$i\gamma(1)$-perfect graphs with girth at least six *

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Abstract

Let k be a nonnegative integer, and let $\gamma(G), i(G)$ denote the domination number and the independent domination number of a graph G, respectively. The so-called $i\gamma(k)$-perfect graphs consist of all such graphs G in which $i(H) - \gamma(H) \leq k$ hold for every induced subgraph H of G. This concept introduced by I. Zverovich in [5] generalizes the well-known domination perfect graphs. He conjectured that $i\gamma(k)$-perfect graphs also have a finite forbidden induced subgraphs characterization as the case for domination perfect graphs. Recently, Dohmen, Rautenbach and Volkmann obtained such a characterization for all $i\gamma(1)$-perfect forests. In this paper, we characterize the $i\gamma(1)$-perfect graphs with girth at least six.

1 Introduction

All graphs considered in this paper are finite, undirected and without loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. A set $D \subseteq V(G)$ is called a dominating set if every vertex of $V(G) - D$ is adjacent to an element of D. The domination number $\gamma(G)$ is the minimum cardinality among all dominating sets of G. An independent set is a set of pairwise non-adjacent vertices of G. The independent domination number $i(G)$ is the minimum cardinality among all independent dominating sets of G.

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It is known that $\gamma(G) \le i(G)$ (see [3]). Summer and Moore [4] define a graph G to be domination perfect if $\gamma(H) = i(H)$ for every induced subgraph H of G. Many papers have devoted to characterize such graphs. In 1995, I. Zverovich and V. Zverovich obtained a complete characterization of the domination perfect graphs by using the forbidden induced subgraphs. For detailed information the reader may refer to the paper [6].

I. Zverovich [5] generalized this concept by introducing the so-called $i\gamma(k)$-perfect graphs for any nonnegative integer k. A graph G is said to be $i\gamma(k)$-perfect if $i(H) - \gamma(H) \le k$ for every induced subgraph H of G. Note that an $i\gamma(0)$-perfect graph is a domination perfect graph. He conjectured that for any positive integer k the $i\gamma(k)$-perfect graphs can be characterized by a finite number of forbidden induced subgraphs, as in the case for domination perfect graphs. The first step toward this direction had been done by Dohmen, Rautenbach and Volkmann in [1], although they doubted the possibility of a complete characterization. In their paper, they presented a sufficient condition for a graph to be $i\gamma(1)$-perfect, and they characterized all $i\gamma(1)$-perfect forests. For this purpose, they defined four classes of graphs as follows.

Definition 1 *For* $1 \le i \le 4$, *let* H_i *be the graphs as in Fig.1.* $\mathcal{H}_i$ *are the graph classes by adding an arbitrary set of edges between pairs of vertices* x, y *such that* x, y *belong to different sets among the sets* A, B, C, D.

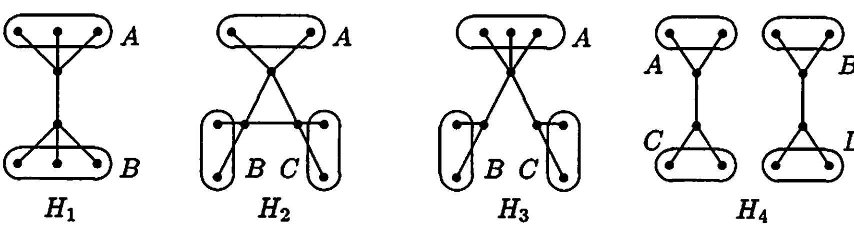

Fig. 1.

Theorem 1 ([1], Theorem 2.3) *If a graph* G *does not contain any graph in* $\mathcal{H}_i$ $(i = 1, 2, 3, 4)$ *as an induced subgraph, then* G *is* $i\gamma(1)$*-perfect.*

Theorem 2 ([1], Theorem 2.4) *A forest* T *is* $i\gamma(1)$*-perfect if and only if it does not contain one of the graphs* H_1, H_3 *or* H_4 *as an induced subgraph.*

In this paper, we consider graphs with girth at least six, where the girth of a graph G is the length of the shortest cycle if G contains a cycle, otherwise the girth of G is defined to be infinite. We will give a complete characterization for a graph with girth at least six to be $i\gamma(1)$-perfect in similar way. In fact, the following result is proved.

Theorem 3 *A graph G with girth at least six is $i\gamma(1)$-perfect if and only if G does not contain one of $F_i (1 \leq i \leq 10)$ (see Fig. 2) as an induced subgraph.*

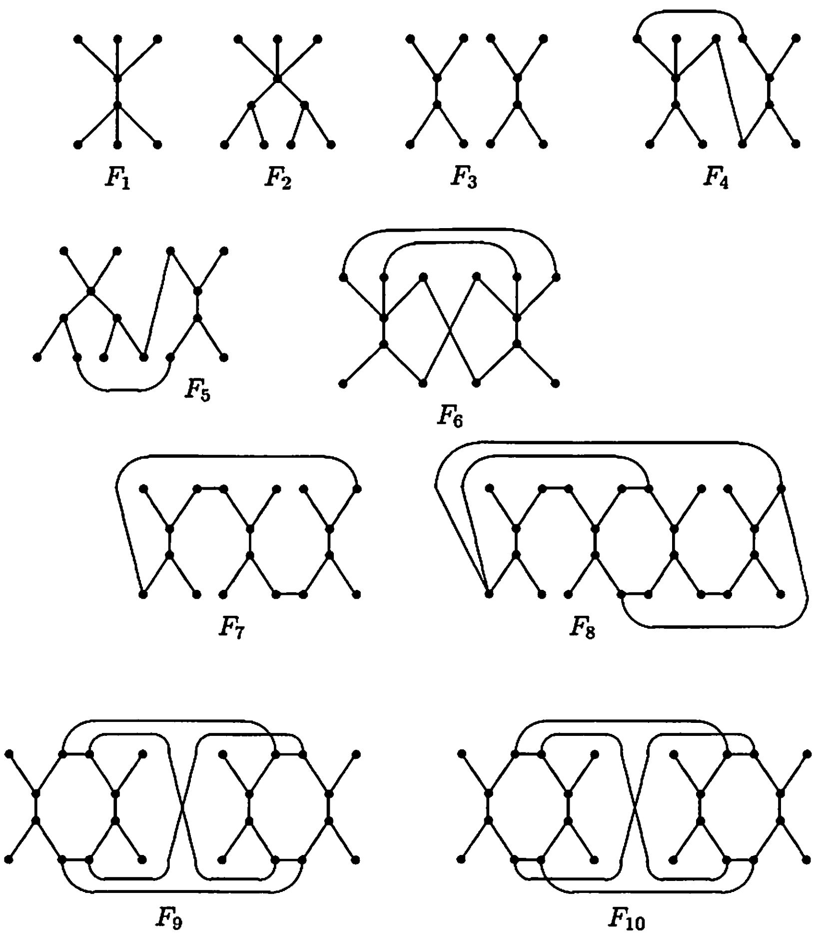

Fig. 2.

We will prove Theorem 3 in the next section. Before that, we give some notations and state an useful lemma proved by Dohmen, Rautenbach and Volkmann. The set $N(x)$ is the neighborhood of the vertex x, $N[x] := N(x) \cup \{x\}$ and $d(x) = |N(x)|$ is the degree of x. For $X \subseteq V(G)$ let $N(X) := \cup_{x \in X} N(x)$ and $N[X] := \cup_{x \in X} N[x]$. For $x \in X \subset V(G)$, the private neighborhood $P(x, X)$ of x with respect to X is defined as $P(x, X) := N[x] \backslash N[X \backslash \{x\}]$. The subgraph of G induced by $X \subset V(G)$ is denoted by $G[X]$. A set of edges is called independent if any two edges in this set share no common end. A graph that is not $i\gamma(k)$-perfect is said to be $i\gamma(k)$-imperfect, and a minimal $i\gamma(k)$-imperfect graph is an $i\gamma(k)$-imperfect graph in which all proper induced subgraphs are $i\gamma(k)$-perfect.

Lemma 1 ([1], Lemma 2.1) *For any nonnegative integer k, let G be a minimal $i\gamma(k)$-imperfect graph and D a dominating set of G with $|D| = \gamma(G)$. Then, (i) $G[D]$ contains no isolated vertex; (ii) For every $x \in D$ the private neighborhood $P(x, D)$ satisfies $\gamma(G[P(x, D)]) \geq 2$. Especially, $P(x, D)$ contains two non-adjacent vertices.*

2 Proof of theorem 3

It is straightforward to verify that for each $1 \leq i \leq 10, F_i$ is a minimal $i\gamma(1)$-imperfect graph with girth at least six, which implies the 'only if' part.

In the below, we assume that G is a graph with girth at least six and G does not contain one of F_i $(1 \leq i \leq 10)$ as an induced subgraph. We come to show that G is $i\gamma(1)$-perfect. By our assumption, we can obtain two simple facts.

Fact 1. For any two adjacent vertices x, y of G, at least one of x, y has degree at most three.

Fact 2. For a vertex $x \in V(G), N(x)$ is independent. If $d(x) \geq 5$, then at most one vertex in $N(x)$ has degree at least three.

Since the girth of G is at least six, $N(x)$ is independent. If G contains two adjacent vertices with degree at least four, then G contains an induced subgraph F_1; if G has a vertex x with degree at least five and $N(x)$ contains two vertices with degree at least three, then G contains an induced subgraph F_2. That implies Fact 1 and Fact 2.

For our purpose, we may assume that G is minimal $i\gamma(1)$-imperfect. Then, $i(G) \geq \gamma(G) + 2$. We proceed to deduce a contradiction. Let D be a dominating set of G with $|D| = \gamma(G)$. By lemma 1, $G[D]$ contains no isolated vertex, and for each vertex $x \in D$ the private neighborhood $P(x, D)$ contains at least two vertices. If $G[D]$ has a path $P = x_1x_2x_3x_4$, then both x_2, x_3 have degree at least four, contradicting to Fact 1. Thus, each path of $G[D]$ contains at most three vertices. So, each component of $G[D]$ is isomorphic to K_2 or $K_{1,k}(k \geq 2)$. Moreover, if there is one component which is isomorphic to $K_{1,k}(k \geq 3)$, then G contains a vertex with degree five and its neighborhood contains two vertices whose degrees are at least three, contradicting to Fact 2. Hence, each component of $G[D]$ is isomorphic to K_2 or $K_{1,2}$.

In the following, let t denote the number of the components of $G[D]$, and let $C \subset V(G) - D$ denote the set of the vertices which are adjacent to at least two vertices of D, and $P = V(G) - (D \cup C)$.

Definition 2 *For* $1 \leq i \leq 5$, *let* J_i *be the graph as in Fig. 3.*

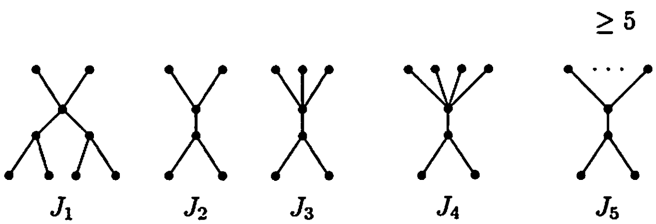

Fig.3

For a component T of $G[D]$, we call the subgraph induced by $T \cup (N(T) \cap P)$ as a **branch** of G. By Fact 1 and Fact 2, each branch of G is isomorphic to a graph J_i for some $1 \leq i \leq 5$. If a branch B of G is isomorphic to J_i for some $1 \leq i \leq 5$, then we define this branch as type J_i. Obviously, G has t branches, which are pairwise disjoint. We denote by $\mathcal{B}$ the set of the branches of G.

If G has only one branch, then G consists exactly of this branch, and we can easily choose an independent dominating set of G with $\gamma(G) + 1$ vertices, a contradiction. Hence, $t \geq 2$. Moreover, for any two distinct branches B_1, B_2 of G, there is at least one edge joining them. For otherwise, G has an induced subgraph F_3, a contradiction.

Note that, by Fact 1 and Fact 2, for a branch B of type J_1, each vertex $x \in V(B) \cap D$ has no neighbor in C; for a branch B of type $J_i (i \geq 3)$, the vertex $x \in V(B) \cap D$ whose private neighborhood $P(x, D)$ contains exactly two vertices also has no neighbor in C; for a branch of type J_2, at most one vertex of $V(B) \cap D$ may have neighbors in C. For convenience, we introduce more notation.

For a branch B of type J_1, the vertex of $V(B) \cap D$ whose degree is 4 is called w-vertex, and the other two vertices in $V(B) \cap D$ are called u_1-vertex and u_2-vertex of B, respectively. Correspondingly, we call the neighborhood of the w-vertex in P as the W-part, the neighborhood of the u_i-vertex in P the U_i-part ($i = 1, 2$) of B, respectively. For a branch B of type $J_i (i \geq 3)$, the vertex in $V(B) \cap D$ whose degree is at least 4 and its neighborhood in P are called u-vertex and U-part of B, respectively. Another vertex in $V(B) \cap D$ and its neighborhood in P are called w-vertex and W-part of B, respectively. For a branch B of type J_2, if $V(B) \cap D$ contain a vertex whose degree is at least 4, then we call this vertex and its private neighborhood in P as u-vertex and U-part, another vertex in $V(B) \cap D$ and its neighborhood in P as w-vertex and W-part of B, respectively. Otherwise, each vertex in $V(B) \cap D$ is called w-vertex, its private neighborhood in P is called W-part of B. We remark that the W-part of any branch and the U_i-part ($i = 1, 2$) of any branch of type J_1 contains exactly two vertices.

Assertion 1 *For a branch B of type $J_i (i \geq 2)$ of G, each vertex $x \in V(G) - V(B)$ is adjacent to at most one vertex of B.*

Proof It is clear by noting that each path in B contains at most four vertices and that G has girth at least six. ■

Assertion 2 *For any two distinct branches B_1, B_2 of G, let X_1 be the W-part of B_1 and X_2 the W-part of B_2, then there is at most one edge joining X_1 and X_2.*

Proof Let x_1, x_2 be w-vertices of B_1, B_2 which correspond to X_1, X_2, respectively. If there are at least two edges joining X_1 and X_2, then the girth condition implies that there are exactly two independent edges joining X_1, X_2. Suppose that $z_1 \in X_1, z_2 \in X_2$ and z_1, z_2 are not adjacent. Let $D' = (D - \{x_1, x_2\}) \cup \{z_1, z_2\}$. Then, D' is a dominating set of G such that

$|D'| = \gamma(G)$. Note that $G[D']$ contains isolated vertices z_1, z_2, contradicting lemma 1. ■

Remark We can similarly deduce that, there is at most one edge joining the U_i-part ($i = 1, 2$) of a branch of type J_1 and the W-part of another branch, and for two distinct branches B_1, B_2 of type J_1 there is at most one edge joining the U_i-part ($i = 1, 2$) X_1 of B_1 and the U_i-part ($i = 1, 2$) X_2 of B_2.

Let B be a branch of type $J_i (i \geq 2)$ of G, and X the W-part of B which corresponds to $x \in V(B) \cap D$, and y another vertex of $V(B) \cap D$. For any $B' \in \mathcal{B} - \{B\}$, we say that B' has **the property (Q) with respect to** (B, X), if it satisfies one of the following conditions (I) - (III).

(I) If B' is of type J_2 and both vertices in $V(B') \cap D$ have degree 3, then there is one edge joining $V(B')$ and X.

(II) If B' is of type J_1, then either there is one edge joining the W-part of B' and X, or there is one edge joining X and each U_i-part ($i = 1, 2$) of B'.

(III) If B' is of type $J_i (i \geq 2)$ such that one vertex of $V(B') \cap D$ has degree at least 4, then let x', y' be a w-vertex and an u-vertex of B', respectively. Either (1) there is one edge joining the W-part of B' and X; or (2) there is one edge joining the U-part of B' and X when $d(y') = 4$ and $N(y) \cap N(y') \neq \emptyset$; or (3) there are two edges joining $N(y') - \{x'\}$ and X when either $d(y') = 4$ and $N(y) \cap N(y') = \emptyset$ hold, or $d(y') = 5$ and $N(y) \cap N(y') \neq \emptyset$ hold.

Assertion 3 *For each vertex x of D, $d(x) \leq 4$.*

Proof Clearly, for a branch B of type J_1, each vertex of $V(B) \cap D$ has degree at most 4. We assume that the assertion is not true, let B_0 be a branch of type $J_i (i \geq 2)$ of G such that $V(B_0) \cap D$ contains a vertex with degree at least 5. Let x_0, y_0, X_0 be a w-vertex, an u-vertex, a W-part of B_0, respectively. $Y_0 = N(y_0) - \{x_0\}$. Then, $d(y_0) \geq 5$. We distinguish two cases to deduce a contradiction.

(1) $d(y_0) \geq 6$. Then, Y_0 contains at least five vertices.

We will show that any branch $B' \in \mathcal{B} - \{B_0\}$ has the property (Q) with respect to (B_0, X_0).

First let B' be a branch of type J_2 and both vertices in $V(B') \cap D$ have degree 3. Assume that there is no edge joining X_0 and $V(B')$, we check the edges between Y_0 and $V(B')$. By assertion 1, there are at most four edges joining Y_0 and $V(B')$. If there are at most three edges joining them, then Y_0 has two vertices which are not adjacent to any vertices of B', and thus G contains an induced subgraph F_3, a contradiction. If there are four edges joining them, then Y_0 has one vertex which has no neighbor in B'. By choosing two more vertices from Y_0 such that their neighbors in B' belong to different W-parts of B', we obtain an induced subgraph F_4 of G, a contradiction. Hence, there must be one edge joining X_0 and $V(B')$.

Next let B' be a branch of type $J_i (i \geq 2)$ and one vertex of $V(B') \cap D$ has degree at least 4. Let x', y' be a w-vertex and an u-vertex of B', respectively. Clearly, $N(y_0) \cap N(y')$ contains at most one vertex. If B' does not have (Q) with respect to (B_0, X_0), by deleting $N(X_0) \cap V(B')$ from B', then for each case we can obtain an induced subgraph of B' that is isomorphic to J_2 and that there is no edge joining $X_0 \cup \{x_0, y_0\}$ and this subgraph. Thus, we can similarly deduce that G contains an induced subgraph F_3 or F_4, a contradiction.

Finally let that B' is a branch of type J_1. If B' does not have (Q) with respect to (B_0, X_0), by deleting $N(X_0) \cap V(B')$, then we can also obtain an induced subgraph that is isomorphic to J_2 and that there is no edge joining $X_0 \cup \{x_0, y_0\}$ and this subgraph. Thus, G contains an induced subgraph F_3 or F_4, a contradiction.

Summarizing above, we have that any branch $B' \in \mathcal{B} - \{B_0\}$ has (Q) with respect to (B_0, X_0).

Now we choose an independent set D_1 in G. For the branch B_0 we put the vertices of X_0 and y_0 into D_1. For any branch B' of type J_2 that both vertices in $V(B') \cap D$ have degree 3, let $x \in V(B') \cap D$ correspond to one W-part of B' which joins one edge to X_0, then we put another vertex y of $V(B') \cap D$ in D_1. For any branch of type $J_i (i \geq 2)$ that one vertex in $V(B') \cap D$ has degree at least 4, if there is one edge joining its W-part and X_0, then we put its u-vertex in D_1; otherwise, we put its w-vertex in D_1. For a branch of type J_1, if there is one edge joining its W-part and X_0, we put both u_1-vertex and u_2-vertex of this branch in D_1; otherwise, we put its w-vertex in D_1. We denote by t' the number of branches of type J_1 with latter choice. Clearly, D_1 is independent. Let

$A = (V(G) - D_1) \cup N(D_1)$. For clarity, we denote by t_1 the number of the branches of type J_1 and $t_2 = t - t_1$. Then, $\gamma(G) = 3t_1 + 2t_2$ and $|D_1| = 3+(t_2-1)+2(t_1-t')+t' = 2t_1+t_2-t'+2$. By direct checking, we can deduce that $|A| = 2t'+(t_1-t')+(t_2-1) = t_1+t_2+t'-1$. Let D_2 be an independent dominating set of $G[A]$. Then, $D_1 \cup D_2$ is an independent dominating set of G. Clearly, $|D_2| \le |A|$, and thus $|D_1| + |D_2| \le 3t_1 + 2t_2 + 1 = \gamma(G) + 1$, a contradiction.

(2) $d(y_0) = 5$. Then, Y_0 contains exactly four vertices.

If all branches in $\mathcal{B} - \{B_0\}$ has (Q) with respect to (B_0, X_0), then we can similarly choose an independent dominating set of G with at most $\gamma(G) + 1$ vertices, a contradiction. Let B_1 be the branch which does not have (Q) with respect to (B_0, X_0).

If B_1 is a branch of type $J_i (i \ge 2)$ that one vertex in $V(B') \cap D$ has degree at least 4, then there is no edge joining its W-part and X_0. Let x_1, y_1 be a w-vertex and an u-vertex of B_1, respectively. We distinguish three subcases to deduce a contradiction.

(a) If $d(y_1) = 4$ and $N(y_1) \cap N(y_0) \ne \emptyset$, by girth condition of G, then there is no edge joining the U-part of B_1 and Y_0. If there is no edge joining U-part of B_1 and X_0, then G has an induced subgraph F_3, a contradiction. If $d(y_1) = 5$ and $N(y_1) \cap N(y_0) \ne \emptyset$, then we can similarly deduce that G has an induced subgraph F_3, a contradiction.

(b) Assume that $d(y_1) = 4$ and $N(y_1) \cap N(y_0) = \emptyset$. If there is exactly one edge joining X_0 and $N(y_1) - \{x_1\}$, by checking the edges joining Y_0 and B_1, we can obtain an induced subgraph F_3 or F_4 or F_6 in G, a contradiction. If there is no edge joining $N(y_1) - \{x_1\}$ and X_0, we can deduce that G has an induced subgraph F_3 or F_4, a contradiction.

(c) For remaining case, as there are at most two edges joining $N(y_1) - \{x_1\}$ and X_0, we can similarly deduce that G has an induced subgraph F_3 or F_4 or F_6, a contradiction.

If B_1 is a branch of type J_1, then we can similarly deduce that B_1 has (Q) with respect to (B_0, X_0), a contradiction.

Thus, B_1 is a branch of type J_2 that both vertices in $V(B') \cap D$ have degree 3. Moreover, if there are at most three edge joining Y_0 and B_1, then G contains an induced subgraph F_3 or F_4, a contradiction. Hence, there are four edges joining Y_0 and B_1.

Now we check the edge joining B_1 and the branches in $\mathcal{B} - \{B_0, B_1\}$. We can similarly show that, if $B' \in \mathcal{B} - \{B_0, B_1\}$ is type of J_1, then either there is one edge joining its W-part and B_1 or there is one edge joining each U_i-part ($i = 1, 2$) of B' and B_1; if $B' \in \mathcal{B} - \{B_0, B_1\}$ is type of $J_i (i \geq 2)$, then either there is one edge joining one W-part of B' and B_1, or there are at least $|N(y') - \{x'\}| - 1$ edges joining $N(y') - \{x'\}$ and B_1, where x', y' are a w-vertex and an u-vertex of B', respectively. Thus, we can similarly extend $(V(B_1) - D) \cup \{x_0\}$ to an independent dominating set of G with at most $\gamma(G) + 1$ vertices, a contradiction. This prove assertion 3. ■

By assertion 3, G contains only the branches of types J_1, J_2, J_3.

Assertion 4 *Each vertex in C is adjacent to exactly two vertices in D.*

Proof Let c be any vertex in C. Since each vertex in $N(c) \cap D$ has degree 4, by Fact 1, c has degree at most 3. If c has three neighbors y_1, y_2, y_3 in D, then c is not adjacent to any vertex in $V(G) - \{y_1, y_2, y_3\}$. For $i = 1, 2, 3$ let B_i be the branch of G containing y_i. Let $x_1, Y_1 = \{z_1, z_2\}$ be a w-vertex and an U-part of B_1, respectively. We first show that $\mathcal{B} = \{B_1, B_2, B_3\}$. For otherwise, let $B' \in \mathcal{B} - \{B_1, B_2, B_3\}$. First let B' be the type of J_2. By assertion 2, there are at most two edges joining Y_1 and B'. If there is at most one edge joining Y_1 and B', then G a subgraph F_3 induced by $V(B') \cup \{y_2, y_3, c, y_1, x_1, z_1\}$, where z_1 is not adjacent to any vertex of B', a contradiction. If there are two edges joining Y_1 and B', then the two edges are incident to two vertices of $V(B')$ from different parts, otherwise we can similar deduce a contradiction as in the proof of assertion 1. Thus, G contains a subgraph F_4 induced by $V(B') \cup \{x_1, y_1, z_1, z_2 c, y_2, y_3\}$, a contradiction. If B' is type of J_3 or J_1, we can similarly deduce a contradiction. Hence, $\mathcal{B} = \{B_1, B_2, B_3\}$. Since there is no edge joining any two U-parts of B_1, B_2, B_3, we have that either G has in independent dominating set with 7 vertices or G contains an induced subgraph F_7, a contradiction. This prove assertion 4. ■

Assertion 5 *G has no branch of type J_1.*

Proof By contradiction. For the purpose, we assume that G contains at least one branch of type J_1. We distinguish two cases.

(1) G has at least two branches of type J_1. We first check the edges joining among the U_i-parts of any two distinct branches of type J_1. Let B_1, B_2 be any two branches of type J_1. If there exist one U_i-part of B_1 and one U_i-part of B_2 such that no edge join them, by assertion 1 and remark, then G has an induced subgraph F_3 or F_4, a contradiction. So, there is one edge joining each U_i of B_1 and each U_i-part of B_2. Moreover, if the U_i-parts of B_1 has only two vertices s_1, s_2 which have neighbors in U_i-parts of B_2, then their neighbors in U_i-parts of B_2 contain at least three vertices. For otherwise, G has a cycle of length 4, a contradiction. Thus, at least one of B_1, B_2, saying B_2, satisfies that its U_i-parts contains at least three vertices which have neighbors in U_i-parts of B_1.

Let B_1, B_2 be two branches of type J_1 as above. We choose an independent set D' as follow. We put the w-vertex and all vertices of U_i-parts of B_1 in D', and put w-vertex of B_2 in D'. Note that $V(B_2) - N(D')$ contains at most one vertices. For any branch $B' \in \mathcal{B} - \{B_1, B_2\}$ of type J_1, we put w-vertex of B' in D'. By above statement, $V(B') - N(D')$ contains at most two vertices. For any branch B' of type J_2 that both two vertices in $V(B') \cap D$ have degree 3, it is easy to see that there is one edge joining $V(B')$ and each U_i-part of B_1. For a vertex $x \in V(B') \cap D$, if $N(x)$ has no neighbor in the U_i-parts of B_1, then we put x in D'. Otherwise we choose any one vertex of $V(B') \cap D$ and put it in D'. Clearly, $V(B') - N(D')$ contains at most one vertex. For any branch B' of type $J_i (i = 2, 3)$ of G that one vertex of $V(B') \cap D$ has degree 4, let x', y' be the w-vertex and the u-vertex of B', respectively. It is easy to see that for each U_i-part of B either there is one edge joining the W-part of B' and this part, or there are at least two edges joining $N(y') - \{x'\}$ and this part. For the former case, we put u-vertex of B' in D', for latter case we put w-vertex of B' in D'. Note that $V(B') - N(D')$ also contains at most one vertex.

Let A denote the subgraph induced by $V(G) - N(D')$ and D'' be an independent dominating set of A. Then, $D' \cup D''$ is an independent dominating set of G. Clearly, $|D''| \leq |A|$. A simple count shows that $|D' \cup D''| \leq \gamma(G) + 1$, a contradiction.

(2) G has only one branch B of type J_1. Note that, if there is a branch of type J_2 or J_3 such that both two vertices of one W-part of this branch have neighbors in U_i-parts of B, then we can similarly obtain an independent

dominating set of G with at most $\gamma(G)+1$ vertices, a contradiction. Thus, for any branch of type J_2 or J_3, any W-part has at most one vertex which has neighbors in U_i-parts of B. Now we claim that for any branch B' of type J_2 or J_3, there is one W-part of B' such that one vertex of this part has two neighbors in U_i-parts of B.

For that B' is type of J_2 such that both two vertices in $V(B') \cap D$ has degree 3, if claim is not true, by noting that G has no induced subgraph F_5, then there are two independent edges joining two W-parts of B' and two U_i-parts of B, and one more edge joining $V(B')$ and W-part of B. Let X be the W-part of B' such that both two vertices have neighbors in B, and $y_1 \in V(B) \cap D$ correspond to the U_i-part which has no neighbors of X. We choose W-part of B and the U_i-part of B which has a neighbor of X and y_1 and put them in D'. Then, we can similarly extend D' to an independent dominating set with at most $\gamma(G)+1$ vertices, a contradiction. For B' is type of $J_i (i \geq 2)$ such that one vertex of $V(B') \cap D$ has degree 4, let x', y' be a w-vertex and an u-vertex of B'. If claim is not true, then there are two edges joining $N(y') - \{x'\}$ and one U_i-part of B and one edge joining $N(y') - \{x'\}$ and W-part of B. Then, we can similarly deduce that G contains an independent dominating set with at most $\gamma(G)+1$ vertices, a contradiction.

Hence, the claim is true. If G contains only two branches, then it is easy to see that G has an independent dominating set with 6 vertices, a contradiction. Otherwise, let B_1, B_2 be any two branches in $\mathcal{B} - \{B\}$. By claim, for $i = 1, 2$, $V(B_i)$ has one W-part which contains a vertex s_i which has one neighbor in each U_i-part of B. If each vertex of U_i-parts of B is adjacent to s_1 or s_2, by choosing w-vertex of B and the W-part of B_i containing s_i for $i = 1, 2$, then G has a dominating set D' with $\gamma(G)$ vertices such that $G[D']$ contains isolated vertices. By lemma 1, a contradiction. Thus, s_1, s_2 share exactly one common neighbor in the U_i-parts of B. Let U_1-part of B contain the common neighbor a of s_1, s_2. Then, their neighbors in U_2-part of B are distinct. Now for any branch $B' \in \mathcal{B} - \{B, B_1, B_2\}$, let s' be the vertex in one W-part of B' which has one neighbor in each U_i-part of B. If s' is adjacent to a, then s' will share two common neighbors with s_1 or s_2, and thus G has a cycle of length 4, a contradiction. Thus, s' must be adjacent to the vertex $a' \in U_1 - \{a\}$. By putting the w-vertex of B and a', s_1, s_2 in D', we can similarly extend D' to an independent dominating set of G with at most $\gamma(G)+1$ vertices, a contradiction. This prove assertion 5. ■

Assertion 6 $C = \emptyset$

Proof Otherwise, let $c \in C$ and B_1, B_2 be two branches of type J_2 of G such that $y_1 \in N(c) \cap V(B_1) \cap D, y_2 \in N(c) \cap V(B_2) \cap D$. Let x_1, x_2, X_1, X_2 be the w-vertices and W-parts of B_1, B_2, respectively. Since there is no edge joining U-part of B_1 and U-part of B_2, either B_1 has (Q) with respect to (B_2, X_2), or B_2 has (Q) with respect to (B_1, X_1). Suppose that B_1 has (Q) with respect to (B_2, X_2). If any branch $B' \in \mathcal{B} - \{B_1, B_2\}$ has (Q) with respect to (B_2, X_2), then G has an independent dominating set with at most $\gamma(G)+1$ vertices, a contradiction. Thus, there exists one branch $B_3 \in \mathcal{B} - \{B_1, B_2\}$ which does not have (Q) with respect to (B_2, X_2).

We first assume that B_3 has one W-part X_3 such that there is one edge joining X_3 and U-part of B_2 and one edge joining $s \in X_3$ and c. Note that the union of $\{y_1, c, s, y_2, x_2\}$ and U-part of B_2 induce a subgraph which isomorphic to J_3. By assertion 2, for each branch in $\mathcal{B} - \{B_1, B_2, B_3\}$ there is one edge joining one W-part and s. If there is one edge joining B_1 and X_3, by putting X' and x_2 in D', then we can extend D' to an independent dominating set of G with at most $\gamma(G)+1$ vertices, a contradiction. If there is no edge joining B_1 and X_3, then there are edges joining another part Y of B_3 and X_1. Moreover, B_3 has (Q) with respect to (B_1, X_1), for otherwise G contains an induced subgraph F_4, a contradiction. Since $d(c) \leq 3$, all branches in $\mathcal{B} - \{B_1, B_2, B_3\}$ have (Q) with respect to (B_1, X_1). Thus, if B_2 also has (Q) with respect to (B_1, X_1), then we similarly deduce a contradiction. If B_2 does not have (Q) with respect to (B_1, X_1), then we can obtain an induced subgraph F_7 from B_1, B_2, B_3, a contradiction.

Otherwise, let x_3, y_3 be a w-vertex and an u-vertex of B_3, respectively. Then, there are three independent edges joining $N(y_3) - \{x_3\}$ and $N(y_2) - \{x_2\}$. For this case, we can similarly deduce a contradiction as above. This prove assertion 6. ■

By assertion 5 and 6, G contains only branches of type J_2, J_3 and $V(G) = D \cup P$.

Assertion 7 *(1) Let B be a branch of type J_3 and X the W-part of B, then there is exactly one branch B' in $\mathcal{B} - \{B\}$ which does not have (Q)*

with respect to (B, X). *(2) For each vertex* $p \in P, d(p) \leq 4$.

Proof (1) As above shown, there is at least one branch in $\mathcal{B}-\{B\}$ which does not have (Q) with respect to (B, X). If there are $B_1, B_2 \in \mathcal{B}-\{B\}$ which do not have (Q) with respect to (B, X), then, for each $i = 1, 2$, either there are two edges joining one W-part of B_i and U-part of B, or there are three edges joining the U-part of B_i and U-part of B. For $i = 1, 2$, let $x_i \in V(B_i) \cap D$ not correspond to this part of B_i. We put the U-part and w-vertex of B and $\{x_1, x_2\}$ in D_1. Then, D_1 dominates $V(B) \cup V(B_1) \cup V(B_2)$, and thus we can extend D_1 to a dominating set D' of G with $\gamma(G)$ vertices. Note that $G[D']$ contains isolated vertices, a contradiction.

(2) If $p \in P$ has degree at least 5, then $d(x) = 3$ for $x \in N(p) \cap D$. Let B be the branch of G containing x, p and $N(x) = \{p, q, y\}$ and $y \in V(B) \cap D$. If all branches in $\mathcal{B}-\{B\}$ has (Q) with respect to (B, X), where $X = \{p, r\}$ is one W-part of B, then we can deduce a contradiction as above. So, there is one branch B' which does not have (Q) with respect to (B, X). Note that $(\{r, y, x, p\} \cup N(p)) - (N(p) \cap V(B'))$ induce a graph which is isomorphic to J_3 or J_4, there is one edge joining $V(B') - N(p)$ and $N(p) - \{x\}$, implying that $N(p) - \{x\}$ contains a vertex with degree at least 3, contradicting to Fact 2. Hence, $d(x) \leq 4$ for each vertex $x \in P$. ■

To complete the proof, we distinguish two cases.

Case 1 G contains at least one branch of J_3. For this case, we distinguish three subcases.

Subcase 1.1 G has two branches B_1, B_2 of type J_3 such that there are three independent edges joining their U-parts.

For $i = 1, 2$, let x_i, y_i, X_i, Y_i be a w-vertex, an u-vertex, a W-part and an U-part of B_i, respectively. Note that $\mathcal{B}-\{B_1, B_2\} \neq \emptyset$. If there is one branch in $\mathcal{B}-\{B_1, B_2\}$ which does not have (Q) with respect to (B_1, X_1), then we can similarly deduce a contradiction as in the proof of assertion 7(1). Thus, each branch in $\mathcal{B}-\{B_1, B_2\}$ has (Q) with respect to both (B_1, X_1) and (B_2, X_2). Then, there is no edge joining X_1, X_2.

If there is one branch $B' \in \mathcal{B}-\{B_1, B_2\}$ such that one W-part X' of B' satisfies $X' \subset N(X_1) \cup N(X_2)$, let y' correspond to another part of B', then $X_1 \cup X_2 \cup \{y_1, y_2, y'\}$ dominates $V(B_1) \cup V(B_2) \cup V(B')$. Thus, G

contains an independent dominating set with at most $\gamma(G)+1$ vertices, a contradiction. If any branch $B' \in \mathcal{B} - \{B_1, B_2\}$ has one W-part X' such that there are two edges joining a vertex of X' and $X_1 \cup X_2$, then $\mathcal{B} - \{B_1, B_2\}$ has at least two branches. For $i = 1, 2$, let $B_i' \in \mathcal{B} - \{B_1, B_2\}$ and X_i' be W-part of B_i' such that $s_i \in X_i'$ are adjacent to two vertices in $X_1 \cup X_2$. If $X_1 \cup X_2 \subset N(s_1) \cup N(s_2)$, then we can obtain a dominating set D' of G such that $|D'| = \gamma(G)$ and $G[D']$ contains isolated vertices, a contradiction. As G has girth at least six, s_1, s_2 share exactly one common neighbor. We assume that $X_2 \subset N(s_1) \cup N(s_2)$ and $p \in X_1$ is adjacent to both s_1, s_2. Let $q \in X_1 - \{p\}$. Then, all branches in $\mathcal{B} - \{B_1, B_2, B_1', B_2'\}$ have one W-part which contains a neighbor of q. From that, it is easy to see that G contains an independent dominating set with at most $\gamma(G)+1$ vertices, a contradiction.

Hence, $\mathcal{B} - \{B_1, B_2\}$ has a branch B' which has no W-part X' such that there are two edges joining X' and $X_1 \cup X_2$.

If B' is type of J_3, without loss of the generality, we can assume that there is one edge joining its W-part X' and X_2, and two edges joining its U-part Y' and X_1. Let x', y' be a w-vertex and an u-vertex of B' and $a_3 \in Y' - N(X_1)$. If a_3 is adjacent to one vertex of Y_1, then $X_1 \cup Y_1 \cup \{x_2, x'\}$ dominates $V(B_1) \cup V(B_2) \cup V(B')$, and thus $i(G) \leq \gamma(G)+1$, a contradiction. If a_3 is adjacent to a vertex a_2 in Y_2, let $a_1 \in Y_1 \cap N(a_2)$, then $(Y_1 - \{a_1\}) \cup \{a_2\} \cup X_1 \cup \{x_2, x'\}$ dominates $V(B_1) \cup V(B_2) \cup V(B')$, and thus, a contradiction. Thus, a_3 has no neighbor in $Y_1 \cup Y_2$. By assertion 1, 2, it is easy to check that among B_1, B_2, B' there is at most one more edge joining X', Y_1 and at most two more independent edges joining Y', Y_2. And thus, by deleting one vertex properly from each of Y_1, Y_2, Y', we can obtain an induced subgraph F_7, a contradiction. If B' is type of J_2, we can similarly deduce a contradiction.

Subcase 1.2 G has two branches B_1, B_2 of type J_3 such that there are two independent edges joining U-part of B_1 and W-part of B_2.

We assume that there are no such two branches of type J_3 as in subcase 1.1. For $i = 1, 2$, let x_i, y_i, X_i, Y_i be a w-vertex, an u-vertex, a W-part and an U-part of B_i, respectively. Note that $\mathcal{B} - \{B_1, B_2\} \neq \emptyset$ and each branch in $\mathcal{B} - \{B_1, B_2\}$ has (Q) with respect to (B_1, X_1). By assertion 7(1), there is exactly one branch B_3 in $\mathcal{B} - \{B_1, B_2\}$ which does not have (Q) with respect to (B_2, X_2). So, there are two independent edges joining one W-part X_3 of B_3 and Y_2. Let Y_3 be another part of B_3. If there is no edge joining X_1 and X_3, then we can similarly deduce that either $i(G) \leq \gamma(G) + 1$ or G has an induced subgraph F_7, a contradiction. Thus, there is one edge joining X_3, X_1. Let $X_1 = \{p_1, q_1\}$ and $p_1 \in N(X_3) \cap X_1$. If any branches in $\mathcal{B} - \{B_1, B_2, B_3\}$ has one W-part which contains a neighbor of q_1, then we can similarly obtain $i(G) \leq \gamma(G) + 1$, a contradiction. Thus, there exists one branch B_4 in $\mathcal{B} - \{B_1, B_2, B_3\}$ such that no W-part of B_4 contain a neighbor of q_1.

If B_4 is of type J_3 such that there are two independent edges joining its U-part Y_4 and X_1, then, by assertion 7(1), there is one edge joining its W-part X_4 and X_2. Let $p_3 \in N(p_1) \cap X_3, a_4 \in N(p_1) \cap Y_4$. Note that $\{q_1, x_1, y_1, p_1, p_3, a_4\}$ induces a subgraph J_2. Let $a_2 = N(p_3) \cap Y_2$. Then, there is one edge joining this subgraph and $V(B_2) - \{a_2\}$. It is easy to see that only q_1 or a_4 may have neighbors in $V(B_3) - \{a_2\}$. Let $N(p_3) \cap D = \{x_3\}, y_3 \in (V(B_3) \cap D) - \{x_3\}$. Clearly, q_1 is not adjacent to any vertex in X_2. If q_1 is adjacent to a vertex in $Y_2 - \{a_2\}$, then $X_2 \cup Y_2 - \{a_2\} \cup \{p_3, y_1, y_3\}$ dominates $V(B_1) \cup V(B_2) \cup V(B_3)$, and thus $i(G) \leq \gamma(G) + 1$ vertices, a contradiction. If q_1 is adjacent to a_2, then G has a cycle of length 5. Thus, q_1 has no neighbor in $V(B_2)$. If a_4 is adjacent to a vertex in X_2, then $X_2 \subset N(a_4) \cup N(X_4)$, and thus $X_4 \cup Y_4 \cup \{y_1, y_2\}$ dominates $V(B_1) \cup V(B_2) \cup V(B_4)$, and hence $i(G) \leq \gamma(G) + 1$, a contradiction. Thus, a_4 has a neighbor in $Y_2 - \{a_2\}$. Let x_4, y_4 be a w-vertex and an u-vertex of B_4, respectively and $c_4 \in Y_4 - N(X_1)$. Then, $\{y_1, x_2, y_3, x_4, q_1, a_4, c_4\} \cup Y_2 - N(a_4)$ dominates $\cup_{i=1}^{4} V(B_i)$. Now, if any branch B' in $\mathcal{B} - \{B_1, B_2, B_3, B_4\}$ has one W-part X' containing a neighbor of q_1, then we can obtain that $i(G) \leq \gamma(G) + 1$, a contradiction. If B' has one W-part X' containing a neighbor p_1, then we can similarly deduce that $p' \in N(p_1) \cap X'$ is adjacent to a vertex in $V(B_2) - \{a_2\}$. If p' is adjacent to a vertex in X_2, then $\{q_1, x_1, y_1, p_3, a_4, p'\} \cup V(B_2) - \{a_2\}$ induces a subgraph F_4, a contradiction. Thus, p' is adjacent to a vertex in $Y_2 - \{a_2, N(a_4) \cap Y_2\}$. By noting that $d(p_1) \leq 4$, all branches in $\mathcal{B} - \{B_1, B_2, B_3, B_4, B'\}$ have one W-part which contains a neighbor of q_1. Thus, we can deduce that $i(G) \leq \gamma(G) + 1$, a contradiction. If there are two edges joining U-part of B' and X_1, we can similarly deduce that $i(G) \leq \gamma(G) + 1$, a contradiction.

Thus, B_4 is type of $J_i (i = 2$ or $3)$ such that there is one edge joining its one W-part X_4 and p_1. Denote $N(p_1) \cap X_4$ by p_4 and let Y_4 be another part of B_4. We still use the notation for B_1, B_2, B_3. If p_4 is adjacent to one vertex in $Y_2 - \{a_2\}$, then we can similarly deduce a contradiction as above. Thus, there is one edge joining p_4 and X_2. Now we check the edges joining B_3 and B_4. Note that there is no edge joining X_3 and X_4, for otherwise G has a dominating set D' such that $|D'| = \gamma(G)$ and $G[D']$ has isolated vertices, a contradiction. If B_3 is type J_3, then we can similarly deduce a contradiction as above. If B_4 is type of J_3, then there are two edges joining Y_4 and X_3 or Y_3. For either case we can easily deduce a contradiction. So, both B_3, B_4 are type of J_2. Now, if there is one edge joining X_3 and Y_4 or X_4 and Y_3, then we can similarly as above deduce that $i(G) \leq \gamma(G) + 1$, a contradiction. So, there is only one edge between $V(B_3)$ and $V(B_4)$ which joins Y_3, Y_4. Let $a_3 \in Y_3, a_4 \in Y_4$ and a_3, a_4 be adjacent. Let $c_2 \in Y_2 - N(X_3)$. If c_2, a_4 are not adjacent, then we can deduce that G contains an induced subgraph F_7, a contradiction. Thus, c_2, a_4 are adjacent. Let $x_4 \in V(B_4) \cap D$ correspond to X_4 in B_4. If $d(p_1) = 3$, then $X_3 \cup Y_4 \cup \{y_1, q_1, x_2, x_3, x_4\}$ dominates $\cup_{i=1}^{4} V(B_i)$, and thus $i(G) \leq \gamma(G)+1$, a contradiction. If there is one branch $B_5 \in \mathcal{B} - \{B_1, B_2, B_3, B_4\}$ such that $p_5 \in N(p_1) \cap V(B_5)$, then we can similarly deduce that B_5 is type J_2 and p_5 has one neighbor in X_2. Let X_5 be W-part of B_5 containing p_5 and Y_5 another W-part of B_5. Similarly, there is only one edge between $V(B_3)$ and $V(B_5)$ which joins Y_3 and Y_5. Let $a_5 \in N(Y_3) \cap Y_5$. Then, c_2, a_5 are adjacent. Then, we can extend $\{y_1, q_1, x_2, c_2, y_3\} \cup X_3$ to an independent dominating set of G with at most $\gamma(G) + 1$ vertices, a contradiction.

Subcase 1.3 For any two distinct branches of type J_3 of G, there is one edge joining their W-parts.

Then, G contains at most two branches of J_3. For otherwise, let B_1, B_2, B_3 be three branches of type J_3 of G. As there are three edges joining among their W-parts, it is easy to verify that there are three vertices which dominate their W-parts. Thus, we can extend that to a dominating set D' of G such that $|D'| = \gamma(G)$ and $G[D']$ contains isolated vertices, a contradiction.

First we assume that G has two branches B_1, B_2 of type J_3. For $i = 1, 2$, let x_i, y_i, X_i, Y_i be w-vertex, u-vertex, W-part and U-part of B_i, respectively. By assertion 7(1), for $i = 1, 2$ there is one branch B_i' of such that

B_i' does not have (Q) with respect to (B_i, X_i). By direct checking, we can show that B_1', B_2' are distinct. For $i = 1, 2$, we assume that there are two edges joining the W-part Y_i' of B_i' and Y_i, and X_i' is another W-part of B_i'. Let $X_1 = \{p_1, q_1\}, X_2 = \{p_2, q_2\}$ and p_1, p_2 be adjacent. If there is one edge joining $V(B_2')$ and q_1, as any branch in $\mathcal{B} - \{B_1, B_2, B_2'\}$ has (Q) with respect to (B_2, X_2), then $i(G) \leq \gamma(G) + 1$, a contradiction. We can similarly deduce that no edge joins $V(B_1')$ and q_2. As there is one edge joining $X_1' \cup Y_1'$ and X_2 and one edge joining $X_2' \cup Y_2'$ and X_1, by symmetry, we consider three cases.

(i) There is one edge joining Y_1' and p_2 and one edge joining Y_2' and p_1. For this case, if there is one edge joining X_2' and Y_1', let $a_1' \in N(p_2) \cap Y_1'$. If $|N(a_1')| \geq 4$, then for any branch in $\mathcal{B} - \{B_1, B_2, B_1', B_2'\}$ there is one edge joining its W-part and q_2, and thus we can extend $Y_1' \cup \{q_2\}$ to an independent dominating set with at most $\gamma(G) + 1$ vertices, a contradiction. If $Y_1' \subset N(X_2') \cup N(X_2)$, we can extend $X_2 \cup X_2'$ to an independent dominating set with at most $\gamma(G) + 1$ vertices, a contradiction. We can similarly deduce that there is no edge joining Y_1' and Y_2'. Thus, by symmetry, there is only one edge between $V(B_1')$ and $V(B_2')$ which joins X_1' and X_2'. Let $a_1 \in Y_1 - N(Y_1')$. Note that if there is no edge joining a_1 and X_2', then G contains an induced subgraph F_7, a contradiction. Otherwise, a_1 is adjacent to a vertex in X_2'. Let $a_2 \in Y_2 - N(Y_2')$. Similarly, we have that a_2 is adjacent to a vertex in X_1'. Let $p_1' \in X_1', p_2' \in X_2'$ be adjacent and $x_1' \in N(p_1') \cap D, x_2' \in N(p_2') \cap D$. If a_1 is not adjacent to p_2', then we can extend $X_1' \cup Y_1' \cup \{a_1\}$ to an independent dominating set with at most $\gamma(G)+1$ vertices, a contradiction. Thus, a_1 is adjacent to p_2' and a_2 is adjacent to p_1'. Now, it is easy to verify that $\{x_1, p_1, N(p_1) \cap Y_2', p_2, x_2, N(p_2) \cap Y_1'\}$ and $\{a_1, p_2', x_2', p_1', x_1', a_2\}$ induce a subgraph F_3, a contradiction.

(ii) There is one edge joining X_1' and p_2 and one edge joining Y_2' and p_1. We can similar deduce that there is no edge joining $V(B_1')$ and Y_2'. If there is one edge joining X_2', Y_1', let $V(B_1') \cap D = \{x_1', y_1'\}$ and $Y_1' = \{a_1', b_1'\}$, and $c_1 \in Y_1, p_2' \in X_2'$ such that $\{c_1, p_2'\} \subset N(a_1')$, then $\{x_1', y_1', a_1', b_1', p_2', c_1\}$ induce a subgraph J_2. It is easy to see that there is no edge joining this subgraph and X_2. Thus, there are two edges joining Y_2 and $\{p_2', c_1\}$. Then, $a_2 \in Y_2 - N(Y_2')$ is adjacent to p_2', and hence $i(G) \leq \gamma(G) + 1$, a contradiction. Thus, there is only one edge between $V(B_1')$ and $V(B_2')$ which joins X_1' and X_2'. Let $p_1' \in X_1'$ be adjacent to p_2. We can similarly deduce that p'_1 is adjacent to $p'_2 \in X'_2$ and $a_1 \in Y_1 - N(Y'_1)$ is adjacent to p'_2. Let $V(B'_2) \cap D = \{x'_2, y'_2\}$, $Y'_2 = \{a'_2, b'_2\}$ and $a'_2 \in N(p_1), c_2 \in N(a'_2)$ and y'_2 correspond to Y'_2. Note that $\{x'_2, y'_2, b'_2, a'_2, p_1, c_2\}$ induces a subgraph J_2, there is one edge joining this subgraph and B'_1, and thus c_2 is adjacent to one vertex a'_1 in Y'_1. Now, both $Y_2 \cup \{y_2, a'_1, a'_2\}$ and $\{a_1, p'_1, x'_1, p'_2, x'_2, p_2\}$ induces a subgraph J_2. Thus, a_1 is adjacent to one vertex in Y_2. Clearly, a_1 is not adjacent to c_2, for otherwise G has a cycle of length 5, a contradiction. Let $b_2 \in N(a_1) \cap Y_2$. Then, $(V(B_1) - X_1) \cup \{p'_2, b_2\}$ induces a subgraph J_3. So, for any branch B' in $\mathcal{B} - \{B_1, B_2, B'_1, B'_2\}$, if there is no edge joining B' and b_2, then B' contains a neighbor p' of p'_2, and thus p' is adjacent to a'_2. Hence, we can extend $(Y_2 - \{c_2\}) \cup Y'_1$ to an independent dominating set of G with at most $\gamma(G) + 1$ vertices, a contradiction.

(iii) There is one edge joining X'_1 and p_2 and one edge joining X'_2 and p_1. Clearly, there is no edge joining X'_1, X'_2, for otherwise we can deduce that $i(G) \leq \gamma(G) + 1$. If there is one edge joining Y'_1, Y'_2, by assuming $a'_1 \in Y'_1, a'_2 \in Y'_2$ and $a'_1 a'_2 \in E(G)$, then $N(a'_1) \cup N(a'_2)$ induces a subgraph J_2, and thus $N(a'_1) \cup N(a'_2) \cup N(p_1) \cup N(p_2)$ induces a subgraph F_3, a contradiction. For otherwise, we may assume that there is one edge joining Y'_1, X'_2. Let $N(a'_1) \cap (Y_1 \cup X'_2) = \{c_1, q'_2\}$. As there is no edge joining X'_2 and Y_2, if a'_1 has no neighbor in Y'_1, then $V(B_2) \cup \{x'_1, y'_1, a'_1, b'_1, q'_2, c_1\}$ induce a subgraph F_4, otherwise $(V(B_2) - N(a'_1)) \cup \{x'_1, y'_1, a'_1, b'_1, q'_2, c_1\}$ induce a subgraph F_3, a contradiction.

Next we assume that G has only one branch B_1 of type J_3. Let x_1, y_1, X_1, Y_1 be a w-vertex, an u-vertex, a W-part and an U-part of B_1. Let B_2 be the only branch of type J_2 that there are two edges joining one W-part Y_2 of B_2 and Y_1. Let X_2 another W-part of B_2 and $x_2 \in V(B_2) \cap D$ correspond to X_2 and $y_2 \in V(B_2) \cap D$ correspond to Y_2. By assertion 7(2), G has at most 8 branches. Let $X_1 = \{p_1, q_1\}$. If $p_1 \in X_1$ has degree 4, then $N(p_1) \cup N(x_1)$ induce a subgraph J_3, and thus there are two independent edges joining one W-part of B_2 and $N(p_1)$. If G has at least 6 branches, then there is one more branch B' that there are two edges joining B' and this W-part of B_2. By choosing all vertices in $V(B') - D$, we can deduce that $i(G) \leq \gamma(G) + 1$, a contradiction. Thus, G has only 5 branches. Let $p_i \in N(p_1) \cap V(B_i)$ for $i = 3, 4, 5$ and two independent edges join one W-part of B_2 and p_4, p_5. If $\{p_4, p_5\} \subset N(Y_2)$, let $Y_2 = \{a_2, b_2\}$, by noting that both $N(a_2) \cup N(y_2)$ and $N(b_2) \cup N(y_2)$ induce a subgraph J_2, then there are two edges joining the part of B_3 which does not contain p_3 to Y_1, a contradiction. Thus, $\{p_4, p_5\} \subset N(X_2)$. Then, there is no edge joining X_2 and $V(B_3)$, for otherwise we can deduce that $i(G) \leq 11$, a contradiction. So, the edges between B_2, B_3 must join to Y_2. Then, at least one of B_4, B_5, saying B_4, does not have (Q) with respect to (B_2, Y_2). Let $X_3 = \{p_3, q_3\}$ and $Y_3 = \{a_3, b_3\}$ be two W-parts of B_3. Clearly, q_3 has no neighbor in B_3. If p_3 has a neighbor in Y_2, by checking the edges among B_2, B_3, B_4, B_5, we can deduce that either G contains an induced subgraph F_7 or $i(G) \leq 11$, a contradiction. So, there is one edge joining Y_2 and Y_3. Let a_2, a_3 be adjacent. If a_3 is adjacent to a vertex in $V(B_4)$, then this vertex is in the W-part Y_4 of B_4 which does not contain p_4. Then, $N(a_2) \cup N(a_3)$ and $N(p_1) \cup N(p_4)$ induce a subgraph F_3, a contradiction. If b_3 has neighbors in both B_4, B_5, then we can obtain $i(G) \leq 11$, a contradiction. If b_3 has no neighbor in B_4, then there is one edge joining X_3 and Y_4, and thus B_2, B_3, B_4 induce a subgraph F_7, a contradiction. Then, there is one edge joining X_3 and $V(B_5)$. Clearly, this edge must join to the part Y_5 of B_5 which does not contain p_5. If there is no edge joining Y_2 and Y_5, then B_2, B_3, B_5 induce a F_7, a contradiction. Otherwise, by choosing $Y_2 \cup \{b_3\}$, we can deduce that $i(G) \leq \gamma(G) + 1$, a contradiction. Similarly we can show that $d(q_1) \leq 3$. So, each vertex in X_1 has degree at most 3. Thus, G has at most 6 branches.

We still use above notation for B_1, B_2. If G has 6 branches, then $d(p_1) = d(q_1) = 3$. For $i = 3, 4, 5, 6$, let X_i be the W-part of B_i which contains a neighbor of p_1 or q_1, and Y_i another W-part of B_i. Assume that $N(p_1) = \{p_3, p_4, x_1\}$ and $N(q_1) = \{p_5, p_6, x_1\}$. If p_3 is adjacent to one vertex a_2 in Y_2, then $N(p_3) \cup N(a_2)$ induce a subgraph J_2, and thus either $d(p_3) = 4$ or $d(a_2) = 4$. By using above skill, we can similarly deduce a contradiction. So, there is no edge joining Y_2 and $\{p_3, p_4, p_5, p_6\}$. Clearly, there are at most two independent edges joining X_2 and $\{p_3, p_4, p_5, p_6\}$. Thus, we may assume that p_3 is adjacent to a_2 and p_5 is adjacent to b_2. Note that each set of $N(p_1) \cup N(p_3)$ and $N(p_3) \cup N(N(P_3) \cap V(B_3) \cap D)$ induces a subgraph J_2. If B_6 joins one edge to p_3, then $d(p_3) = 4$, and thus we can similarly deduce a contradiction. If B_6 joins one edge to a_2, or two edges to $X_3 - \{p_3\}, p_4$, then we can obtain $i(G) \leq 13$, a contradiction.

If G has 5 branches, for $i = 3, 4, 5$ let X_i be one W-part of B_i which contains a neighbor of p_1 or q_1 and Y_i another W-part of B_i, and x_i correspond to X_i and y_i correspond to Y_i. Assume that $N(p_1) = \{p_3, p_4, x_1\}, N(q_1) = \{p_5, x_1\}$. Then, there is no edge joining X_5 and B_2.

If there is one edge joining Y_5 and Y_2, then there is no edge joining B_5 and X_2. If p_3 is adjacent to one vertex p_2 in X_2, then $N(p_3) \cup N(x_3)$ induce a graph J_2. Thus, there is one edge joining Y_5 and X_3. Clearly, this edge is incident to p_3. Then, $N(p_3) \cup N(x_3)$ induces a graph J_3. If there is one edge joining X_4 and $N(p_3) \cap Y_5$, then either G has an induced subgraph F_1 or $i(G) \leq 11$, a contradiction; otherwise there are two independent edge joining Y_4 and $N(p_3) - \{p_1\}$, then we can also deduce that $i(G) \leq 11$, a contradiction. So, there is no edge joining $\{p_3, p_4\}$ and X_2. We may assume that p_3 joins one edge to Y_2. Then, there is no edge joining Y_2 and B_4. Let $a_5 \in Y_5$ be adjacent to a vertex $a_2 \in Y_2$. If $p_3 \in N(a_2)$, then $N(a_2) \cup N(y_2)$ induce a graph J_3. Thus, there are two edges joining Y_4 and $N(a_2) - \{p_3\}$. Note that there is no edge joining X_2 and X_4, X_2 joins one edge to Y_4. If Y_3 joins one edge to X_4, then B_2, B_4, B_5 induce a subgraph F_7. If X_3 joins one edge to Y_4, we can easily obtain $i(G) \leq 11$ vertices. Thus, Y_3 joins one edge to Y_4. If $N(Y_3) \cap N(Y_5) \cap Y_4 \neq \emptyset$, then we can extend $Y_4 \cup (Y_1 - N(Y_4))$ an independent dominating set with at most 11 vertices; otherwise $N(Y_3) \cap N(Y_1) \cap Y_4 \neq \emptyset$, then we can extend $Y_5 \cup (Y_2 - N(Y_3))$ an independent dominating set with at most 11 vertices, a contradiction. Thus, p_3 is adjacent to b_2. If there is no edge joining B_4 and Y_5, then we can deduce that there are two edges joining Y_4 and Y_1, a contradiction; otherwise, we can extend $X_3 \cup Y_5$ to an independent dominating set with at most 11 vertices, a contradiction.

If $a_5 \in Y_5$ joins one edge only to X_2, then we can similarly deduce that $c_1 \in Y_1 - N(Y_2)$ is adjacent to a_5. If p_3 joins one edge to Y_2, by noting that there is no edge joining X_5 and B_4, then B_4 joins one edge to Y_5. Thus, we can extend $Y_2 \cup Y_5$ to an independent dominating set with at most 11 vertices, a contradiction. So, both p_3, p_4 have no neighbors in Y_2. We may assume that p_3 joins one edge to X_2. We can similarly deduce that $N(Y_5) \cap N(p_3) \cap X_2 \neq \emptyset$. Let $p_2 \in N(Y_5) \cap N(p_3) \cap X_2$. It is easy to see that there is no edge joining X_2 and X_3, and thus no edge joining B_2 and X_3. If $a_4 \in Y_4$ is adjacent to p_2, then $N(p_2) \cup N(x_2)$ induce a graph J_3, and thus, q_1 is adjacent to a_4 or a_5, a contradiction. If a_4 is adjacent to another vertex q_2 in X_2, by noting that c_1 is also adjacent to a_4, and thus we can extend $Y_4 \cup Y_2 \cup \{p_2\}$ to an independent dominating set with at most 11 vertices, a contradiction. So, there is no edge joining X_2, Y_4. Then, there is one edge joining Y_2, Y_4. Note that $N(p_2) \cup N(x_2)$ induce a graph J_2, B_4 joins one edge to a_5. Thus, $N(a_5) - \{c_1\} \cup N(y_5)$ induce a graph J_2. Then, there is one edge joining $V(B_1) - \{c_1\}$ and this subgraph. It is easy to see that this edge must join $Y_1 - \{c_1\}$ and $N(a_5) \cap B_4$, implying G has a cycle of length 5, a contradiction.

If G has only 4 branches, We still use above notation for B_3, B_4. First let $N(p_1) \cap X_3 = \{p_3\}, N(q_1) \cap X_4 = \{p_4\}$. Then, there is no edge joining $X_3 \cup X_4$ and B_2. So, at least one of Y_3, Y_4 joins one edge to X_2, we can similarly as above deduce a contradiction. Hence, we may assume $N(p_1) \cap X_3 = \{p_3\}$ and $N(p_1) \cap X_4 = \{p_4\}$. We assume that B_3 does not have (Q) with respect to (B_2, Y_2). Then, there is one edge joining X_2 and B_3.

If $p_2 \in X_2$ is adjacent to $a_3 \in Y_3$, then $c_1 \in Y_1 - N(Y_2)$ is adjacent to a_3. Now, if B_4 joins one edge to $V(B_2) - N(a_3)$, then we can extend $X_2 \cup Y_2 - \{p_2\} \cup \{a_3\}$ to an independent dominating set with at most 9 vertices, a contradiction. Thus, p_2 has one neighbor in B_4. If p_2 is adjacent to a vertex in Y_4, then we can easily deduce a contradiction. So, p_2 is adjacent to p_4. Thus, the edge between B_3, B_4 must join between X_3, Y_4. If p_3 is adjacent to a vertex in Y_4, then $N(p_3) \cup N(x_3)$ induce a graph J_2, and thus there is one more edge joining Y_2, Y_4, implying $i(G) \leq 9$, a contradiction. Thus, the vertex q_3 is adjacent to Y_4. Moreover, any more edge among B_1, B_2, B_3, B_4 implies $i(G) \leq 9$, a contradiction. Otherwise, by deleting a vertex in $Y_1 - \{c_1\}$ we obtain an induced subgraph F_8, a contradiction.

If there is no edge joining X_2, Y_3, then p_3 is adjacent to a vertex $p_2 \in X_2$. Now, it is easy to see that there is no edge joining X_2, X_4. If there is one edge joining Y_2 and Y_4, then this edge is the only edge between B_2, B_4. Clearly, there is no edge joining X_4, Y_3, for otherwise, G has an induced subgraph F_7. Thus, there exists one edge (or exactly one edge) from Y_4 to Y_3. let $a_2 \in Y_2, a_4 \in Y_4$ be adjacent. If a_4 has one neighbor in Y_3, then $N(a_4) \cup N(a_2))$ induce a graph J_2, which joins no edge to the subgraph induced by $N(p_1) \cup N(p_3)$, implying that G has an induced subgraph F_3, a contradiction. If b_4 has one neighbor, we can deduce that $i(G) \leq 9$ vertices, a contradiction. Hence, there is no edge joining Y_2, Y_4. Clearly, if there is only one edge between B_2, B_4 which joins X_2, Y_4, then we can deduce a contradiction as above. Thus, p_4 is adjacent to a vertex $a_2 \in Y_2$. Note that $N(a_2) \cup N(y_2)$ induce a graph J_2, there is one edge joining a vertex $a_3 \in Y_3$ and $a_1 \in N(a_2) \cap Y_1$. Now, it is easy to see that the edge between B_3, B_4 must join Y_3, Y_4, and thus there is one more edge joining X_2, Y_4. If the edge between Y_3, Y_4 is not incident to a_3, then we can extend $Y_1 \cup Y_3 - \{a_3\}$ to an independent dominating set of G with at most 9 vertices. If the edge between X_2, Y_4 is not incident to p_2, then we can extend $X_3 \cup X_2 - \{p_2\}$ to an independent dominating set of G with at most 9 vertices, a contradiction. Hence, a_3 joins one edge to Y_4 and p_2 joins one edge to Y_4. If these two edges are independent, by direct checking, we deduce that there is no edge joining $N(a_1) \cup Y_1$ and $N(p_2) \cup N(p_3)$, implying that G has an induced subgraph F_3, a contradiction. Thus, these two edges are incident to the same vertex in Y_4. Moreover, there is no edge joining Y_1 and Y_4, for otherwise we have $i(G) \leq 9$ a contradiction. Now, by deleting the vertex in $N(Y_2) \cap Y_1 - \{a_1\}$, we obtain an induced subgraph F_9, a contradiction.

Case 2 All branches in G are type J_2.

Note that if one branch B_1 joins two edge from one W-part to another branch B_2, then we choose all vertices in $V(B_2) \cap P$ and one vertex in $V(B_1) \cap D$. As any branch joins one edge to B_2, $i(G) \leq \gamma(G) + 1$, a contradiction. Hence, there are at most two edges joining any two distinct branches, and if there are two edges, then the two edges must join to different W-part for each branch. Obviously, G has at least 4 branches for otherwise we can easily deduce a contradiction. In the follow, We first show that each vertex in P has degree at most 3.

If not, let X_1 be W-part of a branch B_1 which contains a vertex p_1 with degree 4. For $i = 3, 4, 5$, Let $N(p_1) \cap V(B_i) = \{p_i\}$ and p_i belongs to W-part X_i of B_i. For $i = 1, 3, 4, 5$, let Y_i be another W-part of B_i, and x_i correspond to X_i and y_i correspond to Y_i. Assume that B_2 does not have (Q) with respect to (B_1, X_1). Let x_2 correspond to W-part X_2 of B_2 and y_2 correspond to W-part Y_2 of B_2, respectively. We assume that there is one edge joining Y_1 and Y_2. Note that $N(x_1) \cup N(p_1)$ induces a subgraph J_3, there are two independent edges joining $\{p_3, p_4, p_5\}$ to X_2 or Y_2.

We first assume that there are two independent edges joining $\{p_4, p_5\}$ and X_2. If $\mathcal{B} - \{B_1, \cdots, B_5\} \neq \emptyset$, by noting that $N(p_1) \cup N(p_4) - \{p_3\}$ induces a subgraph J_2, then there is one edge joining B' in $\mathcal{B} - \{B_1, \cdots, B_5\}$.

By Fact 1, $d(p_4) \leq 3$ and $d(p_5) \leq 3$. Then, B' must join one edge to $N(p_4) \cap X_2$. Similarly, B' joins one edge to $N(p_5) \cap X_2$. By above statement, a contradiction. So, $\mathcal{B} - \{B_1, \cdots, B_5\} = \emptyset$. Now, if X_4 joins one edge to Y_5, then, by choosing $X_5 \cup Y_5 \cup (X_2 - N(p_5))$, we can deduce that $i(G) \leq 11$, a contradiction. Similarly, there is no edge joining X_5 and Y_4. As there is no edge joining X_4, X_5, there is one edge joining Y_4, Y_5. Note that there is no edge joining X_4 and $V(B_3)$ and no edge joining X_5 and $V(B_3)$, for otherwise, by choosing $X_4 \cup X_5$, we can also deduce that $i(G) \leq 11$, a contradiction. As Y_3 can join edges to at most one of Y_4, Y_5, we may assume that there is one edge joining X_3 and Y_5. If X_3 joins one edge to B_2, then this edge is incident to p_3. As $d(p_3) < 4$, $p_3 \notin N(Y_5)$. Then, by choosing $Y_5 \cup \{p_3\}$, we can obtain that $i(G) \leq 11$ vertices, a contradiction. If Y_3 joins one edge to X_2, let $p_2 \in N(Y_3) \cap X_2$. If $p_5 \in N(p_2)$, then $\{p_2, p_4\}$ can be extended an independent dominating set with at most 11 vertices, otherwise, $p_4 \in N(p_2)$ and thus $\{p_2, p_5\}$ can be extended an independent dominating set with at most 11 vertices, a contradiction. Thus, there is only one edge between B_2 and B_3 which joins Y_2 and Y_3. If there is only one edge between B_2 and B_5, then B_2, B_3, B_5 induce a subgraph F_7. Thus, there is one edge joining Y_5 and Y_2. If $N(Y_4) \cap N(Y_2) \cap Y_5 \neq \emptyset$, then we can similarly deduce that G has an induced subgraph F_3, otherwise, by choosing $Y_2 \cup Y_5 - N(Y_2) \cap Y_5$, we can deduce that $i(G) \leq 11$, a contradiction.

Next we assume that there are two independent edges joining $\{p_4, p_5\}$ to Y_2. For this case, we can similarly deduce that G has only 5 branches and there is only edge between B_4, B_5 which joins Y_4, Y_5, and there is no edge joining X_4 and B_3 and no edge joining X_5 and B_3. We still assume that there is one edge joining X_3 and Y_5. As B_1, B_4, B_5 join the edges to Y_2, only X_2 joins one edge to B_3. If X_2 joins one edge to X_3, then $p_3 \in N(X_2)$, and thus $p_3 \notin N(Y_5)$. Then, we can similarly deduce that $i(G) \leq 11$ vertices, a contradiction. Thus, X_2 joins one edge to Y_3. As G contains no induced subgraph F_7, Y_3 joins one edge to Y_1. Clearly, Y_2 and Y_3 join edges to same vertex a_1 in Y_1. On other hand, there is one edge joining Y_5 and X_2, and also Y_3 and Y_5 join edges to same vertex p_2 in X_2. Also X_2 and Y_1 has same neighbor in Y_3. If X_3 joins one edge to Y_4, then Y_4 joins one edge to p_2, and thus there are three edge among X_2, Y_4, Y_5, implying a dominating set D' of G with $|D'| = \gamma(G)$ and $G[D']$ contains isolated vertices, a contradiction. Thus, Y_3 joins one edge to Y_4. For this situation, we can easily deduce that $i(G) \leq 11$ vertices, a contradiction. This prove that each vertex in P has degree at most 3.

Let B be a branch of G, $V(B) \cap D = \{x, y\}$, X, Y be two W-parts of B which correspond to x, y, respectively. If each vertex in $X \cup Y$ has degree at most two, then G has at most 5 branches. If X has one vertex p with degree 3, then $\{x, y\} \cup X \cup N(p)$ induce a graph J_2. Let $N(p) = \{x, p_1, p_2\}$ and $p_1 \in V(B_1), p_1 \in V(B_2)$. As there is at least one branch B_3 of G which does not have (Q) with respect to (B, X), B_3 joins one edge to p_1 or p_2. We may suppose that B_3 joins one edge to p_1. Thus, $\{x, p, p_2, p_1\} \cup N(p_1)$ induces a graph J_2. As each branch in $\mathcal{B} - \{B, B_1, B_2, B_3\}$ joins one edge to $N(p_1) \cap V(B_3)$ or p_2, $\mathcal{B} - \{B, B_1, B_2, B_3\}$ has at most two branches. If $\mathcal{B} - \{B, B_1, B_2, B_3\}$ has two branches B_4, B_5, then at least one of B_4, B_5 has (Q) with respect to (B, X). For otherwise, we can deduce that $d(p_1) = 4$ or $d(p_2) = 4$, a contradiction. Let X_4, Y_4 be two parts of B_4 and X_4 join one edge to X. If B_5 joins one edge to $N(p_1) \cap V(B_3)$, then B_4 joins one edge to p_2. Clearly, this edge joins Y_4 and p_2. Let $N(p_1) \cap V(B_1) \cap D = \{x_1\}$. Note that $N(x_1) \cup N(p_1)$ induce a graph J_2, Y_4 joins one edge to $N(x_1) - V(B_1) \cap D$. From that, we can deduce that $i(G) \leq 13$, a contradiction. Thus, B_5 does not have (Q) with respect to (B, X) and B_4 joins one edge to $N(p_1) \cap V(B_3)$. Then, B_5 joins one edge to p_2 and B_4 joins one edge to $N(p_2) \cap V(B_5)$. Note that $N(x_1) \cup N(p_1)$ induce a graph J_2, B_5 joins one edge to $N(x_1) - V(B_1) \cap D$. Let $N(p_2) \cap V(B_2) \cap D = \{x_2\}$. We can similar deduce that B_3 joins one edge to $N(x_2) - V(B_2) \cap D$. From that, we can easily deduce that $i(G) \leq 13$, a contradiction. Hence, G has at most 5 branches.

If G has 5 branches and each vertex in P has degree 2, then we can easily deduce that $i(G) \leq 11$, a contradiction. Assume that P has a vertex with degree 3, we still use above notation. We first assume that B_4 joins one edge from X_4 to X and B_3 joins one edge to p_1. Now we can deduce that Y_4 joins one edge to $N(p_1) \cap V(B_3)$. We denote the part of B_3 containing $N(p_1) \cap V(B_3)$ by X_3 and another part by Y_3. If X_3 joins one edge to Y, then B_2 does not have (Q) with respect to (B_3, X_3). We denote the part of B_2 containing p_2 by X_2 and another part by Y_2. If X_2 joins one edge to Y_3, then this edge is incident to p_2, and thus Y_4 joins one edge to $N(p_2) \cap Y_3$, a contradiction. So, Y_2 joins one edge to Y_3. As G has no induced subgraph F_3, Y_2 joins one edge to Y. From that we can deduce that $i(G) \leq 11$, a contradiction. If Y_3 joins one edge to Y, then Y_4 joins one edge to Y. Clearly, there is no edge joining X_2 and B_3. Now, if Y_2 joins one edge to X_3, we can also deduce that $i(G) \leq 11$, a contradiction. Thus, there is no edge joining B_2 and X_3. We denote $N(X_3) \cap Y_4$ by a_4. Then, a_4 is not adjacent to p_2, for otherwise there is a cycle of length 5. If a_4 is adjacent to another vertex in X_2, then G has an independent dominating set with at most 11 vertices, a contradiction. Thus, a_4 joins one edge to Y_2. Then, another vertex b_4 in Y_4 joins one edge to Y. As B, B_2, B_3 join edges to Y_4, B_2 joins one edge to X_4. Clearly, there is no edge joining X_2, X_4, thus Y_2 joins one edge to X_4. Now we check edges between B_1, B_2. Clearly, there is no edge between X_2 and B_3, thus there is only one edge joining B_1, B_2. If Y_1 joins one edge to X_2, then B_1, B_2, B_3 induce a subgraph F_7, a contradiction. So, Y_1 joins one edge to Y_2, and thus we can obtain an independent dominating set with at most 11 vertices, a contradiction.

Hence, G has 4 branches. If there is only one edge joining any two branches, then it is easy to obtain an induced subgraph F_7. We assume that there are two edges joining B, B_1. If G has a branch B' such that there are two branches that both of them join two edges to B', then G contains an independent dominating set with at most 9 vertices, a contradiction. So, each branch of B_2, B_3 joins only one edge to each of B, B_1. Now, if there is only one edge joining B_2, B_3, then we can deduce that G is isomorphic to F_8. If there are two edges between B_2, B_3, then we can deduce that G is isomorphic to F_9 or F_{10}, a contradiction. This complete the proof of Theorem 3. ∎

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