Alternating unimodal sequences of Whitney numbers

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Abstract. *We consider the lattice of order ideals of the union of an n- element fence and an antichain of size i, whose Hasse diagram turns out to be isomorphic to the i-th extended Fibonacci cube. We prove that the Whitney numbers of these lattices form a unimodal sequence satisfying a particular property, called alternating, we find the maximum level of the same sequence and determine the exact values of these numbers.*

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1 Introduction

We recall [10] that a *ranked poset* is a poset P with a function $r : P \to \mathbf{N}$, called rank, such that $r(y) = r(x) + 1$ for all elements $x, y \in P$ with x covered by y. When P is finite, the function r is usually chosen so that the minimal elements have rank 0. The *height* of P is its maximum rank.

The *Whitney number* $W_k(P)$ of a ranked poset P is the number of elements of rank k and the *rank polynomial* of P is the polynomial $W(P; q)$ which has these numbers as coefficients, i.e.

$$W(P; q) = \sum_{k \geq 0} W_k(P)\, q^k .$$

As it is well known, the set $\mathcal{J}(P)$ of all order ideals of a poset P, ordered by inclusion, is a distributive lattice. If P is finite, then $\mathcal{J}(P)$

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is ranked with height $|P|$ and its rank polynomial is monic with degree $|P|$.

We also recall that the *fence*, or *zigzag poset*, of order n is the poset $Z_n = \{x_1, x_2, \cdots, x_n\}$ in which $x_1 < x_2 > x_3 < \cdots$ are the cover relations. Denote by Z_n^i the poset formed by the union of an n-element fence and an antichain of size i.

It is known [6] that the simple graph underlying the Hasse diagram of $\mathcal{J}(Z_n)$ is isomorphic with the Fibonacci cube Γ_n, which is the graph whose vertices are the binary strings of length n without two consecutive ones and whose edges are the pairs of vertices with unitary Hamming distance.

The extended Fibonacci cubes, introduced by Wu [11], are constructed by the same recursive relation as the Fibonacci cube, but with different initial conditions. For positive i,n, with $n \geq i$, the ith extended Fibonacci cube of order n, denoted by Γ_n^i is an induced subgraph of Q_n, where $V(\Gamma_n^i) = V_n^i$ is defined recursively by the relation

$$V_{n+2}^i = 0V_{n+1}^i + 10V_n^i$$

with initial conditions $V_i^i = B_i, V_{i+1}^i = B_{i+1}$, where B_k denotes the set of binary strings of length k. Thus $\Gamma_i^i = Q_i$, $\Gamma_{i+1}^i = Q_{i+1}$ and in general, when $n \geq i+2$, the vertices of Γ_n^i are (0,1)-strings in which last i positions are vertices of Q_i and the first $n-i$ positions are vertices of Γ_{n-i}. Therefore the first $n-i$ and the following i positions of a vertex of Γ_n^i represent an order ideal of F_{n-i} and an order ideal of an i-element antichain, respectively. As a consequence, we obtain the following Proposition, where for convenience of notation in the rest of the article, we consider Γ_{n+i}^i, $n \geq 0$.

Proposition 1 *The ith-extended Fibonacci cube of order $n+i$, $n \geq 0$, is isomorphic to the Hasse diagram of the lattice of order ideals of Z_n^i.*

Another consequence is that we are able to determine the rank polynomial of $\mathcal{J}(Z_n^i)$, say F_n^i, as product of the corresponding polynomials of $\mathcal{J}(Z_n)$ and $\mathcal{J}(iZ_1)$.

Definition Let $C_m = \{c_0, c_1, ..., c_m\}$ be a unimodal sequence of positive real numbers having maximum at the $\lfloor \frac{m}{2} \rfloor$ position.

We call C_m **right-alternating** (also denoted r-alternating) when, for $n = \lfloor \frac{m}{2} \rfloor$

$$c_n \geq c_{n+1} \geq c_{n-1} \geq c_{n+2} \geq c_{n-2} \geq c_{n+3} \geq ... \geq c_0 \geq c_m$$

and we call C_m **left-alternating** (l-alternating) when

$$c_n \geq c_{n-1} \geq c_{n+1} \geq c_{n-2} \geq c_{n+2} \geq ... \geq c_0 \geq c_m.$$

It is easy to see that C_m is r-alternating if and only if the following two inequalities are satisfied:

$$R_1 : c_{n-j} \leq c_{n+j} \tag{1}$$

and

$$R_2 : c_{n-j} \geq c_{n+j+1} \tag{2}$$

foe every $0 \leq j \leq n$ and $n = \lfloor \frac{m}{2} \rfloor$. Similarly C_m is l-alternating if and only if the following two inequalities hold:

$$L_1 : c_{n-j} \geq c_{n+j} \tag{3}$$

and

$$L_2 : c_{n-j} \leq c_{n+j-1} \tag{4}$$

where $0 \leq j \leq n$.
In the case of m odd, say $m = 2n+1$, $n > 1$, C_m is called **reflective** when

$$c_{n-j} = c_{n+j+1}$$

where $0 \leq j \leq n$.

In this paper we prove that the Whitney numbers of the lattice $\mathcal{J}(Z_n^i)$ of order ideals of the union of an n-element fence and an i- antichain form an alternating unimodal sequence. Moreover we find the maximum level of the same sequence and we determine the exact values of these numbers. These results turn out to be a contribution to the study of unimodality results for various posets.

2 Whitney numbers of the Fibonacci lattices

In [6] it is proved that the simple graph underlying the Hasse diagram of $\mathcal{J}(Z_n)$ is isomorphic to the Fibonacci cube Γ_n. Recall that the rank polynomial of the Fibonacci cube is

$$F_n(q) = \sum_{k=0}^{n} f_{n,k}.q^k$$

where $f_{n,k} = W_k(\mathcal{J}(Z_n))$. It is proved [7] that the polynomials $F_n(q)$ are unimodal, symmetric whenever n is even, having the maximum level in the $\lfloor n/2 \rfloor$-th position. Moreover they satisfy the recurrences

$$F_{2(n+1)}(q) = F_{2n+1}(q) + q^2.F_{2n}(q)$$

and

$$F_{2(n+1)+1}(q) = q.F_{2(n+1)} + F_{2n+1}(q).$$

which imply the following recurrences for the Whitney numbers

$$f_{2n+2,k+2} = f_{2n+1,k+2} + f_{2n,k} \tag{5}$$

and

$$f_{2n+3,k+2} = f_{2n+2,k+1} + f_{2n+1,k+2}. \tag{6}$$

In Fig. 1 we represent the sequences of the coefficients of $F_n(q)$ for the first values of n.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9		
1	1	1										
2	1	1	1									
3	1	2	1	1								
4	1	2	2	2	1							
5	1	3	3	3	2	1						
6	1	3	4	5	4	3	1					
7	1	4	6	7	7	5	3	1				
8	1	4	7	10	11	10	7	4	1			
9	1	5	10	14	17	16	13	8	4	1		
10	1	5	11	18	24	26	24	18	11	5	1	
11	1	6	15	25	35	40	39	32	22	12	5	1

Fig. 1

Proposition 2 *The elements of the sequence* $\{f_{2n+3,k}, 0 \leq k \leq 2n+3\}$ *satisfy the inequalities:*

$$f_{2n+3,n+1-i} \geq f_{2n+3,n+i+2} \tag{7}$$

where $0 \leq i \leq n$.

Proof.

In [7] it is proved that the sequence $\{f_{2n+3,k}\}_k$ is unimodal, with maximum at $k = n+1$. By (6), the inequality (7) is equivalent to

$$f_{2n+2,n-i} + f_{2n+1,n+1-i} \geq f_{2n+2,n+1+i} + f_{2n+1,n+2+i}.$$

Because $n - i = n+1-(i+1)$, by the symmetry of the sequence $f_{2n+2,k}$, it follows $f_{2n+2,n-i} = f_{2n+2,n+i+2}$ and $f_{2n+2,n+1+i} = f_{2n+2,n+1-i}$; then from (5)

$$f_{2n+1,n+2+i} + f_{2n,n+i} + f_{2n+1,n+1-i} \geq f_{2n+1,n+1-i} + f_{2n,n-i-1} + f_{2n+1,n+2+i}$$

Since $f_{2n,n-i-1} = f_{2n,n+i+1}$, it remains

$$f_{2n,n+i} \geq f_{2n,n+i+1}$$

which is satisfied by the unimodality of the sequence $\{f_{2n,k}\}_k$ having maximum at $k = n$. Thus the result follows. □

Proposition 3 *The elements of the sequence* $\{f_{2n+3,k}, 0 \leq k \leq 2n+3\}$ *satisfy the inequalities:*

$$f_{2n+3,n+1-i} \leq f_{2n+3,n+1+i} \tag{8}$$

where $0 \leq i \leq n$.

Proof. By (6), the inequality (8) is equivalent to

$$f_{2n+2,n-i} + f_{2n+1,n+1-i} \leq f_{2n+2,n+i} + f_{2n+1,n+i+1}.$$

By the symmetry of the sequence $\{f_{2n+2,k}\}$, we obtain $f_{2n+2,n+1-(i+1)} = f_{2n+2,n+i+2}$ and $f_{2n+2,n+i} = f_{2n+2,n-i+2}$; then, by (5) and (6)

$$\begin{aligned} & f_{2n+1,n+i+2} + f_{2n,n+i} + f_{2n,n-i} + f_{2n-1,n+1-i} \\ & \leq f_{2n+1,n-i+2} + f_{2n,n-i} + f_{2n,n+i} + f_{2n-1,n+i+1} \end{aligned}$$

and

$$f_{2n,n+i+1} + f_{2n-1,n+i+2} + f_{2n-1,n+1-i} \leq f_{2n,n-i+1} + f_{2n-1,n-i+2} + f_{2n-1,n+i+1}.$$

Applying again (5) we obtain

$$f_{2n-1,n+i+1} + f_{2n-2,n+i-1} + f_{2n-1,n+i+2} + f_{2n-1,n+1-i} \leq$$

$$\leq f_{2n-1,n-i+1} + f_{2n-2,n-i-1} + f_{2n-1,n-i+2} + f_{2n-1,n+1+i}.$$

By the symmetry of $\{f_{2n-2,k}\}_k$ with respect to the $(n-1)$-position it is $f_{2n-2,n+i-1} = f_{2n-2,n-i-1}$. Moreover by Proposition 2 and the unimodality of the sequence $\{f_{2n-1,k}\}$, we have

$$f_{2n-1,n-1-(i-3)} \geq f_{2n-1,n-1+i-3+1} \geq f_{2n-1,n-1+i-2+5},$$

which proves the result. □

Proposition 4 *For n odd, the sequence $\{f_{n,k}\}_k$ is unimodal, has maximum at $\lfloor \frac{n}{2} \rfloor$ and it is r-alternating.*

Proof. It follows from (1), (2) and Propositions 2 and 3. □

3 Rank-polynomial of $\mathcal{J}(\Gamma_n^i)$

Let $F_n^i(q)$ be the rank polynomial of the lattice $\mathcal{J}(\Gamma_n^i)$; thus, by setting $f_{n,k}^i = W_k(\mathcal{J}(Z_n^i)\,)$, we have

$$F_n^i(q) = F_n^{i-1}(q).(1+q) = (\sum_{k=0}^{n} f_{n,k}^{i-1} q^k).(1+q) = \sum_{k=0}^{n} f_{n,k}^i q^k$$

and

$$f_{n,k}^i = f_{n,k}^{i-1} + f_{n,k-1}^{i-1} \tag{9}$$

where $f_{n,0}^i = f_{n,0}^{i-1}$ and $F_n^0(q) = F_n(q)$.

Lemma 1 *The sequence* $\{f_{n,k}^1, 0 \le k \le n\}$ *is unimodal and has maximum at* $k = \lfloor \frac{n+1}{2} \rfloor$; *for n odd it is l-alternating, while for n even it is reflective.*

Proof. The result is true for every $n \le 7$. Assume $n \ge 8$. Recall the sequence $\{f_{n,k}\}_k$ is unimodal and its maximum is at the position k $= \lfloor \frac{n}{2} \rfloor$. Thus

$$f_{n,h} \le f_{n,h+1}$$

for $0 \le h \le \lfloor \frac{n}{2} \rfloor - 1$. Then $f_{n,h-1} \le f_{n,h}$, for $1 \le h \le \lfloor \frac{n}{2} \rfloor - 1$, and by (9) it follows

$$f_{n,h}^1 \le f_{n,h+1}^1.$$

for $0 \le h \le \lfloor \frac{n}{2} \rfloor - 1$.

On the other hand, for $h \ge \lfloor \frac{n}{2} \rfloor + 1$, $f_{n,h-1} \ge f_{n,h}$ and also $f_{n,h} \ge f_{n,h+1}$; then again by (9) it follows

$$f_{n,h}^1 \ge f_{n,h+1}^1$$

Now we prove that, for n odd,

$$f_{n,\lfloor \frac{n}{2} \rfloor}^1 \le f_{n,\lfloor \frac{n+1}{2} \rfloor}^1$$

that is

$$f_{n,\lfloor \frac{n}{2} \rfloor} + f_{n,\lfloor \frac{n}{2} \rfloor - 1} \le f_{n,\lfloor \frac{n+1}{2} \rfloor} + f_{n,\lfloor \frac{n+1}{2} \rfloor - 1}.$$

It suffices to show that the inequality

$$f_{n,\lfloor \frac{n}{2} \rfloor - 1} \le f_{n,\lfloor \frac{n+1}{2} \rfloor}$$

is satisfied, but this follows immediately from Proposition 3.

Now, let $n = 2m$, $m > 1$, be even; then from the condition that $f_{n,k}$ is symmetric, having maximum at $k = m$ we obtain

$$f_{n,m-j} + f_{n,m-j-1} = f_{n,m+j+1} + f_{n,m+j}$$

which is equivalent to

$$f^1_{n,m-j} = f^1_{n,m+j+1}.$$

□

In Fig. 2 we display the sequences of the coefficients of $F^1_n(q)$ for the first values of n.

$i \backslash k$	0	1	2	3	4	5	6	7	8	9
1	1	2	1							
2	1	2	2	1						
3	1	3	3	2	1					
4	1	3	4	4	3	1				
5	1	4	6	6	5	3	1			
6	1	4	7	9	9	7	4	1		
7	1	5	10	13	14	12	8	4	1	

Fig. 2

Denote by R^i_1, R^i_2 and L^i_1, L^i_2 the inequalities (1), (2) and (3), (4) respectively when they involve the sequence of the coefficients of the rank-polynomial of the i-th extended Fibonacci cube.

Theorem 1 *Let $n = 2m+1$ be odd, $m > 1$. Then the sequence $\{f^i_{n,k}; 0 \leq k \leq n, 0 \leq i\}$ is unimodal, has maximum at $k = \lfloor \frac{n+i}{2} \rfloor$ and it is right or left alternating depending on i even or odd respectively.*

Proof. The proof is by induction on i. The result is true for $i = 0, 1$ from Proposition 4 and Lemma 1. Let us assume it is true for every integer lesser than i and we prove for i.
Let $i = 2t$, $t > 0$; then $i-1$ is odd and by assumption, the sequence $\{f^{2t-1}_{n,k}\}$ is unimodal, has maximum at $k = (m+t)$ and it is left-alternating.

First we prove that $\{f^i_{n,k}\}$ is unimodal, having maximum at $k = (m+t)$. For $1 \leq j \leq m+t$ the inequality

$$f^i_{n,m+t-j} \leq f^i_{n,m+t-j+1}$$

is equivalent to the inequality

$$f^{i-1}_{n,m+t-j} + f^{i-1}_{n,m+t-j-1} \leq f^{i-1}_{n,m+t-j+1} + f^{i-1}_{n,m+t-j}$$

which is satisfied by the inductive hypothesis. Moreover by the same reason and the same values of j, it is possible to prove

$$f^{i}_{n,m+t+j} \geq f^{i}_{n,m+t+j+1}.$$

It remains to find an order relation between $f^{i}_{n,m+t}$ and $f^{i}_{n,m+t+1}$. The inequality

$$f^{i}_{n,m+t} \geq f^{i}_{n,m+t+1}$$

is equivalent to

$$f^{i-1}_{n,m+t} + f^{i-1}_{n,m+t-1} \geq f^{i-1}_{n,m+t+1} + f^{i-1}_{n,m+t}$$

which is satisfied by L^{i-1}_{1}'s assumption.
Now we prove the given sequence is right-alternating, that is R^{i}_{1} and R^{i}_{2} are satisfied. We may see that R^{i}_{1} coincides with the inequality

$$f^{i}_{n,m+t-j} \leq f^{i}_{n,m+t+j}$$

that is

$$f^{i-1}_{n,m+t-j} + f^{i-1}_{n,m+t-j-1} \leq f^{i-1}_{n,m+t+j} + f^{i-1}_{n,m+t+j-1}$$

which is satisfied by the assumption of L^{i-1}_{2}.
R^{i}_{2} coincides with the inequality

$$f^{i}_{n,m+t-j} \geq f^{i}_{n,m+t+j+1}$$

equivalent to

$$f^{i-1}_{n,m+t-j} + f^{i-1}_{n,m+t-j-1} \geq f^{i-1}_{n,m+t+j+1} + f^{i-1}_{n,m+t+j}$$

which is satisfied by the assumption of L^{i-1}_{1}.

Now consider the case of i odd, say $i = 2t+1$, $t > 0$. Then $i-1$ is even and by assumption $f^{i-1}_{n,k}$ is unimodal, has maximum in position $m+t$ and is right-alternating.
Proving that $f^{i}_{n,k}$ is unimodal is essentially identical to the case of i even. Our aim now is to prove that the maximum is in position $m+t+1$ and it is l-alternating. Indeed the inequality

$$f^{i}_{n,m+t} \leq f^{i}_{n,m+t+1}$$

is equivalent to

$$f^{i-1}_{n,m+t} + f^{i-1}_{n,m+t-1} \leq f^{i-1}_{n,m+t+1} + f^{i-1}_{n,m+t}$$

which is satisfied by the assumption of R_1^{i-1}. Moreover L_1^i is satisfied; indeed

$$f_{n,m+t+1-j}^i \geq f_{n,m+t+1+j}^i$$

is equivalent to

$$f_{n,m+t+1-j}^{i-1} + f_{n,m+t-j}^{i-1} \geq f_{n,m+t+j+1}^{i-1} + f_{n,m+t+j}^{i-1}$$

which holds by R_2^{i-1}.
Finally L_2^i coincides with the inequality

$$f_{n,m+t+1-j}^i \leq f_{n,m+t+j}^i$$

equivalent to

$$f_{n,m+t+1-j}^{i-1} + f_{n,m+t-j}^{i-1} \leq f_{n,m+t+j}^{i-1} + f_{n,m+t+j-1}^{i-1}$$

satisfied by the assumption of R_1^{i-1}.

□

Notice that the unimodality's property of Theorem 1 is also a consequence of the condition that the product of a log-concave polynomial and a unimodal polynomial remains unimodal.

For example, consider the case of $n = 5$. In Fig. 3 we display the sequences of the values of $f_{5,k}^i$ for the first values of i.

$i \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1	3	3	3	2	1				
1	1	4	6	6	5	3	1			
2	1	5	10	12	11	8	4	1		
3	1	6	15	22	23	19	12	5	1	
4	1	7	21	37	45	42	31	17	6	1

Fig. 3

Theorem 2 *Let $n = 2m$ be even, $m > 1$. Then the sequence $\{f_{n,k}^i; 0 \leq k \leq n, 0 \leq i\}$ is unimodal, has maximum at $k = \lfloor \frac{n+i}{2} \rfloor$ and it is symmetric or reflective depending on i even or odd respectively.*

Proof. The proof of the unimodality and the position of the maximum is practically identical to the odd case; now we want to prove that it is symmetric or reflective depending on i even or odd respectively.

We proceed by induction on the value of i. Let i even, say $i = 2t$, $t > 1$. The condition of simmetry

$$f^i_{n,m+t-j} = f^i_{n,m+t+j},$$

where $0 \leq j \leq m+t$, is equivalent by (9) to the equality

$$f^{i-1}_{n,m+t-j} + f^{i-1}_{n,m+t-j-1} = f^{i-1}_{n,m+t+j} + f^{i-1}_{n,m+t+j-1}$$

By assumption the sequence $f^{i-1}_{n,k}$ is reflective with maximum in positions $m+t-1$ and $m+t$. Thus

$$f^{i-1}_{n,m+t-1-j} = f^{i-1}_{n,m+t+j}$$

and

$$f^{i-1}_{n,m+t-1-(j-1)} = f^{i-1}_{n,m+t+j-1}$$

and the result follows. Now, let us assume that i is odd, say $i = 2t+1$, $t > 0$; our aim is to prove that the sequence $f^i_{n,k}$ is reflective with maximum at $m+t$ and $m+t+1$. The equality

$$f^i_{n,m+t-j} = f^i_{n,m+t+1+j}$$

where $0 \leq j \leq m+t$, is equivalent to the equality

$$f^{i-1}_{n,m+t-j} + f^{i-1}_{n,m+t-j-1} = f^{i-1}_{n,m+t+j+1} + f^{i-1}_{n,m+t+j}.$$

By assumption $f^{i-1}_{n,k}$ is symmetric with maximum at $k = m+t$; this implies $f^{i-1}_{n,m+t-j} = f^{i-1}_{n,m+t+j}$ and $f^{i-1}_{n,m+t-(j+1)} = f^{i-1}_{n,m+t+j+1}$. Then the result follows. □

As an example of the above property, consider the case of $n = 4$. In Fig. 4 we display the sequences of the values of $f^i_{4,k}$ for the first values of i.

$i \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1	2	2	2	1					
1	1	3	4	4	3	1				
2	1	4	7	8	7	4	1			
3	1	5	11	15	15	11	5	1		
4	1	6	16	26	30	26	16	6	1	

Fig. 4

4 Whitney numbers of $\mathcal{J}(Z_n^i)$

The aim of this section is determining an explicit expression for the Whitney numbers of $\mathcal{J}(Z_n^i)$. First we consider a combinatorial interpretation of the Whitney numbers of $\mathcal{J}(Z_{2n+1})$ and find their exact value, using the corresponding values of $\mathcal{J}(Z_{2n})$, calculated in [7].
Recall some basic definitions. A multiset on a set S is a function $\mu : S \to N$, where N is the set of the natural numbers; $\mu(x)$ is the multiplicity of the element $x \in S$; the order of μ is the sum of all the multiplicities, that is

$$ord(\mu) := \sum_{x \in S} \mu(x).$$

A 2-filtering multiset on S is a multiset μ such that $\mu(x) \leq 2$, for each $x \in S$. We write $M_{n,k}^{(3)}$ for the set of all the 2-filtering multisets on $[n] := \{1, 2, ..., n\}$ of order k.

The size of $M_{n,k}^{(3)}$ is given by the *trinomial coefficient* $\binom{n;3}{k}$; clearly, from the combinatorial meaning of these coefficients, it follows the equality

$$(1 + q + q^2)^n = \sum_{k=0}^{2n} \binom{n;\,3}{k} q^k .$$

First we give a combinatorial interpretation of the ideals of the fence Z_{2n+1} in terms of a suitable kind of multisets on $[n+1]$.

For simplicity we write $x_1, \ldots, x_{n+1}$ for the elements of rank 0 of Z_{2n+1} and $y_1, \ldots, y_n$ for those of rank 1, so that y_i covers x_i and x_{i+1} for $1 \leq i \leq n$.

To an ideal I of Z_{2n+1} we associate the multiset μ on $[n+1]$ defined by

$$\mu(i) = \begin{cases} 0 & \text{if } x_i \notin I ,\ y_i \notin I \\ 1 & \text{if } x_i \in I ,\ y_i \notin I \\ 2 & \text{if } x_i \in I ,\ y_i \in I . \end{cases}$$

with the condition that $\mu(x_{n+1}) \neq 2$. Clearly I is an ideal of Z_{2n+1} if and only if μ is a 2-filtering multiset on $[n+1]$ where $\mu(n+1)$ can not be 2. Thus we consider all the multisets 2-filtering on $[n+1]$ of order k which satisfy the condition that if $\mu(i) = 2$, then $\mu(i+1) \neq 0$, where $1 \leq i \leq n$ and we delete all the multisets which have last element equal to 2, which can be considered as all the possible multisets 2-filtering on $[n]$ of order $k-2$. By using the identity proved in [7], we obtain the following

Proposition 5 *The Whitney numbers of $\mathcal{J}(Z_{2n+1})$ have the explicit ex-*

pression

$$f_{2n+1,k} = f_{2n+2,k} - f_{2n,k-2} = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i} \binom{n+1-2i;\,3}{k-2i} (-1)^i - \tag{10}$$

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} \binom{n-2i;\,3}{k-2-2i} (-1)^i. \tag{11}$$

Now let I be an order ideal of Z_n^i of order k. It is easy to see that I is formed by an order ideal of Z_n of order r and an order ideal of i copies of Z_1 of order s, such that $r+s=k$; moreover the number of similar ideals is obtained as a product of the numbers of order ideals of Z_n of order r by the number of order ideals of i copies of Z_1 of order s. This implies that

$$f_{n,k}^i = \sum_{r+s=k} f_{n,r} \cdot \binom{i}{s}$$

where $0 \leq r \leq n$ and $0 \leq s \leq i$. In the case of $n = 2m$ even, $m > 1$,

$$f_{2m,k}^i = \sum_{r+s=k} \left(\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-j}{j} \binom{m-2j;\,3}{r-2j} (-1)^j \right) . \binom{i}{s}$$

while for $n = 2m+1$, we have

$$f_{2m+1,k}^i = \sum_{r+s=k} \left(\sum_{j=0}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m+1-j}{j} \binom{m+1-2j;\,3}{r-2j} (-1)^j \right.$$

$$\left. - \left(\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-j}{j} \binom{m-2j;\,3}{r-2-2j} (-1)^j \right) \right) . \binom{i}{s}.$$

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References

[1] M. Aigner, *"Combinatorial Theory"*, Springer-Verlag, New York, 1979.

[2] I. Beck, *Partial orders and the Fibonacci numbers*, The Fibonacci Quart. **26** (1990), 272-274.

[3] L. Comtet, *"Advanced Combinatorics"*, Reidel, Dordrecht-Holland, Boston, 1974.

[4] T.S. Blyth, J.C. Varlet, *"Ockham algebras"*, Oxford Science Pub. 1994.

[5] E.R. Gansner, *On the lattice of order ideals of an up-down poset*, Discrete Math. **39** (1982), 113-122.

[6] E. Munarini, *The many faces of Fibonacci graphs*, preprint

[7] E. Munarini, N.Zagaglia Salvi, *On the rank polynomial of the lattice of order ideals of fences and crowns*, Discrete Math. 259, 1 - 3 (2002), 163 - 177.

[8] H. M. Pendyala and J. Wu, *Prefix computation on extended Fibonacci cubes*, Congr. Numer. 103 (1994), 202 - 214.

[9] R. Stanley, *"Enumerative Combinatorics"*, Vol. I, Wadsworth and Brooks/Cole, Monterey, CA, 1986.

[10] D. Stanton, D, White, *"Constructive combinatorics"*, Springer-Verlag, New-York, 1986.

[11] J. Wu, *Extended Fibonacci cubes*, IEEE Trans. Parallel and Distributed Systems, 8, n. 12 (1997), 3 - 9.

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