence of a $3\text{-}GDD(2^{\frac{n}{2}})$. By Lemma 1.3(1), there exists a $G\text{-}HD_\lambda(16^{\frac{n}{2}})$ from the known $G\text{-}HD_\lambda(8^3)$. Furthermore, by Lemma 1.4(1), there exists a $G\text{-}GD_\lambda(v)$ from the known $G\text{-}ID_\lambda(16+w,w)$ and $G\text{-}GD_\lambda(16+w)$.

(3) For $n \equiv 3+r \bmod 6$, r=1,2, there exists an $RB[3,1;n-r]$ by [2]. Letting $n-r=6t+3$, the number of the parallel classes of $RB[3,1;n-r]$ is $3t+1$. In order to guarantee $3t+1 \geq r$, it is necessary that "$t \geq 0$ if $r=1$" or "$t \geq 1$ if $r=2$". Furthermore, a $\{3,4\}\text{-}GDD(1^{n-r}r^1)$ can be obtained from $RB[3,1;n-r]$. And, by Lemma 1.3(1), there exists a $G\text{-}HD_\lambda(8^{n-r}(8r)^1)$ by adding the known $G\text{-}HD_\lambda(8^3)$,$G\text{-}HD_\lambda(8^4)$. Thus, by Lemma 1.4(2), a $G\text{-}GD_\lambda(v)$ can be obtained from $G\text{-}ID_\lambda(8+w,w)$ and $G\text{-}GD_\lambda(8r+w)$. As for "$r=2$ and $t=0$", i.e., $G\text{-}GD_\lambda(5\times 8+w)$, which can be obtained from $G\text{-}HD_\lambda(8^5)$, $G\text{-}ID_\lambda(8+w,w)$ and $G\text{-}GD_\lambda(8+w)$.■

Lemma 2.3 *If there exist a $G\text{-}HD_2(8^{2t+1})$ for $t \geq 1$, a $G\text{-}ID_2(8+16,16)$, a $G\text{-}GD_2(9)$ and a $G\text{-}GD_2(16)$, then there exists a $G\text{-}GD_2(v)$ for $v \equiv$ 8,9 (mod 16) and $v \geq 9$.*

Proof. Let v=$8(2t+1)+w$, and $t \geq 1$ (if $w=0$) or $t \geq 0$ (if $w=1$).

For $w=0$ and $t=1$, a $G\text{-}GD_2(24)$ exists from the known $G\text{-}ID_2(8+16,16)$ and $G\text{-}GD_2(16)$.

For $w=0$ and $t \geq 2$, by Lemma 1.4(1), the conclusion follows from the designs $G\text{-}HD_2(8^{2t-1})$ for $t \geq 2$ and $G\text{-}ID_2(8+16,16)$ and $G\text{-}GD_2(16)$.

For $w=1$ and $t \geq 0$, by Lemma 1.4(1), the conclusion follows from the designs $G\text{-}HD_2(8^{2t+1})$ for $t \geq 1$ and $G\text{-}GD_2(9)$. ■

Lemma 2.4[5] *Let positive integer $w<8$, q=3,4,5 and $t \in \{1,2,6,8\}$. If there exist a $G\text{-}HD_\lambda(8^q)$, a $G\text{-}ID_\lambda(8+w,w)$ and a $G\text{-}GD_\lambda(8t+w)$, then there exists a $G\text{-}GD_\lambda(v)$ for $v \equiv w$ (mod 8) and $v \geq 8+w$.*

3 Construction of *HD*

3.1 Using sharply 2-transitive group

Let H be a transformation group acting on n-set N. For any two ordered 2-subsets (x,y) and (x',y') from N, if there exists unique $\xi \in H$ satisfying $(\xi x, \xi y)$=(x',y'), then H is called a ***sharply 2-transitive group*** on N.

Lemma 3.1[3] *Let F_q be a finite field, where q is a prime power. Then, for the multiplication of transformations, all linear transformations on F_q*

$$f_{c,d} : x \longmapsto cx+d \quad \forall x \in F_q$$

form a sharply 2-transitive group on F_q: $L_q=\{f_{c,d} : c \in F_q^*,\ d \in F_q\}$.

Lemma 3.2[5] *Let G be a graph with $2e$ edges. If*

(1) *there exists a mapping f (i.e. vertex labeling) from its vertex set $V(G)$ to the set Z_{2e} such that the induced mapping on its edge set (i.e. edge labeling)*

$$f^*: (x,y) \longmapsto |f(x)-f(y)| \quad \forall x \neq y \in V(G)$$

satisfies $\{f^*(x,y) : x \neq y \in V(G)\} = \{1,1,2,2,\cdots,e-1,e-1\} \bigcup \{0,e\}$,

(2) *G is q-colorable (the coloring set is Q),*

(3) *there exists a sharply 2-transitive group on Q,*

then there exists a G-$HD_2((2e)^q)$, where q is a prime power.

Lemma 3.3 *For graph $G \in \{H_i, D_i, R_j : 1 \leq i \leq 3, 1 \leq j \leq 2\}$, there exists a G-$HD_2(8^q)$ for $q = 3,4,5$.*

Proof. Let $X = Z_8 \times Z_q$, and L_q be the sharply 2-transitive group on a q-set. For the following blocks B and C, (B,C) mod (Z_8, L_q) form the block set of G-$HD_2(8^q)$.

H_1: $B = (1,3,6,2,0,0)$, $C = (2,0,1,2,1,0)$;
H_2: $B = (1,3,6,2,0,3)$, $C = (2,0,1,2,1,1)$;
H_3: $B = (1,3,6,2,0,6)$, $C = (2,0,1,2,1,0)$;
D_1: $B = (1,3,6,2,0,0)$, $C = (2,0,1,0,1,0)$;
D_2: $B = (1,3,6,2,0,3)$, $C = (2,0,1,0,1,1)$;
D_3: $B = (1,3,6,2,0,6)$, $C = (2,0,1,0,1,0)$;
R_1: $B = (1,3,6,2,0,3)$, $C = (2,0,1,0,1,1)$;
R_2: $B = (1,3,6,2,0,6)$, $C = (2,0,1,0,1,0)$. ∎

3.2 Using idempotent symmetric quasi-group

Let I_n be a n-set and $\circ$ be a binary operation on I_n such that the equations $a \circ x = b$ and $y \circ a = b$ are uniquely solvable for every pair of elements a, $b \in I_n$, then $(I_n, \circ)$ is called as a *quasi-group* of order n. A quasi-group is said to be *idempotent* (*symmetric*) if the identity $x \circ x = x$ ($x \circ y = y \circ x$) holds for all $x \in I_n$ ($x, y \in I_n$). It is well known that there exists an idempotent symmetric quasi-group of order v if and only if v is odd.

Lemma 3.4[4] *Let $(I_n, \circ)$ be an idempotent symmetric quasi-group, where $I_n = \{1,2,\cdots,n\}$ and G be a simple graph with e edges. A collection $\mathcal{A} = \{A_{ij} : i,j \in I_n,\ i<j\}$ can be taken as a base of a G-$HD(e^n)$ if and only if the following conditions hold, where $i,j \in I_n$ and $i<j$:*

(1) *For any given block A in $\mathcal{A}$, the differences $d(i, i\circ j)$ and $-d(i\circ j, j)$ both appear or not in A;*

(2) $\{d : \exists d(i,j)\} \bigcup \{d : \exists d(i, i\circ j)\} \bigcup \{d : \exists d(i\circ j, j)\} = Z_e$.

Lemma 3.5 *A D_3-$HD_2(8^{2t+1})$ exists for $t \geq 1$.*

Proof. Let $(I_{2t+1}, \circ)$ be an idempotent symmetric quasi-group, where $I_{2t+1} = \{1,2,\cdots,2t+1\}$ $(t \geq 1)$. Define: $A_{i,j} = (3_{i\circ j}, 2_i, 4_j, 0_i, 0_j, 5_{i\circ j})$, then $\mathcal{A}$=$\{A_{i,j}$ mod (8,−): $1 \le i < j \le 2t+1\}$ form a D_3-$HD(8^{2t+1})$. So, a D_3-$HD_2(8^{2t+1})$ exists for $t \ge 1$ by Lemma 1.3. ■

In this paper, for a block $B = (b_1, b_2, \cdots, b_6)$, $b_k \in Z_n$, $1 \le k \le 6$, $s \ge 0$, denote: $B + s = (b_1 + s, b_2 + s, \cdots, b_6 + s)$ mod n,
$5^s(b_1, b_2, \cdots, b_6) = (5^s b_1, 5^s b_2, \cdots, 5^s b_6)$ mod n,
$B \times m$ means m *times* of the block B for $m > 0$,
$(x, i) + (y, j) = (x + y, i + j)$ mod (n, t), $x, y \in Z_n, i, j \in Z_t$.

4 $\lambda = 2$

In this section, by (**), the scope of order v for the existence of G-$GD_2(v)$ is $v \equiv 0, 1 \pmod 8$. By the known holey designs and recursive constructions in §2 and §3, it is enough to construct a few GDs and IDs with index 2 for some small orders.

Lemma 4.1 *For graph $G \in \{H_i, D_j, R_j : 1 \le i \le 3, 1 \le j \le 2\}$, there exists a G-$GD_2(v)$ for $v \in \{8, 9, 16, 17\}$.*
Proof.
$\underline{v = 8}$ X=$Z_7 \bigcup \{\infty\}$, mod 7.
$H_1 : (\infty, 1, 6, 2, 0, 3)$, $H_2 : (\infty, 1, 6, 2, 0, 5)$, $H_3 : (\infty, 1, 6, 2, 0, 3)$,
$D_1 : (2, 6, \infty, 0, 1, 4)$, $D_2 : (1, 0, \infty, 6, 2, 4)$, $R_1 : (5, 1, 3, 0, \infty, 2)$,
$R_2 : (5, 1, 3, 0, \infty, 2)$.
$\underline{v = 9}$ X=Z_9, mod 9.
$H_1 : (7, 4, 0, 1, 3, 5)$, $H_2 : (7, 4, 0, 1, 3, 2)$, $H_3 : (7, 4, 0, 1, 3, 2)$,
$D_1 : (2, 7, 3, 0, 1, 8)$, $D_2 : (2, 7, 3, 0, 1, 5)$, $R_1 : (8, 4, 1, 0, 2, 7)$,
$R_2 : (8, 4, 1, 0, 2, 7)$.
$\underline{v = 16, 17}$ The designs can be obtained by Lemma 1.1 and Lemma 1.3(2). ■

Lemma 4.2 *There exists a D_3-$GD_2(v)$ for $v \in \{9, 16\}$.*
Proof. $\underline{v = 9:}$ $(2, 7, 3, 0, 1, 5)$ mod 9.
$\underline{v = 16:}$ by Lemma 1.1 and Lemma 1.3(2). ■

Lemma 4.3 *There exists a D_3-$ID_2(8 + 16, 16)$.*
Proof. Take point set $Z_8 \times Z_3$. Denote the element (x, i) of the set $Z_8 \times Z_3$ by x_i. $(4_0, 0_2, 5_0, 0_1, 6_0, 6_2)$ mod $(8, -)$;
$(4_0, 0_2, 3_0, 0_1, 1_0, 4_1) + s_0$ mod $(8, -)$, $0 \le s \le 6$;
$(1_0, 2_2, 4_0, 2_1, 2_0, 0_0)$, $(2_0, 3_2, 5_0, 3_1, 3_0, 1_0)$, $(3_0, 4_2, 6_0, 4_1, 4_0, 2_0)$,
$(4_0, 5_2, 7_0, 5_1, 5_0, 3_0)$, $(5_0, 6_2, 0_0, 6_1, 6_0, 4_0)$, $(6_0, 7_2, 1_0, 7_1, 7_0, 0_2)$,
$(7_0, 0_2, 2_0, 0_1, 0_0, 3_1)$, $(5_0, 0_2, 0_0, 0_1, 6_0, 3_0)$, $(6_0, 1_2, 1_0, 1_1, 7_0, 0_1)$,
$(7_0, 2_2, 2_0, 2_1, 0_0, 0_1)$, $(0_0, 3_2, 3_0, 3_1, 1_0, 0_1)$, $(1_0, 4_2, 4_0, 4_1, 2_0, 5_2)$,
$(2_0, 5_2, 5_0, 5_1, 3_0, 1_0)$, $(3_0, 6_2, 6_0, 6_1, 4_0, 2_0)$, $(4_0, 7_2, 7_0, 7_1, 5_0, 2_0)$,
$(4_0, 1_2, 3_0, 1_1, 2_0, 1_0)$, $(5_0, 2_2, 4_0, 2_1, 3_0, 1_0)$, $(6_0, 3_2, 5_0, 3_1, 4_0, 2_0)$,

$(7_0,4_2,6_0,4_1,5_0,3_0)$, $(0_0,5_2,7_0,5_1,6_0,3_0)$, $(1_0,6_2,0_0,6_1,7_0,5_0)$, $(2_0,7_2,1_0,7_1,0_0,6_0)$, $(3_0,7_2,2_0,7_1,0_0,0_2)$, $(0_0,1_2,3_0,1_1,1_0,0_2)$. ■

Lemma 4.4 *There exists no D_3-$GD_2(8)$.*
Proof. Suppose there exists a D_3-$GD_2(8) = (X, \mathcal{B})$, where $|X|$=8, $|\mathcal{B}|$=7. For each $x \in X$, let x be at a position of d_i-degree in the ith block ($d_i = 0$ means that x doesn't appear in the ith block), then $\sum_{i=1}^{7} d_i = 14$. Let $a = |\{d_i : d_i = 3, 1 \le i \le 7\}|$, $b = |\{d_i : d_i = 1, 1 \le i \le 7\}|$, then $3a + b = 14$, which implies x must appear at pendant vertex at least twice. So, x running over X, the pendants will be occupied $2 \times 8 = 16$ times, which is impossible for the degree-type $1^1 3^5$ of D_3. ■

Theorem A For graph $G \in \{H_i, D_i, R_j : 1 \le i \le 3,\ 1 \le j \le 2\}$, there exists a G-$GD_2(v) \Longleftrightarrow v \equiv 0, 1 \pmod 8$ and $v \ge 8$, except for D_3-$GD_2(8)$.
Proof. From the following table, the existence of G-$GD_2(v)$ for $v \equiv 0, 1 \pmod 8$ can be obtained with the exception D_3-$GD_2(8)$, where $1 \le i \le 3$ and $1 \le j \le 2$.

Graph G	H_i, D_j, R_j,	D_3
G-$GD_2(v)$	$v = 8, 9, 16, 17$ (Lemma 4.1)	$v = 9, 16, v \ne 8$ (Lemma 4.2,4.4)
G-$ID_2(-,-)$		(8+16,16) (Lemma 4.3)
G-$HD_2(-)$	$(8^q) : q = 3, 4, 5$ (Lemma 3.3)	$(8^{2t+1}) : t \ge 1$ (Lemma 3.5)
Conclusion	by Lemma 2.1	by Lemma 2.3

■

5 $\lambda = 4$

In this section, by (**), the scope of order v for the existence of G-$GD_4(v)$ is $v \equiv 0, 1 \pmod 4$ and $v \ge 8$. By the known G-designs, holey designs and recursive constructions in §2 – §4, it is enough to construct a few GDs and IDs with index 4 for some small orders.

The proofs of the following three lemmas appear in Appendix I, which is published in our website: http://qdkang.hebtu.edu.cn (online).

Lemma 5.1 *For graph $G \in \{H_1, H_3, D_1, D_3, R_1, R_2\}$, there exists a G-$ID_2(8+w, w)$. Further there exists a G-$ID_4(8+w, w)$ for $w = 4, 5$, too.*

Lemma 5.2 *For graph $G \in \{H_2, D_2\}$, there exists a G-$ID_4(8+w, w)$ for $w = 4, 5$.*

Lemma 5.3 *For graph $G \in \{H_i, D_i, R_j : 1 \le i \le 3,\ 1 \le j \le 2\}$, there exist G-$GD_4(v)$ for $v \in \{12, 13, 20, 21, 52, 53, 68, 69\}$ and D_3-$GD_4(8)$.*

Theorem B *For graph* $G \in \{H_i, D_i, R_j : 1 \le i \le 3, 1 \le j \le 2\}$, *there exists a* G-$GD_4(v) \iff v \equiv 0,1 \pmod 4$ *and* $v \ge 8$.
Proof. From the following table, the existence of G-$GD_4(v)$ for $v \equiv 4,5 \pmod 8$ can be obtained, where $1 \le i \le 3,\ 1 \le j \le 2$ and $w = 4,5$.

Graph G	H_i, D_i, R_j
G-$GD_4(8m+w)$	$m = 1,2,6,8$ (Lemma 5.3)
G-$ID_4(8r+w, w)$	$r = 1$ (Lemma 5.1,5.2)
G-$HD_2(-) \Longrightarrow G$-$HD_4(-)$	$(8^q) : q = 3,4,5$ (Lemma 3.3)
Conclusion	by Lemma 2.4

Furthermore, by Theorem A, the existence spectrum for G-$GD_4(v)$ will be $v \equiv 0,1 \pmod 4$ and $v \ge 8$, where the unique exception in Theorem A: D_3-$GD_2(8)$ does not exist, but D_3-$GD_4(8)$ exists (see Lemma 5.3). ∎

6 $\lambda = 8$

Lemma 6.0 *Let G be a simple graph, $p, q, r, \alpha, \beta, a, \lambda$ be positive integers, and $a \ge b \ge 0, a \ne 2,6$. If there exist a G-$HD_\lambda(p^1q^1r^1\alpha^1)$ and a G-$HD_\lambda(p^1q^1r^1\beta^1)$, then there exists a G-$HD_\lambda((ap)^1(aq)^1(ar)^1((a-b)\alpha + b\beta)^1)$.*
Proof. It is well known to exist a 4-$GDD(a^4)$ for $a \ne 2,6$. Weight the elements of the 4-$GDD(a^4)$ as follows: Weight every element of three groups among the 4-$GDD(a^4)$ by p, q and r, respectively. For the rest group of the 4-$GDD(a^4)$, each of b elements is weighed by β, other elements are weighed by α. Then there exists a G-$HD_\lambda((ap)^1(aq)^1(ar)^1((a-b)\alpha + b\beta)^1)$ from the known HDs. ∎

By (**), the scope of order v for the existence of G-$GD_8(v)$ is any $v \ge 6$.

6.1 Graphs $H_i, D_j, R_j,\ 1 \le i \le 3,\ 1 \le j \le 2$

Theorem 6.1 *If there exist a G-$GD_8(u)$ for $u \equiv 0,1 \pmod 4$ and $u \ge 8$, a G-$HD_8(2^31^1)$, a G-$HD_8(2^4)$, a G-$HD_8(2^33^1)$ and a G-$GD_8(m)$ for $m \in \{6,7,10,11,14,15,18,19,22,23,31,35,38,46, 47,50,54\}$, then there exists a G-$GD_8(v)$ for $v \equiv 2,3 \pmod 4$ and $v \ge 6$.*
Proof. Let $v = 16t + s$, where $s \in \{4i+2, 4i+3 : i \in Z_4\}$ and $t \ge 0$ (if $s \ge 6$) or $t \ge 1$ (if $s < 6$). First, taking $p = q = r = 2$ and suitable α, β, a, b,

and using Lemma 6.0, we have the following table, $v = 6a + (a-b)\alpha + b\beta$.

α	β	a	b	t	known HD_8	obtained HD_8	v
1	3	$2t$	$t+1$	$t \neq 1,3$	$2^3 1^1, 2^3 3^1$	$(4t)^3(4t+2)^1$	$16t+2$
1	3	$2t$	$t+3$	$t \geq 4$	$2^3 1^1, 2^3 3^1$	$(4t)^3(4t+6)^1$	$16t+6$
2	3	$2t+1$	2	$t \geq 1$	$2^4, 2^3 3^1$	$(4t+2)^3(4t+4)^1$	$16t+10$
1	2	$2t+2$	$2t$	$t \neq 0,2$	$2^3 1^1, 2^4$	$(4t+4)^3(4t+2)^1$	$16t+14$
1	3	$2t+1$	$t-2$	$t \geq 2$	$2^3 1^1, 2^3 3^1$	$(4t+2)^3(4t-3)^1$	$16t+3$
1	3	$2t+1$	t	$t \geq 0$	$2^3 1^1, 2^3 3^1$	$(4t+2)^3(4t+1)^1$	$16t+7$
1	3	$2t+1$	$t+2$	$t \geq 1$	$2^3 1^1, 2^3 3^1$	$(4t+2)^3(4t+5)^1$	$16t+11$
1	3	$2t+3$	$t-3$	$t \geq 3$	$2^3 1^1, 2^3 3^1$	$(4t+6)^3(4t-3)^1$	$16t+15$

Here, the conditions for t guarantee the existence of 4-$GDD(a^4)$ and $a \geq b \geq 0$. As well, the numbers listed in the last column are just all orders $v = 16t + s$ above. By Lemma 1.4(2), in order to obtain these G-$GD_8(v)$ for $v = w^3 n^1$, it is enough to exist G-$GD_8(w)$ and G-$GD_8(n)$ for $m, n \equiv 0, 1, 2 \pmod 4$. From the known conditions, there exists a G-$GD_8(u)$ for any $u \equiv 0, 1 \pmod 4$, $u \geq 8$. Therefore, the following recursions are obtained:

$$G\text{-}GD_8(4t+2) \Longrightarrow \begin{cases} G\text{-}GD_8(16t+2) \text{ for } t \neq 1,3 \\ G\text{-}GD_8(16(t-1)+6) \text{ for } t \geq 5 \\ G\text{-}GD_8(16t+10) \text{ for } t \geq 1 \\ G\text{-}GD_8(16t+14) \text{ for } t \neq 0,2 \\ G\text{-}GD_8(16t+3) \text{ for } t \geq 3 \\ G\text{-}GD_8(16t+7) \text{ for } t \geq 2 \\ G\text{-}GD_8(16t+11) \text{ for } t \geq 1 \\ G\text{-}GD_8(16(t-1)+15) \text{ for } t \geq 4 \end{cases},$$

where some conditions for t are reduced since $u \geq 8$ in G-$GD_8(u)$. It is easy to see that, in order to obtain all G-$GD_8(v)$ for the orders $v \equiv 2, 3 \pmod 4$, we need to construct the following G-$GD_8(v)$:

v=18, 50; 6, 22, 38, 54; 10; 14, 46; 19, 35; 7, 23; 11; 15, 31, 47.

This completes the proof. ■

The proofs of the following two lemmas appear in Appendix I, which is published in our website: http://qdkang.hebtu.edu.cn (online).

Lemma 6.2 *For graph $G \in \{H_i, D_j, R_j : 1 \leq i \leq 3,\ 1 \leq j \leq 2\}$, there exist a G-$HD_8(2^3 1^1)$, a G-$HD_8(2^3 3^1)$ and a G-$HD_8(2^4)$.*

Lemma 6.3 *For graph $G \in \{H_i, D_j, R_j : 1 \leq i \leq 3,\ 1 \leq j \leq 2\}$, there exists a G-$GD_8(v)$ for $v \in \{6, 7, 10, 11, 14, 15, 18, 19, 22, 23, 31, 35, 38, 46, 47, 50, 54\}$.*

6.2 Graph D_3

Lemma 6.4 *There exists a D_3-$ID_8(8t+w, w)$ for t=1, 2 and w=2, 3, 6, 7.*

Proof. We will give a detailed proof in Appendix I, which is published in our website: http://qdkang.hebtu.edu.cn (online). ■

Lemma 6.5 *There exists a D_3-$GD_8(v)$ for* v=7, 10, 11, 14, 15, 18, 19, 22, 23.
Proof. $X=Z_{v-1}\bigcup\{x\}$ for even v, mod $(v-1)$; $X=Z_v$ for odd v, mod v.
$\underline{v=7}$ $(3,2,5,0,1,4),(5,3,4,0,2,1)\times 2$.
$\underline{v=10}$ $(4,2,5,0,1,3),(4,2,5,0,1,x)\times 2$, $(1,3,5,0,x,2)\times 2$.
$\underline{v=11}$ $(6,3,5,0,2,4),(6,3,5,0,2,8),(4,5,8,0,1,7),(6,3,5,0,2,1)\times 2$.
$\underline{v=14}$ $(5,4,8,0,2,11),(5,4,8,0,2,x),(0,1,5,4,x,2),(7,3,6,0,2,x)\times 4$.
$\underline{v=15}$ $(7,3,6,0,2,13),(7,3,6,0,2,4),(7,3,6,0,2,9),(7,3,6,0,2,10)$,
$(7,3,6,0,2,12),(7,3,6,0,2,5),(7,3,6,0,2,1)$.
$\underline{v=18}$ $(x,4,9,3,0,2)$, $(x,2,9,1,0,16)$, $(6,3,8,0,2,1)\times 5$, $(6,3,8,0,2,x)\times 2$.
$\underline{v=19}$ $(9,4,13,3,0,6),(9,2,11,1,0,4),(6,3,8,0,2,1)\times 5$,
$(6,3,8,0,2,18)\times 2$.
$\underline{v=22}$ $(4,8,6,0,2,12),(4,8,6,0,2,10),(6,8,4,0,2,10),(10,2,5,0,9,x)\times 8$.
$\underline{v=23}$ by D_3-$ID_8(16+7,7)$ and D_3-$GD_8(7)$. ■

Lemma 6.6 *There exists no D_3-$GD_\lambda(6)$ for $\lambda\equiv 8 \pmod{16}$.*
Proof. Let $\lambda=16t+8$, $t\geq 0$. Suppose there exists a D_3-$GD_\lambda(6)$=$(X,\mathcal{B})$, where $|X|=6$, and $|\mathcal{B}|=15(2t+1)$. It is easy to see that each $x\in X$ should appear in each block. The degree-type of D_3 is 3^51^1. Let x be at a position of d_i-degree in the ith block, then $\sum_{i=1}^{15(2t+1)} d_i=40(2t+1)$, where $d_i\in\{1,3\}$. Let a=$|\{d_i : d_i=3, 1\leq i\leq 15(2t+1)\}|$, b=$|\{d_i : d_i=1, 1\leq i\leq 15(2t+1)\}|$, then we have

$$\begin{cases} a+b=15(2t+1) \\ 3a+b=40(2t+1)\end{cases} \Longrightarrow \quad 2a=25(2t+1).$$

This is a contradictory equation. ■

Theorem C For graph $G\in\{H_i,D_i,R_j : 1\leq i\leq 3, 1\leq j\leq 2\}$, there exists a G-$GD_8(v)$ for $v\geq 6$, except for $(G,v)=(D_3,6)$.
Proof. From the following three tables, the existence of G-$GD_8(v)$ for $v\equiv 2,3 \pmod 4$ can be gotten, with the exception D_3-$GD_8(6)$, where $1\leq i\leq 3$, $1\leq j\leq 2$, $w=2,3,6,7$.

Graph G	H_i, D_j, R_j
G-$GD_8(4m+2)$	$m\in\{1,2,3,4,5,9,11,12,13\}$
G-$GD_8(4n+3)$	$n\in\{1,2,3,4,5,7,8,11\}$ (Lemma 6.3)
G-$HD_8(-)$	$(2^33^1),(2^4),(2^31^1)$ (Lemma 6.2)
Conclusion	by Theorem 6.1

Graph G	D_3
G-$GD_8(v)$	$v=7,10,11,14,15,18,19,22,23$ (Lemma 6.5)
G-$ID_8(8r+w,w)$	$r=1,2$ (Lemma 6.4)
Conclusion	by Lemma 2.2, 6.6

Furthermore, by Theorem B, the conclusion follows. ■

7 Designs for some small orders

Lemma 7.1 *There exists a* D_3-$GD_\lambda(8) \Longleftrightarrow \lambda > 2,\ 2|\lambda$.
Proof. The necessity follows from (*) in §1 and Lemma 4.4. On the other hand, we know that there exists a D_3-$GD_4(8)$ by Lemma 5.3. Also, we have D_3-$GD_6(8)$ on set $Z_7 \cup \{x\}$:

$$(4,6,x,0,2,3),\ (6,4,x,0,3,1),\ (6,2,1,0,5,3) \bmod 7.$$

Furthermore, for any $\lambda > 2$ and $2|\lambda$, denote $\lambda = 4 \cdot \frac{\lambda}{4}$ ($\lambda \equiv 0 \bmod 4$) or $\lambda = 4 \cdot \frac{\lambda-6}{4} + 6$ ($\lambda \equiv 2 \bmod 4$). So, the conditions are sufficient. ■

Lemma 7.2 *There exists a* D_3-$GD_\lambda(6) \Longleftrightarrow 16|\lambda$.
Proof. The necessity follows from (*) in §1 and lemma 6.6. On the other hand, for $\lambda = 16t$, $t \geq 1$, we have the following constructions on $Z_5 \cup \{x\}$:

$$D_3\text{-}GD_{16}(6):\ (2,3,x,0,1,4)\times 4,\ (4,1,x,0,2,3),\ (1,3,0,2,4,x) \bmod 5.$$

So, the condition is sufficient. ■

8 Conclusion

Proof of Theorem 1.2:
Summarizing Lemma 1.1, Theorems A, B, C and Lemmas 7.1-7.2, we obtain the conclusions. ■

ACKNOWLEDGMENT:
We are grateful to the referees for valuable comments.

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