Super-Arc-Connected and Super-Connected Total Digraphs*

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Abstract: Let D be a strongly connected digraph with order at least two, $T(D)$ denote the total digraph of D, $\kappa(D)$ and $\lambda(D)$ denote the connectivity and arc-connectivity of D, respectively. In this paper we study super-arc-connected and super-connected total digraphs. Following results are obtained:

1. $T(D)$ is super-arc-connected if and only if $D \not\cong \overrightarrow{K_2}$.
2. If $\kappa(D) + \lambda(D) > \delta(D) + 1$, then $T(D)$ is super-connected.

Key words: Total digraph; Max-arc-connected; Super-arc-connected ; Super-connected

1 Introduction

For graph-theoretical terminology and notation not defined here we follow Bondy and Murty [3]. We consider strict digraph D (digraph having no loops and no parallel arcs are allowed) with vertex set $V(D)$ and arc set $A(D)$. For a vertex $v \in V(D)$, we denote the indegree, the outdegree of v, the minimum indegrees and outdegrees in D by $d_D^-(v)$, $d_D^+(v)$, $\delta^-(D)$ and $\delta^+(D)$, respectively. We denote the minimum degree of D by $\delta(D) = min\{\delta^-(D), \delta^+(D)\}$. $\overrightarrow{K_n}$ denotes the complete digraph of order n.

Let $D = (V(D), A(D))$ be a digraph, $|V(D)| = n$, $|A(D)| = m$,$V(D) = \{v_1, v_2, \cdots, v_n\}$. The *line digraph* of D, denoted by $L(D)$, is the digraph with vertex set $V(L(D)) = \{a_{ij} | a_{ij} = (v_i, v_j)$ is an arc in $D\}$, and a vertex a_{ij} is adjacent to a vertex a_{st} in $L(D)$ if and only if $v_j = v_s$ in D. The *total*

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digraph of D, denoted by $T(D)$, is a digraph with vertex-set $V(T(D)) = V(D) \cup A(D)$, there is an arc $(a, b) \in A(T(D))$ from vertex a to vertex b in $T(D)$ if and only if one of following four cases holds:

(1). If $a \in V(D)$ and $b \in V(D)$, then $(a, b) \in A(D)$.

(2). If $a \in V(D)$ and $b \in A(D)$, then a is the tail of arc b in D.

(3). If $a \in A(D)$ and $b \in V(D)$, then b is the head of arc a in D.

(4). If $a \in A(D)$ and $b \in A(D)$, then the head of arc a is the tail of arc b in D.

In fact, the total digraph $T(D)$ can be viewed as $V(T(D)) = V(D) \cup V(L(D))$ and $A(T(D)) = A(D) \cup A(L(D)) \cup A(D, L(D))$, where $A(D, L(D))$ denotes the arcs with one end in $V(D)$ and the other end in $V(L(D))$. For each vertex $v \in V(D)$, $d^+_{T(D)}(v) = 2d^+_D(v)$, since there are $d^+_D(v)$ out-arcs from v to vertices in $V(D)$ and $d^+_D(v)$ out-arcs from v to vertices in $V(L(D))$. For each vertex $a_{ij} \in V(L(D))$, $d^+_{T(D)}(a_{ij}) = d^+_D(v_j) + 1$, since $d^+_{L(D)}(a_{ij}) = d^+_D(v_j)$ and there are exactly one out-arc (a_{ij}, v_j) from a_{ij} to vertices in $V(D)$. Similarly, $d^-_{T(D)}(v) = 2d^-_D(v)$, $d^-_{T(D)}(a_{ij}) = d^-_D(v_i) + 1$.

An *arc-cut* of a strongly connected digraph D is a set of arcs whose remove makes D no longer strongly connected. The *arc-connectivity* $\lambda(D)$ is the minimum cardinality of an arc-cut over all arc-cuts of D. The inequality $\lambda(D) \leq \delta(D)$ is immediate. We call a digraph D *maximally arc-connected*, for short *max-*λ, if $\lambda(D) = \delta(D)$.

For a vertex $v \in V(D)$, denote by $N^+_D(v)$ the set of out-neighbors of v, $N^-_D(v)$ the set of in-neighbors of v, $E^+_D(v)$ the set of out-arcs of v, $E^-_D(v)$ the set of in-arcs of v. A strongly connected digraph D is *super-arc-connected*, for short *super-*λ, if every minimum arc-cut is either $E^+_D(v)$ or $E^-_D(v)$ for some vertex v. The *connectivity* $\kappa(D)$, *max-*κ, *super-*κ are similarly defined.

In [4], Chen characterized super-edge-connected undirected total graphs. In this paper, we study super-arc-connected and super-connected total digraphs.

The following two lemmas [7] will be used in our discussions.

Lemma 1.1. *Let D be a digraph with order at least two. Then D is strongly connected if and only if the line digraph $L(D)$ is strongly connected.*

Lemma 1.2. *Let D be a strongly connected digraph with order at least two. Then the subdigraph with order at least two of $L(D)$ is strongly connected* *if and only if the corresponding arc-induced subdigraph of D is strongly connected.*

2 Super-arc-connected total digraphs

Theorem 2.1. *Let D be a strongly connected digraph with order at least two. Then $T(D)$ is max-λ.*

Proof. Clearly, $\delta(L(D)) = \delta(D) \geq 1$, $\delta(T(D)) = \delta(D) + 1$. In order to prove $T(D)$ is max-λ, it suffices to show that $\lambda(T(D)) \geq \delta(T(D)) = \delta(D)+1$. Let S be a minimum arc-cut of $T(D)$, then there exists a nonempty proper vertex subset $X \subseteq V(T(D))$ such that there is no arc from X to $\overline{X}$ in $T(D) - S$, where $\overline{X} = V(T(D))\backslash X$.

We consider three cases.

Case 1. $X \subseteq V(D)$.

If $|X| = 1$, then $|S| \geq \delta(T(D))$. If $n - 1 \geq |X| \geq 2$, since $D - S$ is no longer strongly connected, and every vertex $v \in X$ has $d_D^+(v) \geq \delta(D)$ out-neighbors in $V(L(D))$, we have

$$|S| \geq \lambda(D) + |X| \cdot \delta(D) > \delta(D) + 1.$$

Finally, if $|X| = n$, then

$$|S| \geq n\delta(D) \geq \delta(D) + 1. \quad (1)$$

Case 2. $X \subseteq V(L(D))$.

If $|X| = 1$, then $|S| \geq \delta(T(D))$. If $\delta(D) \geq |X| \geq 2$, since every vertex $a \in X$ has at least $\delta(L(D)) - (|X| - 1) = \delta(D) - |X| + 1$ out-neighbors in $V(L(D))$ and exactly one out-neighbors in V(D), we have

$$|S| \geq |X|(\delta(L(D)) - |X| + 1) + |X| > \delta(D) + 1.$$

If $m - 1 \geq |X| > \delta(D)$, then

$$|S| \geq \lambda(L(D)) + |X| > \lambda(L(D)) + \delta(D) \geq \delta(D) + 1.$$

Finally, if $|X| = m$, then

$$|S| \geq n\delta(D) \geq \delta(D) + 1. \quad (2)$$

Case 3. $X \cap V(L(D)) \neq \varnothing$ and $X \cap V(D) \neq \varnothing$.

We may suppose that $V(D) \not\subseteq X$ and $V(L(D)) \not\subseteq X$. In fact, in the case that $V(D) \subseteq X$ or $V(L(D)) \subseteq X$, by considering $\overline{X}$, the proof is analogous to the proof of Case 1 or Case 2. For each arc (v_i, v_j) from $X \cap V(D)$ to $\overline{X} \cap V(D)$, if the corresponding vertex $a_{ij} \in \overline{X} \cap V(L(D))$, then (v_i, a_{ij}) is an arc from X to $\overline{X}$; if the corresponding vertex $a_{ij} \in X \cap V(L(D))$, then (a_{ij}, v_j) is an arc from X to $\overline{X}$. Hence, if $\delta(D) = 1$, then $|S| \geq 2\lambda(D)+\lambda(L(D)) = 3 > \delta(D)+1$. Now we consider the case that $\delta(D) \geq 2$. Let $|X \cap V(D)| = n_1$ and $|X \cap V(L(D))| = n_2$.

If $\delta(D) \geq n_1 \geq 1$, then $|S| \geq 2n_1(\delta(D) - n_1 + 1) + \lambda(L(D)) > \delta(D) + 1$. If $\delta(D) \geq n_2 \geq 1$, then $|S| \geq n_2(\delta(L(D)) - n_2 + 1) + 2\lambda(D) > \delta(D) + 1$. If $n - 1 \geq n_1 > \delta(D)$ and $\delta(D)(n_1 - 1) \geq n_2 > \delta(D)$, then

$$
\begin{aligned}
|S| &\geq \lambda(D) + \lambda(L(D)) + \delta(D)n_1 - n_2 \\
&\geq \lambda(D) + \lambda(L(D)) + \delta(D)n_1 - \delta(D)(n_1 - 1) \\
&> \delta(D) + 1.
\end{aligned}
$$

Finally, if $n - 1 \geq n_1 > \delta(D)$ and $m - 1 \geq n_2 > \delta(D)(n_1 - 1))$, then

$$
\begin{aligned}
|S| &\geq \lambda(D) + \lambda(L(D)) + n_2 - n_1 \\
&> \lambda(D) + \lambda(L(D)) + \delta(D)(n_1 - 1) - n_1 \\
&= \lambda(D) + \lambda(L(D)) + n_1(\delta(D) - 1) - \delta(D) \\
&\geq \lambda(D) + \lambda(L(D)) + (\delta(D) + 1)(\delta(D) - 1) - \delta(D) \\
&= \lambda(D) + \lambda(L(D)) + \delta(D)(\delta(D) - 1) - 1 \\
&\geq \delta(D) + 1.
\end{aligned}
$$

We thus conclude that $\lambda(T(D)) \geq \delta(D)+1$. Since $\lambda(T(D)) \leq \delta(T(D)) = \delta(D) + 1$, we have $\lambda(T(D)) = \delta(D) + 1 = \delta(T(D))$. Hence $T(D)$ is max-λ. □

Theorem 2.2. *Let D be a strongly connected digraph with order at least two. Then $T(D)$ is super-λ if and only if $D \not\cong \overrightarrow{K_2}$.*

Proof. It is evident that if $T(D)$ is super-λ then $D \not\cong \overrightarrow{K_2}$. From the proof of Theorem 2.1, the equality $\lambda(T(D)) = \delta(T(D))$ may hold only in the following two cases: (1). $X \subseteq V(D)$ and $|X| = 1$ or $|X| = n$; (2). $X \subseteq V(L(D))$ and $|X| = 1$ or $|X| = m$. If $|X| = 1$, then we are done. If

$|X| = n$ in the first case or $|X| = m$ in the second case, then the equality holds if and only if

$$\delta(D) = 1 \text{ and } |V(D)| = 2 \text{ (see (1) and (2)), i.e. } D \cong \overrightarrow{K_2}.$$

Hence if $D \ncong \overrightarrow{K_2}$, then $T(D)$ is super-λ. The proof is completed. □

3 Super-connected total digraphs

Theorem 3.1. *Let D be a strongly connected digraph with order at least two. Then $\kappa(T(D)) \geq min\{\delta(T(D)), \kappa(D) + \lambda(D)\}$.*

Proof. Let S be a minimum vertex-cut of $T(D)$. Then there exists a nonempty proper vertex subset $X \subsetneq V(T(D))$ such that there is no arc from X to $\overline{X}$ in $T(D) - S$, where $\overline{X} = V(T(D))\backslash(X \cup S)$ and the induced subdigraph of X in $T(D)$ is strongly connected.

We consider three cases.

Case 1. $X \subseteq V(D)$.

If $|X| = 1$, then $|S| \geq \delta(T(D))$. If $n - 1 \geq |X| \geq 2$, by noting that either $N_D^+(X)$ is a vertex cut of D or $N_D^+(X) = V(D)\backslash X$, and no two vertices in X may have a common out-neighbor in $V(L(D))$, we have

$$|S| \geq min\{\kappa(D), n - |X|\} + |X| \cdot \delta(D) > \delta(D) + 1 = \delta(T(D)).$$

Finally, if $|X| = n$, then for every vertex a_{ij} of $V(L(D))$, there exists an arc (v_i, a_{ij}) from $V(D)$ to $V(L(D))$ such that a_{ij} is the head of the arc, so S must contain all the vertices of $V(L(D))$. This case is impossible.

Case 2. $X \subseteq V(L(D))$.

Subcase 2.1. $|X| = 1$. $|S| \geq \delta(T(D))$.

Subcase 2.2. $m - 1 \geq |X| \geq 2$.

Since the vertex-induced subdigraph of X in $L(D)$ is strongly connected and it has at least two vertices, we denote the set of vertices incident with the arcs corresponding to X in D by Y, by Lemma 1.2, then Y is strongly connected and it has at least two vertices. Thus each vertex of Y is the head of one arc from X to Y.

If $\delta(D) = 1$, then

$$|S| \geq min\{\kappa(L(D)), m - |X|\} + 2 = 3 > \delta(D) + 1.$$

If $\delta(D) \geq 2$, we consider the following two subcases:

Subcase 2.2.1. $m - \kappa(L(D)) \geq |X| \geq 2$.

Suppose $a_{ij} \in X$ is a vertex such that there is at least one arc from a_{ij} to $S \cap V(L(D))$. Denote by $d = d^+_{L(D)}(a_{ij})(= d^+_D(v_j) \geq \delta(D))$. Then there are d vertices $\{a_{jt} \in V(L(D)) | t = 1, 2, \cdots, d\}$ such that $(a_{ij}, a_{jt}) \in A(L(D))$. Write $P = \{a_{j1}, a_{j2}, \cdots, a_{jl}\} \subseteq S \cap V(L(D))$ and $Q = \{a_{j,l+1}, a_{j,l+2}, \cdots, a_{jd}\} \subset X \cap V(L(D))$, where $1 \leq l \leq d-1$ (note that if $|P| = d$, then X can not be strongly connected) Thus

$$|Y| \geq |Q| + 1.$$

If $d > \delta(D)$, then

$$|S| \geq |Q| + 1 + |P| > \delta(D) + 1 = \delta(T(D)).$$

Now we consider the case that $d = \delta(D)$.

If $|Y| > |Q| + 1$, then

$$|S| \geq |P| + |Y| > |P| + |Q| + 1 = \delta(D) + 1 = \delta(T(D)).$$

If $|Y| = |Q| + 1$, since $|Q| \leq \delta(D) - 1$, then each vertex $v_h \in Y \setminus v_j$ is the tail of one arc (v_h, v_g) from Y to $V(D) - Y$. Then $a_{hg} \notin P$, but $a_{hg} \in S \cap V(L(D))$ and $(a_{jh}, a_{hg}) \in A(L(D))$. Thus

$$|S| \geq |P| + 2|Q| + 1 > \delta(D) + 1 = \delta(T(D)).$$

Subcase 2.2.2. $m - 1 \geq |X| > m - \kappa(L(D))$.

Since there are $m - |X|$ arcs from $S \cap V(L(D))$ to $V(D)$, then there is a set Y with at most $m - |X|$ vertices in $V(D)$ which are the heads of these arcs. If $|Y| = m - |X|$, since $\delta(D) \geq 2$, there must exist a vertex $v_i \in Y$ which is the head of one arc from X to v_i. Thus,

$$|S| \geq m - |X| + (n - (m - |X|)) + 1 = n + 1 > \delta(D) + 1 = \delta(T(D)).$$

If $|Y| < m - |X|$, the above inequality obviously holds.

Subcase 2.3. $|X| = m$.

For every vertex v_j of $V(D)$, there exists an arc (a_{ij}, v_j) from $V(L(D))$ to $V(D)$ such that v_j is the head of the arc, so S must contain all the vertices of $V(D)$. This case is impossible.

Case 3. $X \cap V(L(D)) \neq \varnothing$ and $X \cap V(D) \neq \varnothing$.

If $n \geq |X \cap V(D)| \geq n - \kappa(D)$ or $m > |X \cap V(L(D))| \geq m - \kappa(L(D))$, then the argument is similar to that of Case 1 or Case 2. If $n - \kappa(D) > |X \cap V(D)| \geq 1$ and $m - \kappa(L(D)) > |X \cap V(L(D))| \geq 1$, since $\kappa(L(D)) \geq \lambda(D)$, then

$$|S| \geq \kappa(D) + \kappa(L(D)) \geq \kappa(D) + \lambda(D).$$

The proof is completed. □

Remark. Let D be a strongly connected digraph with order at least two, and $\kappa(D) + \lambda(D) < \delta(D) + 1$. Let S be a minimum arc-cut set such that every component in $D - S$ has at least two vertices. If there exists a minimum vertex-cut set S' such that the heads(or tails) of all arcs of S are in S', then $\kappa(T(D)) = \kappa(D) + \lambda(D)$.

Corollary 3.2. *Let D be a strongly connected digraph with order at least two. If $\kappa(D) + \lambda(D) \geq \delta(D) + 1$, then $T(D)$ is max-κ.*

Theorem 3.3. *Let D be a strongly connected digraph with order at least two. If $\kappa(D) + \lambda(D) > \delta(D) + 1$, then $T(D)$ is super-κ.*

Proof. From the proof of Theorem 3.1, only when $|X| = 1$, or $n - \kappa(D) > |X \cap V(D)| \geq 1$ and $m - \kappa(L(D)) > |X \cap V(L(D))| \geq 1$, the equality $\lambda(T(D)) = \delta(T(D))$ may hold. If $|X| = 1$, then we are done. If $n - \kappa(D) > |X \cap V(D)| \geq 1$ and $m - \kappa(L(D)) > |X \cap V(L(D))| \geq 1$, then the equality cannot hold when $\kappa(D)+\lambda(D) > \delta(D)+1$. Hence if $\kappa(D)+\lambda(D) > \delta(D)+1$, then $T(D)$ is super-κ. The proof is completed. □

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