$\mathbb{Z}$-Cyclic Directed Moore (2, 6) Generalised Whist Tournament Designs on p elements, where $p \equiv 7 \pmod{12}$.

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1 Introduction

A generalised whist tournament design, as discussed in [1], is a schedule of games for a tournament involving v players to be played in $v-1$ or v rounds (depending on the number of players involved). For the purposes of this work, which will be dealing with tournaments on p elements, where $p \equiv 7 \pmod{12}$ is prime, the tournaments will be arranged into v rounds. A game involves k players in a multi-team game with teams of t players competing, and a round consists of $(v-1)/k$ (or v/k) simultaneous games, with a player playing in at most one of these. The schedule must also be balanced in the sense that each pair of players play together as teammates in $(t-1)$ games, and as opponents in $(k-t)$ games. Such a schedule of games will be denoted by (t, k) $GWhD(v)$.

Here, we will be looking at a specific type of (2, 6) $GWhD(6m+1)$ on $6m+1$ players. (2, 6) $GWhD(v)$ are discussed in [2]. Such a design is a schedule of games (or tables) $(a, b; c, d; e, f)$ involving 3 teams of 2 players competing against each other such that

i. the games are arranged into $6m+1$ rounds each of m games;
ii. each player plays in exactly one game in all but one round;
iii. each player partners every other player exactly once;
iv. each player opposes every other player exactly four times.

If we consider the players as being seated around a circular table, then we can think of $(a, b; c, d; e, f)$ as the ordered block $\{a, c, e, b, d, f\}$. Suppose that $(a, b; c, d; e, f)$ is a game in a (2, 6) $GWhD(6m+1)$. Then we say that the pairs $\{a, b\}$, $\{c, d\}$, $\{e, f\}$ are *partners*. $\{a, c\}$, $\{b, e\}$, $\{d, f\}$ are said to be pairs of *opponents of the first kind*. $\{a, e\}$, $\{b, d\}$, $\{c, f\}$ are said to be pairs of *opponents of the second kind*. $\{a, f\}$, $\{b, c\}$, $\{d, e\}$ are said to be pairs of *opponents of the third kind*. $\{a, d\}$, $\{b, f\}$, $\{c, e\}$ are said to be pairs of *opponents of the fourth kind.* This $(2,6)$ $GWhD(6m+1)$ is described as a *moore* $(2,6)$ $GWhD(6m+1)$ if every player has every other player exactly once as an opponent of the first kind, opponent of the second kind, opponent of the third kind and opponent of the fourth kind. These are called *moore tournaments* because of E. H. Moore's discussion of a similar specialisation of whist tournaments in [7], and were first referred to as such in [5]. If we consider our block $\{a, c, e, b, d, f\}$ as players sitting at a circular table in the given order, then we can refer to c as a's *first left hand opponent*, e as a's *second left hand opponent*, f as a's *first right hand opponent* and d as a's *second right hand opponent.* We can make similar definitions for each of b, c, d, e and f. A *directed* $(2,6)$ $GWhD(6m+1)$ is a $(2,6)$ $GWhD(6m+1)$ in which each player is a first left hand opponent, second left hand opponent, first right hand opponent and second right hand opponent of every other player exactly once.

If the players are elements of $\mathbb{Z}_{6m+1}$, and if the *ith* round is obtained from the initial (first) round by adding $i-1$ to each element (mod $6m+1$), then we say that the tournament is *$\mathbb{Z}$-cyclic.* By convention we always take the initial round to be the round from which 0 is absent. The games (tables)

$$(a_1, b_1; c_1, d_1; e_1, f_1), \ldots, (a_m, b_m; c_m, d_m; e_m, f_m)$$

form the initial round of a $\mathbb{Z}$-cyclic moore $(2,6)$ generalised whist tournament design if

$$\bigcup_{i=1}^{m} \{a_i, b_i; c_i, d_i; e_i, f_i\} = \mathbb{Z}_{6m+1} - \{0\} \tag{A}$$

$$\bigcup_{i=1}^{m} \{\pm(a_i - b_i), \pm(c_i - d_i), \pm(e_i - f_i)\} = \mathbb{Z}_{6m+1} - \{0\} \tag{B}$$

$$\bigcup_{i=1}^{m} \{\pm(a_i - c_i), \pm(b_i - e_i), \pm(d_i - f_i)\} = \mathbb{Z}_{6m+1} - \{0\} \tag{C}$$

$$\bigcup_{i=1}^{m} \{\pm(a_i - e_i), \pm(b_i - d_i), \pm(c_i - f_i)\} = \mathbb{Z}_{6m+1} - \{0\} \tag{D}$$

$$\bigcup_{i=1}^{m} \{\pm(a_i - f_i), \pm(b_i - c_i), \pm(d_i - e_i)\} = \mathbb{Z}_{6m+1} - \{0\} \tag{E}$$

$$\bigcup_{i=1}^{m} \{\pm(a_i - d_i), \pm(b_i - f_i), \pm(c_i - e_i)\} = \mathbb{Z}_{6m+1} - \{0\} \tag{F}$$

These games form a directed (2, 6) $GWhD(6m+1)$ if, in addition to satisfying (A) and (B),

$$\bigcup_{i=1}^{m}\{(c_i - a_i), (e_i - c_i), (b_i - e_i), (d_i - b_i), (f_i - d_i), (a_i - f_i)\}$$
$$= \mathbb{Z}_{6m+1} - \{0\} \quad \text{(G)}$$

$$\bigcup_{i=1}^{m}\{(e_i - a_i), (b_i - c_i), (d_i - e_i), (f_i - b_i), (a_i - d_i), (c_i - f_i)\}$$
$$= \mathbb{Z}_{6m+1} - \{0\} \quad \text{(H)}$$

We shall now show that a $\mathbb{Z}$-cyclic directed moore (2, 6) $GWhD(v)$ exists for all v whenever v is a prime $p \equiv 7 \pmod{12}$, with the definite exception of $p = 7$, and the possible exception of $p = 19$ and $p = 31$.

The proof which follows will involve using ideas first discussed by Buratti in [4], and also used in [3]. The theorem of Weil on multiplicative character sums [6, Theorem 5.41] is used. Here is the statement of Weil's theorem, in which the convention is understood that if ψ is a multiplicative character of $GF(q)$, then $\psi(0) = 0$. Adopting this convention we have $\psi(xy) = \psi(x)\psi(y)$ for all $(x, y) \in GF(q) \times GF(q)$.

Theorem 1.1 *Let ψ be a character of order $m > 1$ of the finite field $GF(q)$. Let f be a polynomial of $GF(q)[x]$ which is not of the form kg^m for some $k \in GF(q)$ and some $g \in GF(q)[x]$. Then we have*

$$\left|\sum_{x \in GF(q)} \psi(f(x))\right| \leq (d-1)\sqrt{q}$$

where d is the number of distinct roots of f in its splitting field over $GF(q)$.

Notation. Any nonzero element k of $\mathbb{Z}_p$ can be expressed as θ^m where θ is a primitive root of p. If $b \mid p-1$ and if $m \equiv a \pmod{b}$, we say that $k \in C_a^b$.

2 The Existence Theorem

We now take a close look at a construction and find the conditions which must be satisfied for it to produce a directed moore (2, 6) $GWhD(6m+1)$.

So let $p = 12t + 7$ be prime and let θ be a primitive root of p. We now present a construction.

Construction 1 $(1, -1; x, -x; x^2, -x^2) \times 1, \theta^6, ..., \theta^{12t}$

It can be seen that this is a suitable construction if x is not a cube since if 1, -1, x, $-x$, x^2 and $-x^2$ are expressed in terms of θ, multiplying by θ^{6i} for appropriate values of i gives all of the nonzero elements of $\mathbb{Z}_p$ as required. First we find the conditions under which this forms a $\mathbb{Z}$-cyclic moore $(2,6)$ $GWhD(6m+1)$. The partner differences are ± 2, $\pm 2x$, $\pm 2x^2$. Looking at these in conjunction with (B) we see that, as above, they give every nonzero element of $\mathbb{Z}_p$ when x is not a cube. The differences involving opponents of the first kind are $\pm(x-1)$, $\pm(x^2+1)$, $\pm x(x-1)$. Considering these in conjunction with (C) allows us to conclude that if $(x^2+1)/x^2(x-1)$ is a cube, then the differences give every nonzero element of $\mathbb{Z}_p$. The differences involving opponents of the third kind are $\pm(x^2+1)$, $\pm(x+1)$, $\pm x(x+1)$. Considering these in conjunction with (E) allows us to conclude that $(x^2+1)/x^2(x+1)$ being a cube suffices. Combining these two pieces of information it can now be seen that $(x-1)/(x+1)$ is a cube. The differences involving opponents of the second kind are $\pm(x^2-1)$, $\pm(x-1)$, $\pm x(x+1)$. But we can now express these as $\pm(x^2-1)$, $\pm(x+1)y$ where $y \in C_0^3$, $\pm x(x+1)$. Considering these in conjunction with (D) it can be seen that it is sufficient to require that

$$(x^2-1)/x^2(x+1) \text{ is a cube}$$
$$\text{i.e. } (x-1)/x^2 \text{ is a cube}$$
$$\text{i.e. } (x+1)/x^2 \text{ is a cube.}$$

The differences involving opponents of the fourth kind are $\pm(x^2-1)$, $\pm(x+1)$, $\pm x(x-1)$. We can express these as $\pm(x^2-1)$, $\pm(x-1)y$ where $y \in C_0^3$, $\pm x(x-1)$. Considering these in conjunction with (F) allows us to see that again it is sufficient for $(x-1)/x^2$ to be a cube.

So we can now see that Construction 1 gives us the initial round tables of a $\mathbb{Z}$-cyclic *moore* $(2,6)$ $GWhD(6m+1)$ when

$$x \text{ is not a cube,}$$
$$(x-1)/x^2 \text{ is a cube i.e. } x(x-1) \text{ is a cube,}$$
$$(x+1)/x^2 \text{ is a cube i.e. } x(x+1) \text{ is a cube,}$$
$$(x^2+1)/x \text{ is a cube i.e. } x^2(x^2+1) \text{ is a cube.}$$

Now we want to look at the conditions under which such a $\mathbb{Z}$-cyclic moore $(2,6)$ $GWhD(6m+1)$ will also be directed. In order to find the conditions under which it would also satisfy (G), the differences we're interested in are

$$\begin{gathered} x-1, \\ x^2-x=x(x-1), \\ -1-x^2=-(x^2+1), \\ -x+1=-(x-1), \\ -x^2+x=-(x^2-x)=-x(x-1), \\ x^2+1. \end{gathered}$$

It can be seen that these are the same as the differences involving opponents of the first kind (as seen above). In order to find the conditions under which this construction would also satisfy (H), the differences we're interested in are

$$\begin{gathered} x^2-1=(x-1)(x+1), \\ -(x+1), \\ -(x^2+x)=-x(x+1), \\ -x^2+1=-(x^2-1)=-(x-1)(x+1), \\ x+1, \\ x+x^2=x(x+1). \end{gathered}$$

Here, it can be seen that these are the same as the differences involving opponents of the second kind (as seen above).

This means that if the conditions are satisfied such that Construction 1 gives us the initial round tables of a $\mathbb{Z}$-cyclic *moore* $(2,6)$ $GWhD(6m+1)$, the resulting design is also directed.

Thus the following theorem is established.

Theorem 2.1 *Let $p = 12t+7$ be prime. If there exists an element x of $\mathbb{Z}_p$ such that x is not a cube, $x(x+1)$ is a cube, $x(x-1)$ is a cube and $x^2(x^2+1)$ is a cube, then a directed moore $(2,6)$ $GWhD(p)$ exists.*

It therefore remains to show that a value of x which satisfies the conditions of Theorem 2.1 can be obtained.

Let χ be the character of order 3 exactly which is defined by

$$\chi(y)=\omega^j \text{ if } y \in C_j^3,$$

where $\omega = e^{\frac{2\pi i}{3}}$.

If we now let $\psi(y) = 1+\chi(y)+\chi(y^2)$ and $\delta(y) = 2-\chi(y)-\chi(y^2)$, then it follows that

$$\psi(y)=\begin{cases} 3 & \text{if } y \in C_0^3; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta(y) = \begin{cases} 0 & \text{if } y \in C_0^3; \\ 3 & \text{otherwise.} \end{cases}$$

Consider the sum,

$$S = \sum_{x \in \mathbb{Z}_p} \delta(x) \, . \, \psi(x(x-1)) \, . \, \psi(x(x+1)) \, . \, \psi(x^2(x^2+1)).$$

After multiplying this out and making the appropriate substitutions (using Theorem 1.1), it can be seen that,

$$S \geq 2p - 288\sqrt{p}$$

It is also clearly the case that,

$$S = 81|A| + 2,$$

where elements in A are of the form given in Theorem 2.1.

Thus, $S > 2$ for $p > p_0 = 20,738$. Since $S = 81|A| + 2$, we may claim that A is not empty for $p > p_0$.

It was then checked by computer that appropriate values of x existed for all primes $p < 20,738$ where $p \equiv 7 \pmod{12}$, excluding $p = 7$, 19, 31, 43, 79, 127, 139, 199, 271, 283, 463. Here, we list (p, x_p) where p is the prime and x_p is a suitable value of x for that prime for all relevant primes $p < 5,000$.

(67,29), (103,46), (151,22), (163,50), (211,32), (223,19), (307,14), (331,42), (367,33), (379,19), (439,39), (487,69), (499,45), (523,132), (547,12), (571,33), (607,13), (619,67), (631,16), (643,115), (691,58), (727,51), (739,44), (751,12), (787,11), (811,23), (823,53), (859,51), (883,15), (907,37), (919,44), (967,47), (991,11), (1039,85), (1051,22), (1063,30), (1087,230), (1123,53), (1171,74), (1231,29), (1279,4), (1291,11), (1303,13), (1327,23), (1399,111), (1423,76), (1447,112), (1459,166), (1471,82), (1483,145), (1531,30), (1543,45), (1567,5), (1579,47), (1627,35), (1663,14), (1699,36), (1723,77), (1747,24), (1759,140), (1783,17), (1831,22), (1867,30), (1879,19), (1951,37), (1987,93), (1999,60), (2011,70), (2083,54), (2131,45), (2143,118), (2179,38), (2203,70), (2239,67), (2251,40), (2287,220), (2311,87), (2347,10), (2371,85), (2383,119), (2467,47), (2503,49), (2539,59), (2551,34), (2647,24), (2659,21), (2671,75), (2683,22), (2707,43), (2719,60), (2731,20), (2767,83), (2791,14), (2803,67), (2851,47), (2887,5), (2971,79), (3019,33), (3067,11), (3079,13), (3163,25), (3187,14), (3259,91), (3271,97), (3307,5), (3319,15), (3331,35), (3343,20), (3391,6), (3463,41), (3499,22), (3511,83), (3547,67), (3559, 74), (3571, 95), (3583, 32), (3607, 176), (3631, 70), (3643, 55), (3691, 59), (3727, 56), (3739, 17), (3823, 118), (3847, 17), (3907, 11), (3919, 25), (3931, 12), (3943, 49), (3967, 17), (4003, 61), (4027, 21), (4051, 37), (4099, 74), (4111, 22), (4159, 200), (4219, 16), (4231, 29), (4243, 116), (4327, 57), (4339, 14), (4363, 143), (4423, 97), (4447, 30), (4483, 66), (4507, 194), (4519, 76), (4567, 45), (4591, 21), (4603, 12), (4639, 109), (4651, 58), (4663, 47), (4723, 14), (4759, 123), (4783, 141), (4831, 14), (4903, 18), (4951, 22), (4987, 71), (4999, 12)

A computer programme was then constructed using the Magma computational algebra package, and the following examples were found for 8 of the 11 remaining values of p.

Example 2.1. A $\mathbb{Z}$-cyclic directed moore $GWhD(43)$ is given by the initial round $(1,39;19,6;36,28) \times 1, 3^6, ..., 3^{36}$.

Example 2.2. A $\mathbb{Z}$-cyclic directed moore $GWhD(79)$ is given by the initial round $(1,15;29,23;55,35) \times 1, 29^6, ..., 29^{72}$.

Example 2.3. A $\mathbb{Z}$-cyclic directed moore $GWhD(127)$ is given by the initial round $(1,63;56,115;17,90) \times 1, 56^6, ..., 56^{120}$.

Example 2.4. A $\mathbb{Z}$-cyclic directed moore $GWhD(139)$ is given by the initial round $(1,95;110,67;4,102) \times 1, 2^6, ..., 2^{132}$.

Example 2.5. A $\mathbb{Z}$-cyclic directed moore $GWhD(199)$ is given by the initial round $(1,147;71,180;70,113) \times 1, 22^6, ..., 22^{192}$.

Example 2.6. A $\mathbb{Z}$-cyclic directed moore $GWhD(271)$ is given by the initial round $(1,30;26,20;77,226) \times 1, 26^6, ..., 26^{264}$.

Example 2.7. A $\mathbb{Z}$-cyclic directed moore $GWhD(283)$ is given by the initial round $(1,156;20,11;121,220) \times 1, 46^6, ..., 46^{276}$.

Example 2.8. A $\mathbb{Z}$-cyclic directed moore $GWhD(463)$ is given by the initial round $(1,366;245,371;316,369) \times 1, 245^6, ..., 245^{456}$.

It was also verified by computer that there is not a $\mathbb{Z}$-cyclic moore $GWhD(7)$, and so it follows that there also isn't a design of this type which is directed in addition.

Thus the following theorem is established.

Theorem 2.2 *A $\mathbb{Z}$-cyclic directed moore* $(2,6)$ $GWhD(p)$ *exists for all primes* $p \equiv 7 \pmod{12}$, $p \geq 43$.

It is shown in [2] that the existence of a homogeneous $(p, 6, 1)$ difference matrix follows from the existence of a $\mathbb{Z}$-cyclic directed $(2, 6)$ $GWhD(p)$.

The following product theorem is also given in [2], and can be used to prove the existence of a great many further generalised whist designs which are $\mathbb{Z}$-cyclic, directed and moore.

Theorem 2.3 *If n_1 and n_2 are positive integers of the form $6m+1$ such that there exist $\mathbb{Z}$-cyclic directed (alt. moore) $(2, 6)$ $GWhD(n_i)$, $i = 1, 2$ and if there exists a homogeneous $(n_1, 6, 1)$ difference matrix then there exists a $\mathbb{Z}$-cyclic directed (alt. moore) $(2, 6)$ $GWhD(n_1 n_2)$.*

Clearly it follows that our $\mathbb{Z}$-cyclic directed moore $(2, 6)$ $GWhD(p)$ can be combined in this way to generate others.

Corollary 2.4
There exists a $\mathbb{Z}$-cyclic directed moore $(2, 6)$ $GWhD(p_1^{\alpha_1} p_2^{\alpha_2} \ldots)$ for all $\alpha_i \geq 1$ and $p_i \equiv 7 \pmod{12}$, $p_i \geq 43$.

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