The planar Ramsey numbers $PR(K_4 - e, K_k - e)$ *

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Abstract

The planar Ramsey number $PR(H_1, H_2)$ is the smallest integer n such that any planar graph on n vertices contains a copy of H_1 or its complement contains a copy of H_2. It is known that the Ramsey number $R(K_4 - e, K_k - e)$ for $k \leq 6$. In this paper we prove that $PR(K_4 - e, K_6 - e) = 16$ and show the lower bounds on $PR(K_4 - e, K_k - e)$.

Keywords: *planar graph; Ramsey number; forbidden subgraph*

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph G with vertex-set $V(G)$ and edge-set $E(G)$, we denote the order and the size of G by $p(G) = |V(G)|$ and $q(G) = |E(G)|$, respectively.

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A graph G will be called an (H_1, H_2)-*graph* if it does not contain a subgraph isomorphic to H_1, and its complement $\overline{G}$ has no subgraph isomorphic to H_2. An $(H_1, H_2; n)$-*graph* is an (H_1, H_2)-graph with order n. The Ramsey number $R(H_1, H_2)$ is the smallest integer n such that there is no $(H_1, H_2; n)$-graph, or equivalently, it is the least positive integer n such that every 2-coloring of the edges of K_n contains a subgraph isomorphic to H_1 in the first color or a subgraph isomorphic to H_2 in the second color.

A graph is said to be *embedded* in a surface S when it is drawn on S so that no two edges intersect. A graph is *planar*, if it can be embedded in the plane; a *plane graph* has already been embedded in the plane. We refer to the regions defined by a plane graph as its *faces*. A face is said to be *incident* with the vertices and edges in its boundary. The *length* of a face is the number of edges with which it is incident. If a face has length α, we say it is an α-*face*. For a plane graph G, let f denote the number of faces, and f_α denote the number of α-faces.

A planar graph G will be called an (H_1, H_2)-*P-graph* if it does not contain a subgraph isomorphic to H_1, and its complement $\overline{G}$ has no subgraph isomorphic to H_2. An $(H_1, H_2; n)$-*P-graph* is an (H_1, H_2)-*P*-graph with order n. The planar Ramsey number $PR(H_1, H_2)$ is the smallest integer n such that there is no $(H_1, H_2; n)$-*P*-graph. So $PR(H_1, H_2) \leq R(H_1, H_2)$.

Let $d(v)$ denote the degree of a vertex $v \in V(G)$, $\delta(G)$ the minimum degree of G. The neighborhood and the closed neighborhood of a vertex $v \in V(G)$ are denoted by $N(v) = \{u \in V(G) | uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. Let $G \cup H$ denote a disjoint sum of G and H, and nG is a disjoint sum of n copies of G. Let $G\langle W \rangle$ denote the subgraph of G induced by $W \subseteq V(G)$.

The Ramsey number $R(K_4 - e, K_6 - e) = 17$ was given by McNamara and Radziszowski[5]. The definition of planar Ramsey numbers was firstly introduced by Walker[9]. Steinberg and Tovey[6] studied the case when both H_1 and H_2 are complete. They proved that

$$\begin{aligned} &PR(K_2, K_k) = k, \\ &PR(K_k, K_2) = k, \quad k \leq 4, \\ &PR(K_3, K_k) = 3k - 3, \\ &PR(K_k, K_l) = 4l - 3, \quad k \geq 4 \text{ and } (k, l) \neq (4, 2). \end{aligned}$$

For a connected graph H_1 with order at least 5, Gorgol[4] proved that

$$PR(H_1, K_k) = 4k - 3.$$

Bielak and Gorgol[1] also proved that

$$PR(C_4, K_5) = 13.$$

Bielak[2] determined that

$$PR(C_4, K_6) = 17.$$

It is shown that

$$PR(C_4, K_7) = 20^{[8]}.$$

Dudek and Ruciński[3] showed that

$$PR(K_4 - e, K_5 - e) = 13.$$

It was shown that

$$PR(K_4 - e, K_5) = 14^{[7]}.$$

In this paper, we study the case that $(H_1, H_2) = (K_4 - e, K_k - e)$. we prove that $PR(K_4 - e, K_6 - e) = 16$ and show the lower bounds on $PR(K_4 - e, K_k - e)$.

For a 3-connected planar graph, Whitney[10] showed that

Whitney's Theorem. A 3-connected planar graph has a unique planar embedding.

Hereafter, we discuss a 3-connected planar graph in its unique planar embedding unless specified otherwise.

2 Preliminary results

Lemma 2.1. If G is a $(K_4 - e, K_k - e; n)$-P-graph, then $\delta(G) \geq n - PR(K_4 - e, K_{k-1} - e)$.
Proof. We prove it by way of contradiction. Assume that $\delta(G) < n - PR(K_4 - e, K_{k-1} - e)$. Let v be a vertex of degree $\delta(G)$ and $H = G - N[v]$, then $p(H) = n - \delta(G) - 1 > n - n + PR(K_4 - e, K_{k-1} - e) - 1 \geq PR(K_4 - e, K_{k-1} - e)$. Since $K_4 - e \not\subseteq H$, we have $K_{k-1} - e \subseteq \overline{H}$. The appropriate $k-1$ vertices of H together with v would yield a $K_k - e$ in $\overline{G}$, a contradiction. So, $\delta(G) \geq n - PR(K_4 - e, K_{k-1} - e)$. □

Lemma 2.2.[7] If G is a planar graph such that $K_4 - e \not\subseteq G$, then
(1) $q(G) \leq \lfloor 12(p(G) - 2)/5 \rfloor$, and
(2) $2q(G) - 4p(G) + 8 \leq f_3 \leq q(G)/3$.

Observation 2.3. If G is a $(K_4 - e, K_3 - e; 3)$-P-graph, then $K_{1,2} \subseteq G$.

Observation 2.4. If G is a $(K_4 - e, K_3 - e; 4)$-P-graph, then $G \cong C_4$.

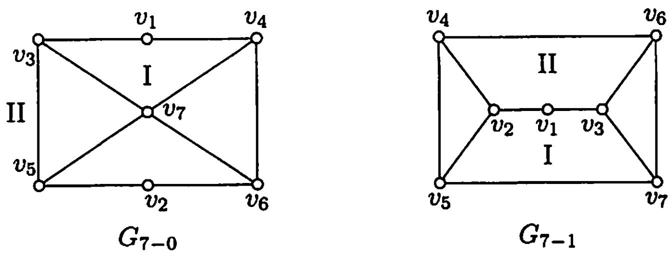


Fig. 2.1 The graphs G_{7-0} and G_{7-1}

Let G_{7-0} and G_{7-1} be the graphs as shown in Fig. 2.1, we have

Lemma 2.5. If G is a $(K_4 - e, K_4 - e; 7)$-P-graph, then $G_{7-0} \subseteq G$ or $G_{7-1} \subseteq G$.

Proof. Since $PR(K_4 - e, K_3 - e) = 5$, by Lemma 2.1, we have $\delta(G) \geq 2$. By Lemma 2.2, we have $q(G) \leq \lfloor 12(7-2)/5 \rfloor = 12$, implying $\delta(G) \leq 3$. So, $2 \leq \delta(G) \leq 3$. Let v be a vertex with degree $\delta(G)$ and $H = G - N[v]$. There are two cases depending on $\delta(G)$.

Case 1. Suppose that $\delta(G) = 2$, then $p(H) = 4$. By Observation 2.4, we have $H \cong C_4$, denoted by $a_1a_2a_3a_4$. Let $N(v) = \{u_1, u_2\}$. If $u_1(u_2)$ is nonadjacent to both a_1 and $a_3(a_2$ and $a_4)$, then v,a_1,a_3(v,a_2,a_4) and $u_1(u_2)$ would yield a $K_4 - e$ in $\overline{G}$, a contradiction. Hence u_1 is adjacent to at least one vertex of $\{a_1, a_3\}$ and at least one vertex of $\{a_2, a_4\}$, say $u_1a_1, u_1a_2 \in E(G)$. And u_2 is adjacent to at least one vertex of $\{a_1, a_3\}$ and at least one vertex of $\{a_2, a_4\}$. Since $K_4 - e \not\subseteq G$, u_2 cannot adjacent to both a_1 and a_2. If $u_1u_2 \notin E(G)$, there are three subcases(see $G_{7.1} - G_{7.3}$ in Fig. 2.2). Hence $G_{7-0} \subseteq G$ or $G_{7-1} \subseteq G$. If $u_1u_2 \in E(G)$, there is only one case(see $G_{7.4}$ in Fig. 2.3). Hence $G_{7-1} \subseteq G$.

Case 2. Suppose that $\delta(G) = 3$, then $p(H) = 3$. By Observation 2.3, we have $K_{1,2} \subseteq H$, that is, H is isomorphic to $K_{1,2}$ or K_3. Let $N(v) = \{u_1, u_2, u_3\}$. Assume $H \cong K_3$. Then since $K_4 - e \not\subseteq G$, we have $|E(G\langle N(v)\rangle)| \leq 1$ and each vertex of $\{u_1, u_2, u_3\}$ is adjacent to at most one vertex of $V(H)$. Thus there is at least one vertex of $\{u_1, u_2, u_3\}$ whose degree is at most 2, a contradiction to $\delta(G) = 3$. Hence, we have $H \cong K_{1,2}$.

Since $\delta(G) = 3$ and $q(G) \leq 12$, we have $11 \leq q(G) \leq 12$. Assume that $q(G) = 12$. Then by Lemma 2.2, we have $f_3 = 4$. Since $K_4 - e \not\subseteq G$, each edge of G belongs to one triangle. Since $|E(G\langle N(v)\rangle)| \leq 1$, we may

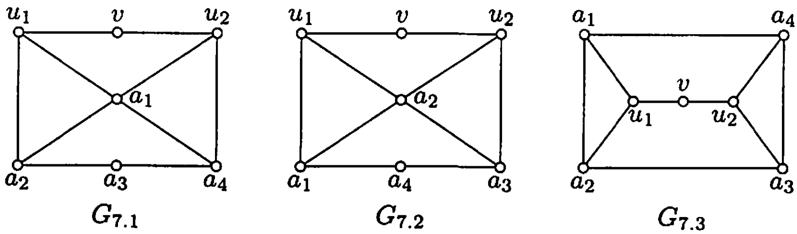


Fig. 2.2 The graphs $G_{7.1}$-$G_{7.3}$.

assume that $u_1u_2, u_1u_3 \notin E(G)$. Then the edge vu_1 does not belong to any triangle, a contradiction. So, we have $q(G) = 11$. By Lemma 2.2, we have $2 \leq f_3 \leq 3$. Let $V(H) = \{a_1, a_2, a_3\}$ and $E(H) = \{a_1a_2, a_2a_3\}$. There are two subcases depending on $|E(G\langle N(v)\rangle)|$.

Case 2.1. Suppose that $|E(G\langle N(v)\rangle)| = 0$. Since $K_4 - e \nsubseteq G$, we have $f_3 = 2$. Hence one vertex of $\{u_1, u_2, u_3\}$ together with a_1 and $a_2(a_2$ and $a_3)$ yield one triangle of G, say $u_1a_1, u_1a_2 \in E(G)$. Then one vertex of $\{u_2, u_3\}$ together with a_2 and a_3 yield the other triangle of G, say $u_2a_2, u_2a_3 \in E(G)$. Since $d(u_3) \geq 3$ and $K_4 - e \nsubseteq G$, u_3 has to be adjacent to both a_1 and a_3, i.e., $G_{7-0} \subseteq G$(see $G_{7.5}$ in Fig. 2.3).

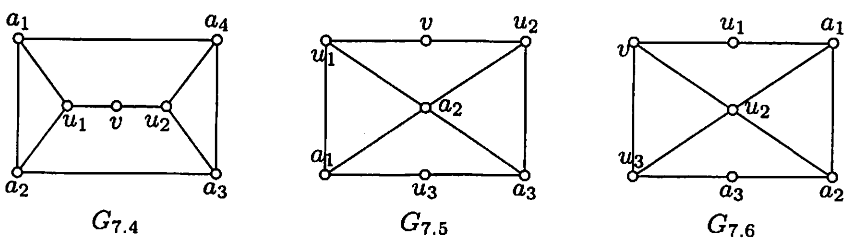


Fig. 2.3 The graphs $G_{7.4}$-$G_{7.6}$

Case 2.2. Suppose that $|E(G\langle N(v)\rangle)| = 1$, say $u_2u_3 \in E(G)$. Since $\delta(G) = 3$ and $K_4 - e \nsubseteq G$, both a_1 and a_3 have to be adjacent to u_1. Then since $K_4 - e \nsubseteq G$, we have $u_1a_2 \notin E(G)$. Since $f_3 \geq 2$, there is at least one vertex of $\{u_2, u_3\}$ together with a_1 and $a_2(a_2$ and $a_3)$ yield a triangle in G, say $u_2a_1, u_2a_2 \in E(G)$. Since $d(u_3) \geq 3$ and $K_4 - e \nsubseteq G$, u_3 has to be adjacent to a_3, i.e., $G_{7-0} \subseteq G$(see $G_{7.6}$ in Fig. 2.3). □

Corollary 2.6. If G is a $(K_4-e, K_4-e; 7)$-P-graph, then it is isomorphic to one graph of $\{G_{7-0}, G_{7-0}+v_1v_2, G_{7-1}, G_{7-1}+v_2v_3\}$ as shown in Fig. 2.1.

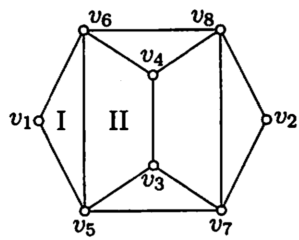


Fig. 2.4 The graph G_{8-0}

Let G_{8-0} be the graph as shown in Fig. 2.4, we can notice that it is a self-complement graph. And we have the following lemma:

Lemma 2.7. If G is a $(K_4-e, K_4-e; 8)$-P-graph, then $G \cong G_{8-0}$.
Proof. Since $PR(K_4-e, K_3-e) = 5$, by Lemma 2.1, we have $\delta(G) \geq 3$. By Lemma 2.2, we have $q(G) \leq \lfloor 12(8-2)/5 \rfloor = 14$, implying $\delta(G) \leq 3$. So, we have $\delta(G) = 3$. Let v be a vertex with degree $\delta(G)$ and $H = G - N[v]$. Then $p(H) = 4$. By Observation 2.4, we have $H \cong C_4$, denoted by $a_1a_2a_3a_4$. Let $N(v) = \{u_1, u_2, u_3\}$. Assume that there is at least one vertex of $\{u_1, u_2, u_3\}$ which is adjacent to two inconsecutive vertices of $\{a_1, a_2, a_3, a_4\}$, say $u_1a_1, u_1a_3 \in E(G)$. Then since $\delta(G) = 3$ and $K_4 - e \nsubseteq G$, each vertex of $\{a_2, a_4\}$ has to be adjacent to at least one vertices of $\{u_2, u_3\}$. In any case, G would contain a subgraph homeomorphic to $K_{3,3}$, a contradiction. Hence each vertex of $\{u_1, u_2, u_3\}$ cannot be adjacent to two inconsecutive vertices of $\{a_1, a_2, a_3, a_4\}$. Since $K_4 - e \nsubseteq G$, it follows $|E(G\langle N(v)\rangle)| \leq 1$. Hence there are two subcases.

Case 1. Suppose that $|E(G\langle N(v)\rangle)| = 0$. Since $\delta(G) = 3$ and $K_4 - e \nsubseteq G$, each vertex of $\{u_1, u_2, u_3\}$ has to be adjacent to two consecutive vertices of $\{a_1, a_2, a_3, a_4\}$, say $u_1a_1, u_1a_2, u_2a_2, u_2a_3, u_3a_3, u_3a_4 \in E(G)$. Now, u_1, u_2, u_3 and a_4 would yield a $K_4 - e$ in $\overline{G}$, a contradiction.

Case 2. Suppose that $|E(G\langle N(v)\rangle)| = 1$, say $u_2u_3 \in E(G)$. Since $d(u_1) \geq 3$ and $K_4 - e \nsubseteq G$, u_1 has to be adjacent to two consecutive vertices of $\{a_1, a_2, a_3, a_4\}$, say $u_1a_1, u_1a_2 \in E(G)$. Since $K_4 - e \nsubseteq G$, u_1 is adjacent neither to a_3 nor to a_4. Therefore since $d(a_3) \geq 3$, a_3 has to be adjacent to

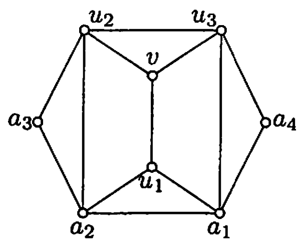


Fig. 2.5 The graph $G_{8.1}$

one vertex of $\{u_2, u_3\}$, say $a_3u_2 \in E(G)$. Suppose that $a_4u_3 \notin E(G)$, then u_1, u_3, a_3 and a_4 would yield a $K_4 - e$ in $\overline{G}$, a contradiction. So, we have $a_4u_3 \in E(G)$. If $u_2a_2(u_3a_1) \notin E(G)$, u_1, a_4, u_2 and $a_2(u_1, a_3, u_3$ and $a_1)$ would yield a $K_4 - e$ in $\overline{G}$, a contradiction. So, we have $u_2a_2, u_3a_1 \in E(G)$, i.e., $G \cong G_{8-0}$(see $G_{8.1}$ in Fig. 2.5). □

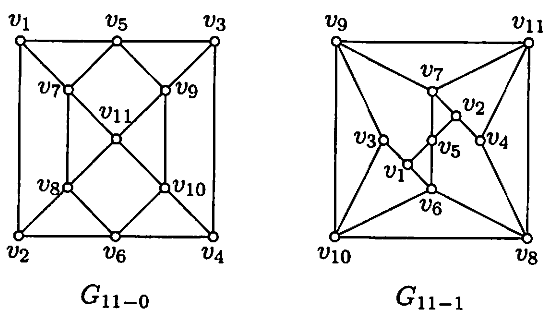


Fig. 2.6 The graphs G_{11-0} and G_{11-1}

Let G_{11-0} and G_{11-1} be the graphs as shown in Fig. 2.6, we have

Lemma 2.8. If G is a $(K_4 - e, K_5 - e; 11)$-P-graph, then $G_{11-0} \subseteq G$ or $G_{11-1} \subseteq G$.

Proof. Since $PR(K_4 - e, K_4 - e) = 9$, by Lemma 2.1, we have $\delta(G) \geq 2$. By Lemma 2.2, we have $q(G) \leq \lfloor 12(11-2)/5 \rfloor = 21$, implying $\delta(G) \leq 3$. Hence $2 \leq \delta(G) \leq 3$. Let v be a vertex with degree $\delta(G)$ and $H = G - N[v]$. There are two cases depending on $\delta(G)$.

Case 1. Suppose that $\delta(G) = 2$, then $p(H) = 8$. By Lemma 2.7, we have $H \cong G_{8-0}$. Let $N(v) = \{u_1, u_2\}$ and $V(H) = \{v_i | \ 1 \leq i \leq 8\}$ as shown in Fig. 2.4. By symmetry it is sufficient to consider that $N[v]$ lie in region I or II. If $N[v]$ lie in region I, then since $K_4 - e \not\subseteq G$, there is at least one edge of $\{u_1v_5, u_1v_6, u_2v_5, u_2v_6\}$ which is not belong to $E(G)$, say $u_1v_5 \notin E(G)$, u_1, v, v_5, v_4 and v_2 would yield a $K_5 - e$ in $\overline{G}$, a contradiction. If $N[v]$ lie in region II, then since $K_4 - e \not\subseteq G$, there is at least one edge of $\{u_1v_3, u_1v_4, u_2v_3, u_2v_4\}$ which is not belong to $E(G)$, say $u_1v_3 \notin E(G)$, u_1, v, v_3, v_8 and v_1 would yield a $K_5 - e$ in $\overline{G}$, a contradiction too.

Case 2. Suppose that $\delta(G) = 3$, then $p(H) = 7$. Let $N(v) = \{u_1, u_2, u_3\}$. Since $K_4 - e \not\subseteq G$, it follows $|E(G\langle N(v)\rangle)| \leq 1$. Without loss of generality, we may assume that $u_1u_2, u_1u_3 \notin E(G)$. Since $d(u_1) \geq 3$ and $K_4 - e \not\subseteq G$, $N[v]$ cannot lie in any triangle of H. By Corollary 2.6, we have H is isomorphic to one graph of $\{G_{7-0}, G_{7-0} + v_1v_2, G_{7-1}, G_{7-1} + v_2v_3\}$. Hence there are two subcases.

Case 2.1. Suppose that H is isomorphic to one graph of $\{G_{7-0}, G_{7-0} + v_1v_2\}$. Let $V(H) = \{v_i | \ 1 \leq i \leq 7\}$ as shown in Fig. 2.1.

Case 2.1.1. Suppose that $H \cong G_{7-0} + v_1v_2$. By symmetry it is sufficient to consider that $N[v]$ lie in region I or II. If $N[v]$ lie in region I, then v_5, v_6, u_1, u_2 and u_3 would yield a $K_5 - e$(or K_5) in $\overline{G}$, a contradiction. Hence $N[v]$ have to lie in region II.

If $|E(G\langle N(v)\rangle)| = 0$, then u_1, u_2, u_3, v_4 and v_6 would yield a $K_5 - e$ in $\overline{G}$, a contradiction. So, we have $|E(G\langle N(v)\rangle)| = 1$, that is, $u_2u_3 \in E(G)$. Since $K_4 - e \not\subseteq G$, u_1 is nonadjacent to at least one vertex of $\{v_3, v_5\}$, say v_3. And v_3 is nonadjacent to at least one vertex of $\{u_2, u_3\}$, say u_2. Then u_1, u_2, v_3, v_4 and v_6 would yield a $K_5 - e$ in $\overline{G}$, a contradiction too.

Case 2.1.2. Suppose that $H \cong G_{7-0}$. Since $d(v_1) \geq 3$ and $d(v_2) \geq 3$, $N[v]$ have to lie in the 6-face of H. There are two subcases depending on $|E(G\langle N(v)\rangle)|$.

Case 2.1.2.1. Suppose that $|E(G\langle N(v)\rangle)| = 0$. Assume that there is at least one vertex of $\{v_3, v_5\}$, say v_3 which is nonadjacent to any vertex of $\{u_1, u_2, u_3\}$. Then u_1, u_2, u_3, v_3 and v_7 would yield a $K_5 - e$ in $\overline{G}$, a contradiction. Hence each vertex of $\{v_3, v_5\}$ is adjacent to at least one vertex of $\{u_1, u_2, u_3\}$. Since $K_4 - e \not\subseteq G$, there is a perfect matching between vertices of $\{v_3, v_5\}$ and two vertices of $\{u_1, u_2, u_3\}$, say $v_3u_1, v_5u_2 \in E(G)$. Similarly, there is a perfect matching between vertices of $\{v_4, v_6\}$ and two vertices of $\{u_1, u_2, u_3\}$. By symmetry, we may assume that $v_4u_1, v_6u_2 \in E(G)$ or $v_4u_1, v_6u_3 \in E(G)$(see $G_{11.1}$ and $G_{11.2}$ in Fig. 2.7). Then since $K_4 - e \not\subseteq G$ and the planarity of G, we have $d(v_1) = 2$, a contradiction.

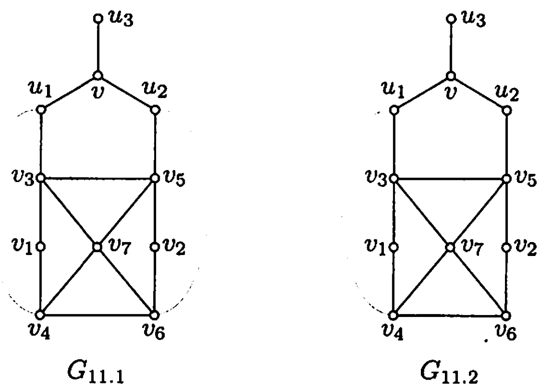


Fig. 2.7 The graphs $G_{11.1}$ and $G_{11.2}$

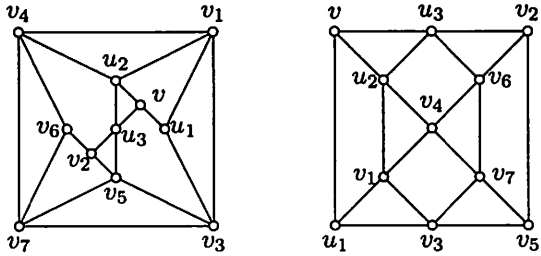
Case 2.1.2.2. Suppose that $|E(G\langle N(v)\rangle)| = 1$, that is, $u_2u_3 \in E(G)$. If $u_2(u_3)$ is nonadjacent to any vertex of $\{v_1, v_2\}$, then v_1, v_2, v_7, v and $u_2(u_3)$ would yield a $K_5 - e$ in $\overline{G}$, a contradiction. Hence each vertex of $\{u_2, u_3\}$ is adjacent to one vertex of $\{v_1, v_2\}$. Since $K_4 - e \nsubseteq G$, there is a perfect matching between vertices of $\{v_1, v_2\}$ and $\{u_2, u_3\}$, say $v_1u_2, v_2u_3 \in E(G)$.

Since $d(u_1) \geq 3$, u_1 is adjacent to at least two vertices of $\{v_1, v_2, v_3, v_5\}$(or $\{v_1, v_2, v_4, v_6\}$). By symmetry it is sufficient to consider that u_1 is adjacent to at least two vertex of $\{v_1, v_2, v_3, v_5\}$. Since $K_4 - e \nsubseteq G$, u_1 is nonadjacent to at least one vertex of $\{v_3, v_5\}$, say v_5.

If $u_1v_1 \notin E(G)$, then v_1, v_5, v_6, u_1 and v would yield a $K_5 - e$ in $\overline{G}$, a contradiction. So, we have $u_1v_1 \in E(G)$. If $u_2v_4 \notin E(G)$, then v_2, v_3, v_4, u_2 and v would yield a $K_5 - e$ in $\overline{G}$, a contradiction. Hence $u_2v_4 \in E(G)$. Assume that u_3 is nonadjacent to any vertex of $\{v_5, v_6\}$, then v_1, v_5, v_6, u_3 and v would yield a $K_5 - e$ in $\overline{G}$, a contradiction. Hence u_3 is adjacent to one vertex of $\{v_5, v_6\}$.

Suppose that $u_3v_5 \in E(G)$. If $u_1v_3 \notin E(G)$, then v_2, v_3, v_4, u_1 and v would yield a $K_5 - e$ in $\overline{G}$, a contradiction. Hence we have $u_1v_3 \in E(G)$, i.e., $G_{11-1} \subseteq G$(see $G_{11.3}$ in Fig. 2.8). Suppose that $u_3v_6 \in E(G)$. If $u_1v_3 \notin E(G)$, then since $d(u_1) \geq 3$, u_1 has to be adjacent to v_2. Now, u_1, u_3, v_4, v_3 and v_5 would yield a $K_5 - e$ in $\overline{G}$, a contradiction. So, we have $u_1v_3 \in E(G)$, i.e., $G_{11-0} \subseteq G$(see $G_{11.4}$ in Fig. 2.8).

Case 2.2. Suppose that H is isomorphic to one graph of $\{G_{7-1}, G_{7-1} + v_2v_3\}$. Let $V(H) = \{v_i | \ 1 \leq i \leq 7\}$ as shown in Fig. 2.1. If $H \cong G_{7-1}$,

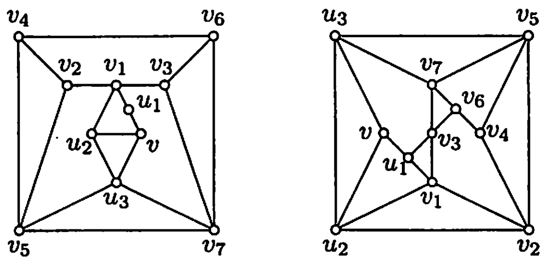


$G_{11.3}$ $G_{11.4}$

Fig. 2.8 The graphs $G_{11.3}$ and $G_{11.4}$

then since $d(v_1) \geq 3$, $N[v]$ have to lie in region I or II. By symmetry it is sufficient to consider that $N[v]$ lie in region I. If $H \cong G_{7-1} + v_2v_3$, then since $d(v_1) \geq 3$, $N[v]$ have to lie in triangle $v_1v_2v_3$ or region I. If $N[v]$ lie in triangle $v_1v_2v_3$, then v_5, v_6, u_1, u_2 and u_3 would yield a $K_5 - e$(or K_5) in $\overline{G}$, a contradiction. Hence $N[v]$ have to lie in region I.

If $|E(G\langle N(v)\rangle)| = 0$, then u_1, u_2, u_3, v_4 and v_6 would yield a $K_5 - e$ in $\overline{G}$, a contradiction. Hence we have $|E(G\langle N(v)\rangle)| = 1$, that is, $u_2u_3 \in E(G)$. Suppose that $v_1u_1 \notin E(G)$. Since $K_4 - e \not\subseteq G$, v_1 is nonadjacent to at least one vertex of $\{u_2, u_3\}$, say u_2. Then u_1, u_2, v_1, v_4 and v_6 would yield a $K_5 - e$ in $\overline{G}$, a contradiction. So, we have $v_1u_1 \in E(G)$.



$G_{11.5}$ $G_{11.6}$

Fig. 2.9 The graphs $G_{11.5}$ and $G_{11.6}$

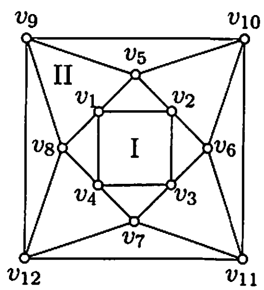
If $u_2(u_3)$ is nonadjacent to any vertex of $\{v_1, v_5\}$, then v_1, v_5, v_6, v and

$u_2(u_3)$ would yield a $K_5 - e$ in $\overline{G}$, a contradiction. Hence each vertex of $\{u_2, u_3\}$ is adjacent to at least one vertex of $\{v_1, v_5\}$. Since $K_4 - e \nsubseteq G$, there is a perfect matching between vertices of $\{v_1, v_5\}$ and $\{u_2, u_3\}$, say $v_1u_2, v_5u_3 \in E(G)$.

If $v_7u_3 \notin E(G)$, then v_1, v_4, v_7, u_3 and v would yield a $K_5 - e$ in $\overline{G}$, a contradiction. Hence we have $v_7u_3 \in E(G)$. If u_1 is adjacent to one vertex of $\{v_5, v_7\}$, say v_7, then v_3, v_4, u_1, u_2 and u_3 would yield a $K_5 - e$ in $\overline{G}$, a contradiction(see $G_{11.5}$ in Fig. 2.9). Hence u_1 is nonadjacent to any vertex of $\{v_5, v_7\}$.

If $H \cong G_{7-1} + v_2v_3$, then we have $d(u_1) = 2$, a contradiction. Hence $H \cong G_{7-1}$. Since $d(u_1) \geq 3$ and $K_4 - e \nsubseteq G$, u_1 is adjacent to just one vertex of $\{v_2, v_3\}$, say v_3. If $v_2u_2 \notin E(G)$, then v_2, v_6, u_1, u_2 and u_3 would yield a $K_5 - e$ in $\overline{G}$, a contradiction. Hence we have $v_2u_2 \in E(G)$, i.e., $G_{11-1} \subseteq G$(see $G_{11.6}$ in Fig. 2.9). □

Corollary 2.9. If G is a $(K_4 - e, K_5 - e; 11)$-P-graph, then it is isomorphic to one graph of $\{G_{11-0}, G_{11-0} + v_2v_3, G_{11-0} + v_5v_6, G_{11-1}\}$ as shown in Fig. 2.6.



G_{12-0}

Fig. 2.10 The graph G_{12-0}

Let G_{12-0} be the graph as shown in Fig. 2.10, we have

Lemma 2.10. If G is a $(K_4 - e, K_5 - e; 12)$-P-graph, then $G \cong G_{12-0}$.
Proof. Since $PR(K_4 - e, K_4 - e) = 9$, by Lemma 2.1, we have $\delta(G) \geq 3$. By Lemma 2.2, we have $q(G) \leq \lfloor 12(12-2)/5 \rfloor = 24$, implying $\delta(G) \leq 4$. Hence $3 \leq \delta(G) \leq 4$. Let v be a vertex with degree $\delta(G)$ and $H = G - N[v]$. There are two cases depending on $\delta(G)$.

Case 1. Suppose that $\delta(G) = 3$, then $p(H) = 8$. By Lemma 2.7, we have $H \cong G_{8-0}$. Let $N(v) = \{u_1, u_2, u_3\}$ and $V(H) = \{v_i | \ 1 \leq i \leq 8\}$ as shown in Fig. 2.4. Since $K_4 - e \not\subseteq G$, it follows $|E(G\langle N(v)\rangle)| \leq 1$. Without loss of generality, we may assume that $u_1u_2, u_1u_3 \notin E(G)$. Therefore since $d(u_1) \geq 3$ and $K_4 - e \not\subseteq G$, $N[v]$ cannot lie in any triangle of H. By symmetry it is sufficient to consider that $N[v]$ lie in region II. Then u_1, u_2, v_1, v_7 and v_8 yield a $K_5 - e$ in $\overline{G}$, a contradiction.

Case 2. Suppose that $\delta(G) = 4$, then $p(H) = 7$. Let $N(v) = \{u_1, u_2, u_3, u_4\}$. Since $K_4 - e \not\subseteq G$, it follows $|E(G\langle N(v)\rangle)| \leq 2$. Therefore since $d(u_i) \geq 4$ and $K_4 - e \not\subseteq G$, $N[v]$ cannot lie in any triangle of H. By Corollary 2.6, we have H is isomorphic to one graph of $\{G_{7-0}, G_{7-0} + v_1v_2, G_{7-1}, G_{7-1} + v_2v_3\}$.

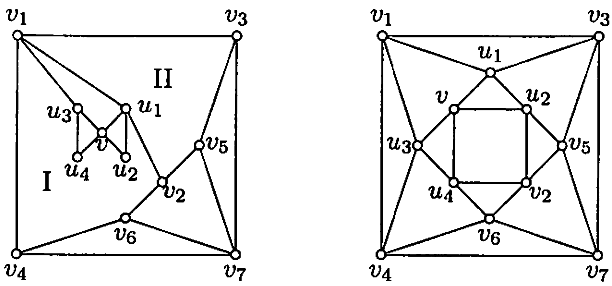
Case 2.1. Suppose that H is isomorphic to one graph of $\{G_{7-0}, G_{7-0} + v_1v_2\}$. Let $V(H) = \{v_i | \ 1 \leq i \leq 7\}$ as shown in Fig. 2.1. If $H \cong G_{7-0} + v_1v_2$, by symmetry, it is sufficient to consider that $N[v]$ lie in region I or II. No matter $N[v]$ lie in which region, there is at least one vertex of $\{v_3, v_4, v_5, v_6\}$ whose degree is 3, a contradiction. Hence we have $H \cong G_{7-1}$. Since $\delta(G) = 4$, $N[v]$ have to lie in the 6-face of H.

Since $\delta(G) = 4$ and $q(G) \leq 24$, we have $q(G) = 24$ and G is a 4-regular graph. By Lemma 2.2, we have $f_3 = 8$. Hence every edge of G belong to one triangle, it is forced that $G\langle N(v)\rangle \cong 2K_2$. Without loss of generality, we may assume that $u_1u_2, u_3u_4 \in E(G)$. Since $d(v_1) = 4$ and $K_4 - e \not\subseteq G$, v_1 has to be adjacent to one vertex of $\{u_1, u_2\}$ and one vertex of $\{u_3, u_4\}$, say $v_1u_1, v_1u_3 \in E(G)$.

If v_2 is adjacent to at least one vertex of $\{u_1, u_3\}$, say u_1. Then there is at least one vertex of $\{v_3, v_4, v_5, v_6\}$ whose degree is 3, a contradiction.(see $G_{12.1}$ in Fig. 2.11). Hence v_2 is adjacent neither to u_1 nor to u_3. Therefore since $d(v_2) = 4$, v_2 has to be adjacent to both u_2 and u_4.

Since $d(v_3) = 4$, v_3 has to be adjacent to just one vertex of $\{u_1, u_2, u_3, u_4\}$. Assume that v_3 is adjacent to one vertex of $\{u_2, u_4\}$, say u_2, then we have $d(u_1) = 3$, a contradiction. Hence v_3 has to be adjacent to one vertex of $\{u_1, u_3\}$, say u_1. Since $d(u_2) = 4$ and $K_4 - e \not\subseteq G$, u_2 has to be adjacent to v_5. Similarly, we have $v_4u_3, v_6u_4 \in E(G)$, i.e., $G \cong G_{12-0}$(see $G_{12.2}$ in Fig. 2.11).

Case 2.2. Suppose that H is isomorphic to one graph of $\{G_{7-1}, G_{7-1} + v_2v_3\}$. Let $V(H) = \{v_i | \ 1 \leq i \leq 7\}$ as shown in Fig. 2.1. No matter $N[v]$ lie in which region, there is at least one vertex of $\{v_1, v_4, v_5, v_6, v_7\}$ whose degree is 3, a contradiction. □



$G_{12.1}$ $G_{12.2}$

Fig. 2.11 The graphs $G_{12.1}$ and $G_{12.2}$

3 The main results

Lemma 3.1. There is no $(K_4 - e, K_6 - e; 16)$-P-graph.
Proof. By contradiction, suppose that G is a $(K_4 - e, K_6 - e; 16)$-P-graph. Since $PR(K_4 - e, K_5 - e) = 13$, by Lemma 2.1, we have $\delta(G) \geq 3$. By Lemma 2.2, we have $q(G) \leq \lfloor 12(16-2)/5 \rfloor = 33$. Hence $\delta(G) \leq 4$. Let v be a vertex with degree $\delta(G)$ and $H = G - N[v]$. There are two cases depending on $\delta(G)$.

Case 1. Suppose that $\delta(G) = 3$, then $p(H) = 12$. By Lemma 2.10, we have $H \cong G_{12-0}$. Let $N(v) = \{u_1, u_2, u_3\}$ and $V(H) = \{v_i | \ 1 \leq i \leq 12\}$ as shown in Fig. 2.10. Since $K_4 - e \not\subseteq G$, it follows $|E(G\langle N(v)\rangle)| \leq 1$. Without loss of generality, we may assume that $u_1u_2, u_1u_3 \notin E(G)$. Since $d(u_1) \geq 3$ and $K_4 - e \not\subseteq G$, $N[v]$ cannot lie in any triangle of H. By symmetry it is sufficient to consider that $N[v]$ lie in region I. Then u_1, u_2, v_5, v_6, v_7 and v_8 would yield a K_6 in $\overline{G}$, a contradiction.

Case 2. Suppose that $\delta(G) = 4$, then $p(H) = 11$. Let $N(v) = \{u_1, u_2, u_3, u_4\}$. By Corollary 2.9, we have H is isomorphic to one graph of $\{G_{11-0}, G_{11-0} + v_2v_3, G_{11-0} + v_5v_6, G_{11-1}\}$.

Case 2.1. Suppose that H is isomorphic to one graph of $\{G_{11-0}, G_{11-0} + v_2v_3, G_{11-0} + v_5v_6\}$. Let $V(H) = \{v_i | \ 1 \leq i \leq 11\}$ as shown in Fig. 2.6. If H is isomorphic to one graph of $\{G_{11-0} + v_2v_3, G_{11-0} + v_5v_6\}$, then no matter $N[v]$ lie in which region, there is at least one vertex of $\{v_1, v_2, v_3, v_4\}$ whose degree is 3, a contradiction. Hence we have $H \cong G_{11-0}$.

Since $d(v_i) \geq 4$ for $1 \leq i \leq 4$, $N[v]$ have to lie in the 6-face of H. Assume

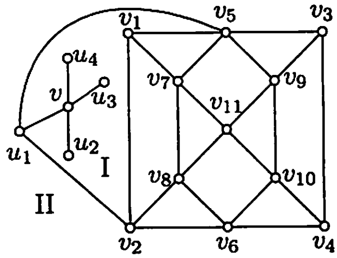


Fig. 3.1

that there is one vertex of $\{u_1, u_2, u_3, u_4\}$ which is adjacent to at least one vertex of $\{v_5, v_6\}$, say $u_1v_5 \in E(G)$. Since $K_4 - e \nsubseteq G$, u_1 is adjacent neither to v_1 nor to v_3. Therefore since $d(u_1) \geq 4$, u_1 has to be adjacent to one vertex of $\{v_2, v_4, v_6\}$, say v_2. Then no matter $N[v] - \{u_1\}$ lie in which region, there is at least one vertex of $\{v_1, v_3\}$ whose degree is 3, a contradiction(see Fig. 3.1). Hence each vertex of $\{u_1, u_2, u_3, u_4\}$ is nonadjacent to any vertex of $\{v_5, v_6\}$. Then since $K_4 - e \nsubseteq G$, $v_5, v_6, v_{11}, u_1, u_2$ and u_3 would yield a $K_6 - e$(or K_6) in $\overline{G}$, a contradiction.

Case 2.2. Suppose that $H \cong G_{11-1}$. Let $V(H) = \{v_i \mid 1 \leq i \leq 11\}$ as shown in Fig. 2.6. No matter $N[v]$ lie in which region, there is at least one vertex of $\{v_1, v_2, v_3, v_4\}$ whose degree is 3, a contradiction. □

Lemma 3.2. If $k \geq 5$, then $PR(K_4 - e, K_k - e) \geq 3k - 2$.

Proof. Let G_{k-1} be a 4-regular planar graph, where

$$\begin{aligned} V(G_{k-1}) = & \{a_i, b_i, c_i : 0 \leq i \leq k-2\}, \\ E(G_{k-1}) = & \{a_ia_{i-1}, a_ia_{i+1}, c_ic_{i-1}, c_ic_{i+1}, b_ia_i, b_ia_{i+1}, b_ic_{i-1}, b_ic_i \\ & : 0 \leq i \leq k-2\} \\ & (\text{ subscrips module } k-1). \end{aligned}$$

For instance, G_4 is isomorphic to G_{12-0} as shown in Fig. 2.10. Let $Y_i = \{a_i, b_i, c_i\}$ for $0 \leq i \leq k-2$. Since no two triangles of G_{k-1} have a common edge, it follows $K_4 - e \nsubseteq G_{k-1}$. Now, we will prove that $K_k - e \nsubseteq \overline{G_{k-1}}$ by contradiction. Suppose that there exists a $K_k - e$ in $\overline{G_{k-1}}$ denoted by H, then $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$. There are two subcases depending on $k-1$.

Case 1. Suppose that $k-1$ is even. We can notice that there are $k-1$ triangles which have no common vertex, say $b_0a_0a_1, b_1c_0c_1, b_2a_2a_3, b_3c_2c_3 \ldots$, $b_{k-3}a_{k-3}a_{k-2}, b_{k-2}c_{k-3}c_{k-2}$, marked them with T in Fig. 3.2. Let S denote the set of these triangles, then $|S| = k-1$. Since $|V(H)| = k$ and

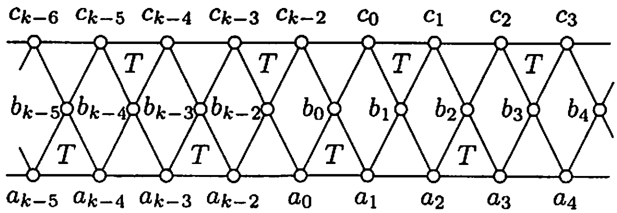


Fig. 3.2 The graph G_{k-1} for $k-1$ being even

$|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, there is just one triangle of S, say $b_0a_0a_1$ whose two vertices belong to $V(H)$. And there is just one vertex which belongs to $V(H)$ in each triangle of $S - \{b_0a_0a_1\}$. By symmetry, it is sufficient to consider that $a_0, b_0 \in V(H)$ or $a_0, a_1 \in V(H)$.

Case 1.1. Suppose that $a_0, b_0 \in V(H)$. Since $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, we have $a_{k-2}, b_{k-2}, c_{k-2} \notin V(H)$(namely $Y_{k-2} \cap V(H) = \emptyset$). It is forced that the remaining vertex of the triangle $b_{k-2}c_{k-3}c_{k-2}$, namely c_{k-3} has to belong to $V(H)$. And since $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, we have $b_{k-3} \notin V(H)$. Hence the remaining vertex of the triangle $b_{k-3}a_{k-3}a_{k-2}$, namely a_{k-3} has to belong to $V(H)$. Then since $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, we have $(Y_{k-2} \cup Y_{k-3}) \cap V(H) = \{a_{k-3}, c_{k-3}\}$. We can prove that $(Y_{k-4} \cup Y_{k-5}) \cap V(H) = \{a_{k-5}, c_{k-5}\}, \ldots, (Y_3 \cup Y_2) \cap V(H) = \{a_2, c_2\}$ by analogy. Therefore since $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, we have $b_1, c_1 \notin V(H)$. So, the remaining vertex of the triangle $b_1c_1c_0$, namely c_0 has to belong to $V(H)$. Hence, we have $\{a_0, b_0, c_0\} \subseteq V(H)$, that is, $|E(G_{k-1}\langle V(H)\rangle)| = 2$, a contradiction.

Case 1.2. Suppose that $a_0, a_1 \in V(H)$. Since $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, we have $b_1 \notin V(H)$. Hence there is just one vertex of $\{c_0, c_1\}$ which belongs to $V(H)$.

Case 1.2.1. Suppose that $c_0 \in V(H)$. Since $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, we have $Y_{k-2} \not\subseteq V(H)$. It is forced that the remaining vertex of the triangle $b_{k-2}c_{k-3}c_{k-2}$, namely c_{k-3} has to belong to $V(H)$. And since $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, we have $b_{k-3} \notin V(H)$. Hence the remaining vertex of the triangle $b_{k-3}a_{k-3}a_{k-2}$, namely a_{k-3} has to belong to $V(H)$, i.e., $(Y_{k-2} \cup Y_{k-3}) \cap V(H) = \{a_{k-3}, c_{k-3}\}$. We can prove that $(Y_{k-4} \cup Y_{k-5}) \cap V(H) = \{a_{k-5}, c_{k-5}\}, \ldots, (Y_3 \cup Y_2) \cap V(H) = \{a_2, c_2\}$ by analogy. Hence, we have $\{a_0, a_1, a_2\} \subseteq V(H)$, that is, $|E(G_{k-1}\langle V(H)\rangle)| = 2$, a contradiction.

Case 1.2.2. Suppose that $c_1 \in V(H)$. Since $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, we have $Y_2 \not\subseteq V(H)$. It is forced that the remaining vertex of the triangle $b_2a_2a_3$, namely a_3 has to belong to $V(H)$. And since $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, we have $b_3 \notin V(H)$. Hence the remaining vertex of the triangle $b_3c_2c_3$, namely c_3 has to belong to $V(H)$, i.e., $(Y_2 \cup Y_3) \cap V(H) = \{a_3, c_3\}$. We can prove that $(Y_4 \cup Y_5) \cap V(H) = \{a_5, c_5\}, \ldots, (Y_{k-3} \cup Y_{k-2}) \cap V(H) = \{a_{k-2}, c_{k-2}\}$ by analogy. Hence, we have $\{a_0, a_1, a_{k-2}\} \subseteq V(H)$, that is, $|E(G_{k-1}\langle V(H)\rangle)| = 2$, a contradiction.

Case 2. Suppose that $k-1$ is odd. Since $|V(H)| = k$ and $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, there exists a Y_i, say Y_0 whose two vertices belong to $V(H)$. By symmetry, it is sufficient to consider that $a_0, b_0 \in V(H)$ or $a_0, c_0 \in V(H)$. If $a_0, b_0 \in V(H)$, then since $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, we have $Y_{k-2} \cap V(H) = \emptyset$. If $a_0, c_0 \in V(H)$, then since $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, we have $Y_1 \cap V(H) = \emptyset$ or $Y_{k-2} \cap V(H) = \emptyset$, say $Y_{k-2} \cap V(H) = \emptyset$. We can notice that there are $k-2$ triangles which have no common vertex in $G_{k-1}\langle V(G_{k-1}) - Y_{k-2}\rangle$, namely $b_0a_0a_1, b_1c_0c_1, b_2a_2a_3, b_3c_2c_3 \ldots, b_{k-4}a_{k-4}a_{k-3}, b_{k-3}c_{k-4}c_{k-3}$, marked them with T in Fig 3.3. Let S denote the set of these triangles, then $|S| = k-2$. And since $|E(G_{k-1}\langle V(H)\rangle)| \leq 1$, there is at most one triangle of S whose two vertices belong to $V(H)$. So, we have $|V(H)| \leq k-1$, a contradiction.

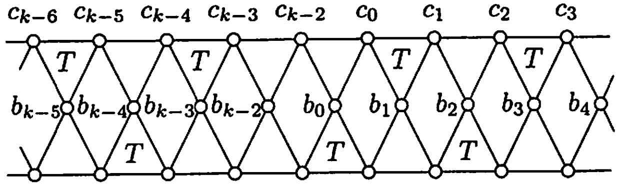


Fig. 3.3 The graph G_{k-1} for $k-1$ being odd

From Case 1-2, we have that the assumption does not hold, that is, $K_k - e \not\subseteq \overline{G_{k-1}}$. So, G_{k-1} is a $(K_4 - e, K_k - e; 3k-3)$-P-graph, i.e., $PR(K_4 - e, K_k - e) \geq 3k-2$. □

Lemma 3.3. If $k \geq 3$, then $PR(K_4 - e, K_k - e) \geq 3k + \lfloor (k-2)/4 \rfloor - 5$.
Proof. Let G_{13-0} be the graph shown in Fig. 3.4. It was proved that G_{13-0} is a $(K_4 - e, K_5; 13)$-P-graph in [7]. Suppose that G is a planar graph which is a union of $\lfloor (k-2)/4 \rfloor$ copies of G_{13-0} and $(k - 4 \times \lfloor (k-2)/4 \rfloor - 2)$ copies of a triangle, then $K_4 - e \not\subseteq G$. Since $K_5 \not\subseteq \overline{G_{13-0}}$, the cardinality of independent set of G is at most $4 \times \lfloor (k-2)/4 \rfloor + (k - 4 \times \lfloor (k-2)/4 \rfloor - 2) =$

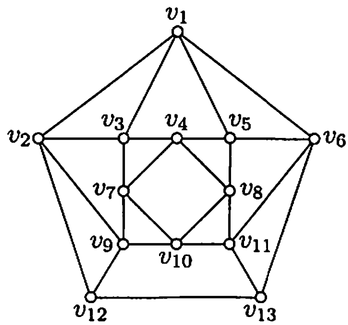


Fig. 3.4.

$k-2$, that is, $K_{k-1} \not\subseteq \overline{G}$. So, we have $K_k - e \not\subseteq \overline{G}$. Hence G is a $(K_4 - e, K_k - e; n)$-P-graph, where $n = 13 \times \lfloor (k-2)/4 \rfloor + 3 \times (k - 4 \times \lfloor (k-2)/4 \rfloor - 2) = 3k + \lfloor (k-2)/4 \rfloor - 6$. The lemma holds. □

By Lemma 3.2 and 3.3, we have

Theorem 3.4. If $k \geq 5$, then $PR(K_4 - e, K_k - e) \geq \max\{3k-2,\ 3k + \lfloor (k-2)/4 \rfloor - 5\}$.

By Lemma 3.1 and Theorem 3.4 setting $k = 6$, we have

Theorem 3.5. $PR(K_4 - e, K_6 - e) = 16$.

By Dudek and Ruciński[3] and Theorem 3.5, we have

$$\begin{aligned} PR(K_4 - e, K_3 - e) &= 5, \\ PR(K_4 - e, K_4 - e) &= 9, \\ PR(K_4 - e, K_5 - e) &= 13, \\ PR(K_4 - e, K_6 - e) &= 16. \end{aligned}$$

The problem of determining the values of $PR(K_4 - e, K_k - e)$ is still remaining open for $k \geq 7$.

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