Vertex-distinguishing total coloring of graphs *

Zhongfu Zhang[1,2] Pengxiang Qiu[1†] Baogen Xu[3] Jingwen Li[4] Xiangen Chen[2] Bing Yao[2]

1. Institute of Applied Mathematics, Lanzhou Jiaotong University, Lanzhou 730070,P.R.China

2. College of Mathematics and Information Science, Northwest Normal University,Lanzhou 730070,P.R.China

3.Department of Mathematics, East China Jiaotong University,Nanchang 330013,P.R.China

4. College of Information and Electrical Engineering, Lanzhou JiaoTong University,Lanzhou 730070,P.R.China

Abstract

Let G be a simple and connected graph of order $p \geq 2$. A proper k-total-coloring of a graph G is a mapping f from $V(G) \bigcup E(G)$ into $\{1, 2, \cdots, k\}$ such that every two adjacent or incident elements of $V(G) \bigcup E(G)$ are assigned different colors. Let $C_f(u) = f(u) \bigcup \{f(uv) \mid uv \in E(G)\}$ be the neighbor color-set of u, if $C_f(u) \neq C_f(v)$ for any two vertices u and v of $V(G)$, we say f a vertex-distinguishing proper k-total-coloring of G, or a k-VDT-coloring of G for short. The minimal number of all over k-VDT-colorings of G is denoted by $\chi_{vt}(G)$, and it is called the $VDTC$ chromatic number of G. For some special families of the complete graph K_n, complete bipartite graph $K_{m,n}$, path P_m and circle C_m etc., we get their $VDTC$ chromatic numbers and propose a conjecture in this article.

Keywords proper edge coloring, vertex-distinguishing, proper total coloring, chromatic number

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1. INTRODUCTION

A proper edge-coloring of G is called vertex-distinguishing (see [1],[2],[4] and [7]) if for any two distinct vertices u and v of G the set of colors assigned to the edges incident to u differs from the set of colors assigned to the edges incident to v. The minimal number of colors required for a vertex-distinguishing proper edge-coloring of G is called the vertex-distinguishing

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†Corresponding author(E-mail: qpx0304@yahoo.com.cn)

proper edge-coloring chromatic number of G (or observability), and is denoted by $\chi'_s(G)$.

All the graph mentioned in this paper are simple and connected. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G respectively. Let $n_d = n_d(G)$ denote the number of vertices of degree d of G with $\delta(G) \leq d \leq \Delta(G)$. It is clear that $\binom{\chi'_s(G)}{d} \geq n_d$ for all d with $\delta(G) \leq d \leq \Delta(G)$. The following conjecture is given in [4] .

Conjecture 1.[4] Let G be a graph and let k be the minimum integer such that $\binom{k}{d} \geq n_d$ for all d such that $\delta(G) \leq d \leq \Delta(G)$, Then $\chi'_s(G) = k$ or $k+1$.

In [9], the adjacent-vertex-distinguishing proper edge-coloring of G is proposed. In [10],Hamed H. proved $\Delta+300$ is a bound on the adjacent vertex distinguishing edge chromatic number.Let f be a proper edge-coloring of G from $E(G)$ into $\{1,2,\cdots,k\}$ and let $C_a(u) = \{f(uv) \mid uv \in E(G)\}$ be the incident color-set of u, if $C_a(u) \neq C_a(v)$ for any two adjacent vertices u and v of $V(G)$, then the edge-coloring f is called an adjacent-vertex-distinguishing proper k-edge-coloring of G, or a k-$AVDE$-coloring of G for short. The minimal number of such k for all k-$AVDE$-colorings of G is called the $AVDEC$ chromatic number of G, and it is denoted by $\chi'_{as}(G)$. The following conjecture is proposed by Zhang et al. in [9].

Conjecture 2. Let G be a connected graph of order not less than three, and G be not the circle C_5 of the length being 5, then $\Delta(G) \leq \chi'_{as}(G) \leq \Delta(G)+2$.

In [11], $D(\beta)$-vertex-distinguishing proper edge-coloring is proposed. Let $G(V,E)$ be a connect graph with order at least 3, k, β are positive integers and f is a mapping from $E(G)$ to $\{1,2,\cdots,k\}$. For any $u \in V(G)$, the set $\{f(uv)|uv \in E(G), v \in V(G)\}$ is denoted by $C(u)$. If (1) for any $uv, vw \in E(G), u \neq w$, we have $f(uv) \neq f(vw)$; (2) for any $u,v \in V(G), 0 < d(u,v) \leq \beta$, we have $C(u) \neq C(v)$, then f is called a k-$D(\beta)$-vertex-distinguishing proper edge-coloring of graph G(k-$D(\beta)$-VDPEC of G in brief) and the number $\chi'_{\beta\text{-}vd}(G) = \min\{k \mid G$ has a k-$D(\beta)$-VDPEC$\}$ is called the k-$D(\beta)$-vertex-distinguishing edge-chromatic number of G, where $d(u,v)$ denotes the distance between u and v in G. The following conjecture is proposed by Zhang et al. in [11].

Conjecture 3. For simple connected graph G with $|V(G)| \geq 3$. Suppose n_i denote the maximum number of vertices with degree i and the distance of every such two vertices are no more than β. Let $\mu_\beta(G) =$

$\min\{\theta | \binom{\theta}{i} \geq n_i, \delta(G) \leq i \leq \Delta(G)\}$. Then for $\beta \geq 2$, we have $\mu_\beta(G) \leq \chi'_{\beta\text{-}vd}(G) \leq \mu_\beta(G) + 1$.

Definition 1[8]**.** A proper k-total-coloring of G is a mapping f from $V(G) \cup E(G)$ to $\{1, 2, \cdots, k\}$ such that any two adjacent or incident elements of $V(G) \cup E(G)$ are assigned different colors. Let $C_f(u) = f(u) \cup \{f(uv) | uv \in E(G)\}$ be the neighbor color set of u, if $C_f(u) \neq C_f(v)$ for any two adjacent vertices u and v of $V(G)$, we say that f is a adjacent-vertex-distinguishing proper k-total-coloring of G, or a k-AVDTC of G for short. The minimal number such that G has k-AVDTC is denoted by $\chi_{at}(G)$, and it is called adjacent-vertex-distinguishing total chromatic number of G.

Conjecture 4[8]**.** Let G be a connected graph of order n (≥ 2), then $\chi_{at}(G) \leq \Delta(G) + 3$.

We have proved that the Conjecture 4 is true for some particular families of cycles, complete graphs, complete bipartite graphs, fans, wheels and trees in [7] and obtained their adjacent-vertex-distinguishing total chromatic numbers.

It's interesting that $\chi'_s(C_5) = \chi'_{as}(C_5) = 5$ and $\chi_{at}(C_5) = 4$. In Definition 1, we get rid of the restrictive condition "adjacent", so a new definition is provided in the following Definition 2.

Definition 2. In the definition 1, if $C_f(u) \neq C_f(v)$ for any two distinct vertices u and v of $V(G)$, then f is called a k-VDTC of G. We call that $\overline{C}_f(u) = C \setminus C_f(u)$ is the complement color set of u (notice that $C_f(u) \neq C_f(v)$ if and only if $\overline{C}_f(u) \neq \overline{C}_f(v)$). The minimal number such that G has a k-VDTC is denoted by $\chi_{vt}(G)$, and it is called vertex-distinguishing total chromatic number of G.

Let G and H are two disjoint simple graphs, the join graph $G \vee H$ of both G and H has $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$.

In this article, we will determine the vertex-distinguishing total chromatic numbers for complete graph K_n, complete bipartite graph $K_{m,n}$, wheel W_n, fan F_n, double star S_{2n}, the join graph $P_n \vee P_n$, the join graph $P_n \vee C_n$, the join graph $C_n \vee C_n$, the connected graph with $\Delta(G) = 2$ and the 2-order n-sperate tree. The other terminologies and marks refer to [5] and [6].

2. MAIN RESULTS

We will use a number $k_T(G) = \min\{n | \binom{n}{d+1} \geq n_d, \delta(G) \leq d \leq \Delta(G)\}$ in the following discussion. A particular instance, however, is that $\chi_{vt}(K_n) = \chi'_{as}(K_n)$ according to Theorem 2.2 in [8], so we have the result below.

THEOREM 1 Let K_n be a complete graph of order n (≥ 3), then $\chi_{vt}(K_n) = n+1$ for $n \equiv 0 \pmod 2$; and $\chi_{vt}(K_n) = n+2$ for $n \equiv 1 \pmod 2$.

THEOREM 2 Let $K_{1,n}$ be a star of order $n+1$($n \geq 2$), then $\chi_{vt}(K_{1,n}) = 4$ when $n = 2$; and $\chi_{vt}(K_{1,n}) = n+1$ when $n \geq 3$.

Proof. When $n = 2$, $k_T(K_{1,2}) = 3$, so $\chi_{vt}(K_{1,2}) \geq 4$. It is easy to find a 4-$VDTC$-coloring of $K_{1,2}$. The detail of the proof is omitted here.

When $n \geq 3$, $k_T(K_{1,n}) = n+1$, so it is natural that $\chi_{vt}(K_{1,n}) \geq n+1$. A $(n+1)$-$VDTC$-coloring f can be given as this: Let v_0 denote the center of $K_{1,n}$ and let $v_1, v_2, \cdots, v_n$ denote n vertices of degree one of $K_{1,n}$. We set $f(v_0) = 0$; $f(v_i) = i$ for $1 \leq i \leq n$; $f(v_0v_i) = i+1$ for $1 \leq i \leq n-1$; and $f(v_0v_n) = 1$. It is obviously that f is a $(n+1)$-$VDTC$-coloring of $K_{1,n}$, the proof is completed. ∎

THEOREM 3 For $m > n \geq 2$, then $\chi_{vt}(K_{m,n}) = m+2$.

Proof. Since $k_T(K_{m,n}) = m+2$ for $m > n \geq 2$, so $\chi_{vt}(K_{m,n}) \geq m+2$. Now, we prove that $K_{m,n}$ has a $(m+2)$-$VDTC$-coloring. Let $V(K_{m,n}) = \{u_1, u_2, \cdots, u_m; v_1, v_2, \cdots, v_n\}$ be the vertex set of G and $E(K_{m,n}) = \{u_iv_j \mid 1 \leq i \leq m; 1 \leq j \leq n\}$ be the edge set of G, we directly make a coloring f as this: $f(u_i) = i$ for $1 \leq i \leq m$; $f(v_j) = m+2$ for $1 \leq j \leq n$; and $f(u_iv_j) \equiv i+j \mod (m+1)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

It is not difficult to verify that f is a $(m+2)$-$VDTC$-coloring of $K_{m,n}$, the proof has been finished. ∎

THEOREM 4 For $n \geq 2$, then $\chi_{vt}(K_{n,n}) = 4$ when $n = 2$; and $\chi_{vt}(K_{n,n}) = n+3$ when $n \geq 3$.

Proof. Since $k_T(K_{2,2}) = 4$ for $n = 2$, it is easy to get a 4-$VDTC$-coloring of $K_{2,2}$ such that $\chi_{vt}(K_{2,2}) = 4$. From $k_T(K_{n,n}) = n+3$ when $n \geq 3$, so there exists that $\chi_{vt}(K_{n,n}) \geq n+3$. Next we only prove that $\chi_{vt}(K_{n,n}) \leq n+3$. Using labelling directly, a $(n+3)$-$VDTC$-coloring f of $K_{n,n}$ is as follows.

Let $V(K_{n,n}) = \{u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n\}$, $E(K_{n,n}) = \{u_iv_j \mid i = 1, 2, \cdots, n; j = 1, 2, \cdots, n\}$ and let $C = \{1, 2, \cdots, n+1, 0, n+3\}$. We set that $f(u_i) = i$ and $f(v_i) = n+3$ for $1 \leq i \leq n$; and $f(u_iv_j) \equiv i+j \pmod{n+2}$ for $1 \leq i, j \leq n$. From Definition 2, there are the color complement sets $\overline{C}(u_1) = C \setminus C_f(u_1) = \{0, n+3\}$; $\overline{C}(u_i) = C \setminus C_f(u_i) = \{i-1, n+3\}$ with $2 \leq i \leq n$; $\overline{C}(v_1) = C \setminus C_f(v_1) = \{1, 0\}$; and $\overline{C}(v_j) = C \setminus C_f(v_j) = \{j-1, j\}$ with $2 \leq j \leq n$. It shows that f is really a $(n+3)$-$VDTC$-coloring of $K_{n,n}$, as desired. ∎

THEOREM 5 Let W_n be a wheel of order $n+1$, then $\chi_{vt}(W_n) = 5$ when $n = 3$ and $\chi_{vt}(W_n) = n + 1$ when $n \geq 4$.

Proof. When $n = 3$, $W_n = K_4$, from the Theorem 1 the result is right. From $k_T(W_n) = n + 1$ when $n \geq 4$, so $\chi_{vt}(W_n) \geq n + 1$. Now, we come to prove $\chi_{vt}(W_n) \leq n + 1$ by giving a $(n + 1)$-$VDTC$-coloring of W_n.

When n=4, it is easy to get a 5-$VDTC$-coloring of W_4.

When $n \geq 5$, let the vertex set $V(W_n) = \{v_0, v_1, v_2, \cdots, v_n\}$ and the edge set

$$E(W_n) = \{v_0v_i \mid 1 \leq i \leq n\} \bigcup \{v_iv_{i+1} \mid 1 \leq i \leq n-1\} \bigcup \{v_nv_1\}.$$

We build a coloring f as this: $f(v_0) = n + 1$; $f(v_i) = i$ for $1 \leq i \leq n$; $f(v_0v_i) = i - 1$ for $2 \leq i \leq n - 1$; $f(v_iv_{i+1}) = i + 2$ for $1 \leq i \leq n - 1$; and $f(v_0v_1) = n, f(v_{n-1}v_n) = 1, f(v_nv_1) = 2$. It is obviously that f is a $(n + 1)$-$VDTC$-coloring of W_n, so the result has been proved. ∎

THEOREM 6 Let F_n be a fan of order $n + 1$, then $\chi_{vt}(F_n) = 5$ when $n = 3$ and $\chi_{vt}(F_n) = n + 1$ when $n \geq 4$.

Proof. For the fan F_2, so $F_2 = K_3$, it is proved in Theorem 1. When $n = 3$, $k_T(F_3) = 5$, and it is obviously to get a 5-$VDTC$-coloring of F_3. When $n \geq 4$, there is $k_T(F_n) = n + 1$. Now it only needs to delete the color of the edge v_nv_1 and the edge v_nv_1 of W_n in the proof of Theorem 5, and then we get a $(n + 1)$-$VDTC$-coloring of F_n. ∎

For a graph $G = (V, E)$, if $V(G) = \{u_0, u_1, \cdots, u_n; v_0, v_1, \cdots, v_n\}$ and $E(G) = \{u_0u_i, 1 \leq i \leq n\} \bigcup \{v_0v_i, 1 \leq i \leq n\} \bigcup \{u_0v_0\}$, we call G as a regular double star of order $2(n + 1)$, especially denote it by S_{2n}.

THEOREM 7 Let S_{2n} be a regular double star on $2(n + 1)$ vertices, then $\chi_{vt}(S_{2n}) = n + 3$.

Proof. When $n = 1$, it is obvious. There is $\chi_{vt}(S_{2n}) \geq n + 3$ from $k_T(S_{2n}) = n + 3$. When $n \geq 2$, we give a $(n + 3)$-$VDTC$-coloring f of S_{2n} to verify $\chi_{vt}(S_{2n}) \leq n + 3$. The detail is such as: $f(u_0) = n + 1, f(u_{n-1}) = n+2, f(u_n) = n+3$ and $f(u_i) = i+2$ for $1 \leq i \leq n-2$; $f(v_0) = n+3, f(v_i) = i+1$ for $1 \leq i \leq n$; and $f(u_0v_0) = n+2, f(u_0u_i) = f(v_0v_i) = i$ for $1 \leq i \leq n$. Hence, f obviously is a $(n+3)$-$VDTC$-coloring of S_{2n}. The proof has been finished. ∎

THEOREM 8 For $n \geq 3$, then $\chi_{vt}(P_n \vee P_n) = k_t(P_n \vee P_n)$.

Proof. It is easy to prove the result when $2 \leq n \leq 8$. For $n \geq 9$, we set a path $P_n = u_1u_2 \cdots u_n$ and another path $P'_n = v_1v_2 \cdots v_n$, and then make a proper total coloring f as this: (1) $f(u_i) = i$ $(1 \leq i \leq n)$;

(2) $u_1u_2, u_2u_3, \cdots, u_{n-1}u_n$ are colored in turn by colors $n+4, n+5$; (3) $f(u_iv_j) = i+j \pmod{(n+3)}, (1 \le i,j \le n)$; (4) $f(v_1) = 0$; (5) the edges $v_2, v_3, \cdots, v_n$ are assigned alternately with colors $n+5$ and $n+4$; and (6) $f(v_iv_{i+1}) = i$, $(1 \le i \le n-1)$. The set $C = \{1, 2, \cdots, n, n+1, n+2, 0, n+4, n+5\}$ just collects all of the colors used above. About the color complement set $\overline{C}(u)$ of every vertex u of $P_n \vee P_n$, we have

(a1) $\overline{C}(u_1) = \{0, n+2, n+4\}$;

(a2) $\overline{C}(u_n) = \{n-1, n-2, n+4\}(n \equiv 0 \pmod 2)$ or $\overline{C}(u_n) = \{n-1, n-2, n+5\}(n \equiv 1 \pmod 2)$;

(a3) $\overline{C}(u_i) = \{i-2, i-1\}$, $2 \le i \le n-1$;

(b1) $\overline{C}(v_1) = \{n+2, n, n+5\}$ and $\overline{C}(v_n) = \{n-2, n, n+4\}$;

(b2) When $n \equiv 0 \pmod 2$, there are $\overline{C}(v_i) = \{i-2, n+4\}$ for $2 \le i \le n-1$ and $i \equiv 0 \pmod 2$, and $\overline{C}(v_i) = \{i-2, n+5\}$ for $3 \le i \le n-2$ and $i \equiv 1 \pmod 2$; and

(b3) When $n \equiv 1 \pmod 2$, there are $\overline{C}(v_i) = \{i-2, n+4\}$ for $2 \le i \le n-2$ and $i \equiv 0 \pmod 2$, and $\overline{C}(v_i) = \{i-2, n+5\}$ for $3 \le i \le n-1$ and $i \equiv 1 \pmod 2$.

Therefor, the coloring f is a $(n+5)$-$VDTC$-coloring of $P_n \vee P_n$. ■

THEOREM 9 For $n \ge 3$, then $\chi_{vt}(P_n \vee C_n) = k_t(P_n \vee C_n)$.

Proof. When $3 \le n \le 6$, it is easy to prove the result. For $n \ge 7$, let $P_n = u_1u_2 \cdots u_n$ be the path of length n, $C_n = v_1v_2 \cdots v_nv_1$ be the circle of length n and $C = \{1, 2, \cdots, n, n+1, n+2, 0, n+4, n+5\}$ be the color set. We make a mapping f from $V(P_n \vee C_n) \cup E(P_n \vee C_n)$ into C as this: (1) $f(u_i) = i$ $(1 \le i \le n)$; (2) Coloring the edges $u_1u_2, u_2u_3, \cdots, u_{n-1}u_n$ by colors $n+5, n+4$ in turn; (3) Coloring the edges $v_1v_2, v_2v_3, \cdots, v_{n-1}v_n$ by colors $1, 2, \cdots, n-1$ in order; (4) $f(v_1) = 0$; (5) $f(v_nv_1) = n+5(n \equiv 1 \pmod 2)), f(v_nv_1) = n+4(n \equiv 0 \pmod 2))$; (6) Coloring the vertices $v_2, v_3, \cdots, v_n$ by colors $n+5, n+4$ in turn; (7) $f(u_iv_j) = i+j \pmod{(n+3)}$ $(1 \le i,j \le n)$.

We work out of all complement color sets on $V(P_n \vee C_n)$ as follows.

(a1) $\overline{C}(u_1) = \{n+2, 0, n+4\}$;

(a2) $\overline{C}(u_n) = \{n-2, n-1, n+5\}(n \equiv 1 \pmod 2)$ or $\overline{C}(u_n) = \{n-2, n-1, n+4\}(n \equiv 0 \pmod 2)$;

(a3) $\overline{C}(u_i) = \{i-2, i-1\}$, $(2 \le i \le n-1)$;

(b1) $\overline{C}(v_1) = \{n+2, n+5\}$ and $\overline{C}(v_n) = \{n-2, n\}$;

(b2) When $n \equiv 1 \pmod 2$, there are $\overline{C}(v_i) = \{i-2, n+4\}(2 \le i \le n-1(i \equiv 0 \pmod 2)))$ and $\overline{C}(v_i) = \{i-2, n+5\}(3 \le i \le n-2(i \equiv 1 \pmod 2)))$;

(b3) When $n \equiv 0 \pmod 2$, there are $\overline{C}(v_i) = \{i-2, n+4\}(2 \le i \le n-2(i \equiv 0 \pmod 2)))$ and $\overline{C}(v_i) = \{i-2, n+4\}(3 \le i \le n-1(n \equiv 1 \pmod 2)))$.

Hence, it is easy to see that f is a $(n+5)$-$VDTC$-coloring of $P_n \vee C_n$. We have completed the proof. ■

THEOREM 10 For $n \geq 3$,there is that $\chi_{vt}(C_n \vee C_n) = k_t(C_n \vee C_n)$.

Proof. When $n = 3$, there is $C_3 \vee C_3 = K_6$, and then $\chi_{vt}(K_6) = 7$. When $n = 4$, it is similar to get the result. For $n \geq 5$, let $u_1u_2 \cdots u_nu_1$ and $v_1v_2 \cdots v_nv_1$ denote two cycles respectively, and let $C = \{1, 2, \cdots, n, n+1, n+2, 0, n+4, n+5\}$ be the color set we will use. A mapping f from $V(C_n \vee C_n) \cup E(C_n \vee C_n)$ into C is given as this: (1) $f(u_i) = i$ $(1 \leq i \leq n)$; (2) Coloring the edges $u_1u_2, u_2u_3, \cdots, u_{n-1}u_n, u_nu_1$ by colors $n+5, n+4$ in turn; (3) $f(u_nu_1) = 0$; (4) Coloring the edges $v_1v_2, v_2v_3, \cdots, v_{n-1}v_n$ by colors $1, 2, \cdots, n-1$ in order; (5) $f(v_1) = 5$; (6) Coloring the vertices $v_2, v_3, \cdots, v_n$ by colors $n+5, n+4$ in turn; (7) $f(v_nv_1) = n+5(n \equiv 1 \pmod 2)), f(v_nv_1) = n+4(n \equiv 0 \pmod 2))$; (8) $f(u_iv_j) = i+j \pmod{(n+3)}$ $(1 \leq i, j \leq n)$. We can compute the complement color sets as follows.

(a1) $\overline{C}(u_1) = \{n+2, n+4\}$;

(a2) $\overline{C}(u_n) = \{n-2, n+5\}(n \equiv 1 \pmod 2))$, $\overline{C}(u_n) = \{n-2, n+4\}(n \equiv 0 \pmod 2))$;

(a3) $\overline{C}(u_i) = \{i-2, i-1\}$, $(2 \leq i \leq n-1)$;

(b1) $\overline{C}(v_1) = \{n+2, n+5\}$ and $\overline{C}(v_n) = \{n-2, 0\}$;

(b2) When $n \equiv 1 \pmod 2$, there are $\overline{C}(v_i) = \{i-2, n+4\}(2 \leq i \leq n-1(i \equiv 0 \pmod 2)))$ and $\overline{C}(v_i) = \{i-2, n+5\}(3 \leq i \leq n-2(i \equiv 1 \pmod 2)))$; and

(b3) When $n \equiv 0 \pmod 2$, there are $\overline{C}(v_i) = \{i-2, n+4\}(2 \leq i \leq n-2(i \equiv 0 \pmod 2)))$ and $\overline{C}(v_i) = \{i-2, n+4\}(3 \leq i \leq n-1(n \equiv 1 \pmod 2)))$.

Thus, the coloring f is a $(n+5)$-VDTC of $C_n \vee C_n$. ∎

3. VERTEX DISTINGUISHING TOTAL CHROMATIC NUMBERS OF PATH AND CYCLE

The combinations of choosing three distinct numbers from $\{1, 2, \cdots, n\}$ form a subset, denoted by $\mathcal{A}_n^3$, and it is called the combination of three numbers, or 3-combination for short. Every element of $\mathcal{A}_n^3$ is written as the form of (a, b, c) where $1 \leq a < b < c \leq n$. We are able to arrange the elements of $\mathcal{A}_n^3$ by a certain defined order which will be introduced as follows.

For any two elements (t_1, t_2, t_3) and (l_1, l_2, l_3) of $\mathcal{A}_n^3$, if $t_2 = l_2$ and $t_3 = l_3$ then we say that they are in the same team. Let (t_1, t_2, t_3) and (l_1, l_2, l_3) be in the same team, we arrange (t_1, t_2, t_3) before (l_1, l_2, l_3) if and only if $t_1 > l_1$. If (t_1, t_2, t_3) and (l_1, l_2, l_3) are not in the same team and $t_3 \neq l_3$, then we arrange (t_1, t_2, t_3) before (l_1, l_2, l_3) if and only if $t_3 < l_3$; If (t_1, t_2, t_3) and (l_1, l_2, l_3) are not in the same team and $t_3 = l_3$, then we arrange (t_1, t_2, t_3) before (l_1, l_2, l_3) if and only if $t_2 < l_2$. Such arrangement given above is called the coloring order of $\mathcal{A}_n^3$. Let (t_1, t_2, t_3) be before (l_1, l_2, l_3), we say that (t_1, t_2, t_3) is adjacent to (l_1, l_2, l_3) if there is no another (s_1, s_2, s_3) such that (t_1, t_2, t_3) is before (s_1, s_2, s_3) and (s_1, s_2, s_3) is before (l_1, l_2, l_3).

For example, we arrange the elements of $\mathcal{A}_6^3$ according to the definition of the coloring order, the result is as: 123, 124, 234, 134, 125, 235, 135, 345, 245, 145, 126, 236, 136, 346, 246, 146, 456, 356, 256, 156.

LEMMA 1 For a 3-combination $\mathcal{A}_n^3$ in the coloring order, it may change appropriately the order of numbers in every (a, b, c) of $\mathcal{A}_n^3$ such that the new 3-combination sequence has the three properties below.

(1) If (t_1, t_2, t_3) is adjacent to (l_1, l_2, l_3), then $t_3 = l_1$;

(2) If (t_1, t_2, t_3) is adjacent to (l_1, l_2, l_3), then $t_2 \neq l_2$; and

(3) Let (a_1, a_2, a_3) be the first term and (b_1, b_2, b_3) be the last one in the new 3-combination sequence, then $a_1 = 1$ and $b_3 = 1$.

Proof. For $\mathcal{A}_4^3$, there are $(1,3,2), (2,1,4), (4,2,3)$ and $(3,4,1)$. When $n = 5$, we have $(1,3,2)$, $(2,1,4)$, $(4,2,3)$, $(3,4,1)$, $(1,5,2)$, $(2,3,5)$, $(5,1,3)$, $(3,4,5)$, $(5,2,4)$ and $(4,5,1)$. Suppose that the lemma is right for $\mathcal{A}_{k-1}^3 (k \geq 6)$. We prove the lemma by the mathematical induction.

For the set $\{1, 2, \cdots, k\}$, after arranging the elements of $\mathcal{A}_k^3$ by the coloring order, we delete the latter $(k-2)$ teams (the 3-combination in these teams are exactly contained number k), then exactly get the result of arranging the elements in $\mathcal{A}_{k-1}^3$ by the coloring order. According to induce assumption, by modifying the numbers in each 3-combination appropriately, the lemma is right. We will prove the lemma is also right by adding the latter $(k-2)$ teams, and the algorithm of 3-combination by adding the latter $(k-2)$ teams is given as follows.

[1] Adding the first three teams as the forms below.

$$(1, k, 2), (2, 3, k), (k, 1, 3), (3, 4, k), (k, 2, 4), (4, 1, k).$$

[2] Assuming the teams $1, 2, 3, \cdots, l-1$ have been already added $(l \leq k-3)$ after appropriately changing the order of numbers in 3-combination such that the teams $1, 2, 3, \cdots, l-1$ have the three properties. Now we add the lth team.

If the last 3-combination in which the order is already arranged as the end number l. When l is even, then adding 3-combination of the lth team as: $(l, l+1, k)$, $(k, l-1, l+1)$, $(l+1, l-2, k), (k, l-3, l+1)$, $(l+1, l-4, k), \cdots$, $(l+1, 2, k)$, $(k, 1, l+1)$; When l is odd, then adding 3-combination of the lth team as: $(l, l+1, k)$ $(k, l-1, l+1)$, $(l+1, l-2, k), (k, l-3, l+1)$, $(l+1, l-4, k), \cdots$, $(k, 2, l+1)$, $(l+1, 1, k)$.

If the last 3-combination in which the order is already arranged as the end number k. When l is even, then adding 3-combination of the l team as: $(k, l, l+1)$, $(l+1, l-1, k)$, $(k, l-2, l+1)$, $(l+1, l-3, k)$, $(k, l-4, l+1) \cdots$, $(k, 2, l+1)$, $(l+1, 1, k)$; When l is odd, then adding 3-combination of the l team as: $(k,l,l+1)$, $(l+1,l-1,k)$, $(k,l-2,l+1),(l+1,l-3,k)$, $(k,l-4,l+1),\cdots$, $(l+1,2,k)$, $(k,1,l+1)$.

[3] At last, adding the $(k-2)th$ team. The way is as the same as [2]. It only needs to change the last 3-combination $(k,1,k-1)$ to $(k,k-1,1)$ or $(k-1,1,k)$ to $(k-1,k,1)$.

We have finished the proof. ∎

It is easy to see that $\chi_{vt}(P_1)=1, \chi_{vt}(P_2)=3, \chi_{vt}(P_3)=4$. According to Lemma 1, we obtain the following result immediately.

THEOREM 11 For a path P_m of order m (≥ 4), if $\binom{n-1}{3} < m-2 \leq \binom{n}{3}$, then $\chi_{vt}(P_m)=n$.

LEMMA 2 Let $\mathcal{A}_n^3 (n \geq 5)$ be a 3-combination number set and a positive integer m satisfy that $\binom{n-1}{3} < m \leq \binom{n}{3}$, there always exits m 3-combinations in $\mathcal{A}_n^3$ such that they may be arranged in the same line. In this line, there are three properties as follows.

(1) If (t_1,t_2,t_3) is adjacent to (l_1,l_2,l_3), then $t_3=l_1$;

(2) If (t_1,t_2,t_3) is adjacent to (l_1,l_2,l_3), then $t_2 \neq l_2$; and

(3) Let (a_1,a_2,a_3) be the first term and (b_1,b_2,b_3) be the last one in the line, then $a_1=1$ and $b_3=1$.

Proof. When $n=5$, then $4 < m \leq 10$, and we have the following cases.

When $m=5$, (1,3,2)(2,1,4)(4,2,3)(3,1,4)(4,5,1);

When $m=6$, (1,3,2)(2,1,4)(4,2,3)(3,4,1)(1,2,5)(5,4,1);

When $m=7$, (1,3,2)(2,1,4)(4,2,3)(3,4,1)(1,5,2)(2,3,5)(5,4,1);

When $m=8$, (1,3,2)(2,1,4)(4,2,3)(3,4,1)(1,5,2)(2,5,3)(3,1,5)(5,4,1);

When $m=9$, (1,3,2)(2,1,4)(4,2,3)(3,4,1)(1,5,2)(2,5,3)(3,1,5)(5,3,4)(4,5,1);

When $m=10$, (1,3,2)(2,1,4)(4,2,3)(3,4,1)(1,5,2)(2,3,5)(5,1,3)(3,4,5)(5,2,4)(4,5,1).

Let $r=m-\binom{n-1}{3}$and $n \geq 6$. According to the method in the proof of Lemma 1, we can arrange the 3-combinations of $\mathcal{A}_{n-1}^3$ by the coloring order (the number in each 3-combination also has order), the new sequence obtained is denoted by $\mathcal{R}$. The last 3-combination in this sequence is $(n-2,n-1,1)$ or $(n-1,n-2,1)$. Now adding r 3-combinations involving n to $\mathcal{R}$ in $\mathcal{A}_n^3$, then we get another sequence $\mathcal{R}_r$ of $\binom{n-1}{3}+r$ 3-combinations such that $\mathcal{R}_r$ possesses the three properties of lemma.

When $r=1$, if the last term of $\mathcal{R}$ is $(n-2,n-1,1)$, then we modify $\mathcal{R}+\{(1,n-1,n)\}$ as: changing the order of numbers in the last term

$(n-2, n-1, 1)$ of $\mathcal{R}$ as $(n-2, 1, n-1)$, and changing the order of numbers in $(1, n-1, n)$ to $(n-1, n, 1)$; If the last term of $\mathcal{R}$ is $(n-1, n-2, 1)$, then we modify $\mathcal{R} + \{(1, n-2, n)\}$ as this: changing the order of numbers of the last term $(n-1, n-2, 1)$ in $\mathcal{R}$ to $(n-1, 1, n-2)$, and changing the order of numbers of $(1, n-2, n)$ as $(n-2, n, 1)$, the new sequence obtained has the three properties of lemma.

When $r = 2$, we modify $\mathcal{R} + \{(1, 2, n), (1, n-1, n)\}$ as: the order of numbers of the last term in $\mathcal{R}$ keeps in unchanged; the order of numbers of the 3-combination involving $(1, 2, n)$ keeps in unchanged; and the order of numbers of the 3-combination involving $(1, n-1, n)$ is changed into $(n, n-1, 1)$. Hence, the new sequence obtained has the three properties of lemma.

When $r = 3$, we modify $\mathcal{R} + \{(1, 2, n), (2, 3, n), (1, n-1, n)\}$ as: changing the order of numbers of $(1, 2, n)$ to $(1, n, 2)$, the order of numbers of $(2, 3, n)$ is unchanged, the order of numbers of $(1, n-1, n)$ is changed into $(n, n-1, 1)$, the new sequence obtained has the three properties of lemma.

When $r = 4$, we modify $\mathcal{R} + \{(1, 2, n), (2, 3, n), (1, 3, n), (1, n-1, n)\}$ as this: changing the order of numbers of $(1, 2, n)$, $(2, 3, n)$, $(1, 3, n)$, $(1, n-1, n)$ into $(1, n, 2), (2, 3, n)$, $(1, n-2, n), (n, n-1, 1)$ respectively.

When $r \geq 5$, according to the order of 3-combination, and the order of numbers in each 3-combination which are pointed in Lemma 1, adding $(r-1)$ 3-combinations they contain n in $\mathcal{A}_n^3$ to the end of $\mathcal{R}$ gradually.

If the last number of the $(r-1)th$ 3-combination is n, then adding again the 3-combination $(n, n-1, 1)$ at last position. The new sequence obtained has the three properties of lemma.

If the last number of the $(r-1)th$ 3-combination has been added s $(s \neq n)$, and the $(r-1)th$ 3-combination belongs to the $(s-1)th$ team which contained n in $\mathcal{A}_n^3$ (note that the number of 3-combinations contained n in $\mathcal{A}_n^3$ is total $\frac{1}{2}(n-1)(n-2)$, and there are $n-2$ teams, according to the order given from Lemma 1, the last number of the jth team in 3-combination is n or $j+1$), and it is not the last 3-combination of the $(s-1)th$ team, then we add $(s, n, 1)$ at last, so the new sequence formed has the three properties of lemma.

If the last number of the $(r-1)th$ 3-combination which is added is $s (s \neq n)$, and when the $r-1$ 3-combination belongs to the $(s-1)th$ team contained n in $\mathcal{A}_n^3$, and it is the last term of the $(s-1)th$ team, then we delete the last the $(r-1)th$ 3-combination, and adding two 3-combinations $(n, 2, n-1)$ and $(n-1, n, 1)$ again, the new sequence formed has the three properties of lemma.

Through the discussion above, the Lemma is proved. ∎

THEOREM 12 Let C_m be a cycle of order m. If $\binom{n-1}{3} < m \leq \binom{n}{3}$ for $n \geq 4$, then $\chi_{vt}(C_m) = 5$ when $n = 3$ and $\chi_{vt}(C_m) = n$ when $n \geq 4$.

Proof. When $m = 3$,it is easy to prove that $\chi_{vt}(C_m) = 5$. When $m \geq 4$, the result follows Lemma 2. ∎

VERTEX DISTINGUISHING TOTAL CHROMATIC NUMBERS OF 2-ORDER n-SEPARATE TREE

The m-order n-separate tree $T_{m,n}$ is a typical model of mathematics in many areas of the computing, social, natural, and information science etc. $T_{m,n}$ is described as follows.

(1) $V(T_{1,n}) = \{v\} \cup \{v_{i_1} \mid 1 \leq i_1 \leq n\}$ and $E(T_{1,n}) = \{vv_{i_1} \mid 1 \leq i_1 \leq n\}$);

(2) $V(T_{2,n}) = V(T_{1,n}) \bigcup\limits_{i_1=1}^{n} \{v_{i_1 i_2} \mid 1 \leq i_2 \leq n\}$ and

$$E(T_{2,n}) = E(T_{1,n}) \bigcup_{i_1=1}^{n} \{v_{i_1} v_{i_1 i_2} \mid 1 \leq i_2 \leq n\};$$

....

(m) Let $M = i_1, i_2, \cdots, i_{m-1}$, we have

$V(T_{m,n}) = V(T_{m-1,n}) \bigcup\limits_{M=1}^{n} \{v_{i_1 i_2 \cdots i_{m-1} i_m} \mid 1 \leq i_m \leq n\}$ and

$E(T_{m,n}) = E(T_{m-1,n}) \bigcup\limits_{M=1}^{n} \{v_{i_1 i_2 \cdots i_{m-2} i_{m-1}} v_{i_1 i_2 \cdots i_{m-1} i_m} \mid 1 \leq i_m \leq n\}$.

It is clearly that $T_{1,n} = K_{1,n} = S_n$, $T_{m,1} = P_{m+1}$, the $VDTC$ chromatic numbers about them have already been given (see Theorem 2 and Theorem 10). For $m = 2$, we have that $k_t(T_{2,n}) = n + 3$ for $1 \leq n \leq 4$ and $k_t(T_{2,n}) = n + \lceil \frac{n}{2} \rceil$ for $n \geq 5$ where $\lceil x \rceil$ denotes the minimum integer no less than x.

THEOREM 13 For a 2-order n-separate tree $T_{2,n}$, $\chi_{vt}(T_{2,n}) = K_t(T_{2,n})$.

Proof. When $n = 1$, it follows Theorem 11. When $n = 2, 3$, it is easy to give a $(n+3)$-$VDTC$-coloring of $T_{2,n}$, so we omit the detail of proof. When $n = 4$, let $C = \{1, 2, \cdots, 6, 0\}$, we directly give a total coloring f of $T_{2,n}$ as this: $f(v) = 1$; $f(vv_i) = i + 1$ for $i = 1, 2, 3, 4$; $f(v_i) = i + 2$ for $i = 1, 2, 3, 4$; $f(v_i v_{ij}) = i + j + 2 \pmod 7$ for $i, j = 1, 2, 3, 4$; $f(v_{1j}) = j + 4 \pmod 7$ for $j = 1, 2, 3, 4$; $f(v_{2j}) = j + 6 \pmod 7$ for $j = 1, 2, 3, 4$; $f(v_{3j}) = j + 1$ for $j = 1, 2, 3$; $f(v_{34}) = 1$; $f(v_{41}) = 4$; $f(v_{42}) = 5$; $f(v_{43}) = 3$; $f(v_{44}) = 4$. Therefore f is a 7-$VDTC$-coloring of $T_{2,4}$.

For $n \geq 5$, let $C = \{1, 2, \cdots, n + \lceil \frac{n}{2} \rceil - 1, 0\}$ be the color set, we set g as: $g(v) = 1$; $g(vv_i) = i + 1$ for $1 \leq i \leq n$; $g(v_i) = i + 2$ for $1 \leq i \leq n$; and $g(v_i v_{ij}) = i + j + 2 \pmod{(n + \lceil \frac{n}{2} \rceil)}$ for $1 \leq i, j \leq n$. According to $\binom{n+\lceil \frac{n}{2} \rceil}{2} \geq n^2 (n \geq 5)$(the equality holds if and only if $n = 6$) and $n + \lceil \frac{n}{2} \rceil$ colors on the edges $v_i v_{ij}$ $(1 \leq i, j \leq n)$, appropriately coloring v_{ij}

such that $\overline{C}_g(v_{i_1j_1}) \neq \overline{C}_g(v_{i_2j_2})$ when $v_{i_1j_1} \neq v_{i_2j_2}$. And there are that $\overline{C}_g(v_i) = \{i\}(1 \leq i \leq n)$ and $\overline{C}_g(v) = \{n + \lceil \frac{n}{2} \rceil\}$. Thus, g is a $(n + \lceil \frac{n}{2} \rceil)$-$VDTC$-coloring of $T_{2,n}$. ∎

A BOUND ON THE VERTEX DISTINGUISHING TOTAL CHROMATIC NUMBER

LEMMA3[2] For simple graph G without isolated edges and with at most one isolated vertex,we have

$$\chi'_s(G) \leq |V(G)| + 1.$$

THEOREM 14 For simple graph G,we have

$$\chi_{vt}(G) \leq |V(G)| + 2.$$

Proof When $|V(G)| = 1$,it is obviously true.

When $|V(G)| = 2$,let $w \notin V(G)$,let G^* denote $G \vee \{w\}$.It follows from definition3 that G^* is a connected graph with $|V(G^*)| \geq 3$.And it follows from lemma3 that

$$\chi'_s(G^*) \leq |V(G^*)| + 1 = |V(G)| + 2.$$

Let f^* be a $|V(G^*)|+1$-vertex distinguishing edge coloring of G^*.Let $f(v) = f^*(wv), v \in V(G); f(uv) = f^*(uv), uv \in E(G)$.Then f is a $(|V(G)| + 2)$-vertex distinguishing total coloring of G.It follows that $\chi_{vt}(G) \leq |V(G)|+2$.

CONJECTURE AND OPEN PROBLEMS

Conjecture. Let G be a connected graph of order n (≥ 2), then $k_T(G) \leq \chi_{vt}(G) \leq k_T(G) + 1$.

Open Problems.

(1)What type of G does have $\chi_{vt}(G) = k_T(G) + 1$?

(2) Let $C^I_{vt}(n) = \{G \mid |V(G)| = n, \chi_{vt}(G) = k_T(G)\}$ and $C^{II}_{vt}(n) = \{G \mid |V(G)| = n, \chi_{vt}(G) = k_T(G) + 1\}$. Whether is it true for

$$\lim_{n\to\infty} \frac{|C^{II}_{vt}(n)|}{|C^I_{vt}(n)|} = 0 \ ?$$

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