NORM OF WEIGHTED COMPOSITION OPERATORS FROM BLOCH SPACE TO H_μ^∞ ON THE UNIT BALL

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Abstract

We calculate the norm of weighted composition operators uC_φ from the Bloch space to the weighted space $H_\mu^\infty(B)$ on the unit ball B.

1. INTRODUCTION

Let B be the open unit ball in $\mathbb{C}^n$, $H(B)$ the class of all holomorphic functions on the unit ball and $H^\infty = H^\infty(B)$ the space of all bounded holomorphic functions on B. For an $f \in H(B)$ we denote $\nabla f = (\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n})$.

The Bloch space $\mathcal{B} = \mathcal{B}(B)$, consists of all $f \in H(B)$ such that

$$b(f) = \sup_{z\in B}(1 - |z|^2)\,|\nabla f(z)| < \infty.$$

The Bloch space with the norm $\|f\|_{\mathcal{B}} = |f(0)| + b(f)$, becomes a Banach space.

The little Bloch space $\mathcal{B}_0$ is a subspace of $\mathcal{B}$ consisting of all $f \in \mathcal{B}$ such that $(1 - |z|^2)|\nabla f(z)| \to 0$, as $|z| \to 1$.

The weighted space $H_\mu^\infty = H_\mu^\infty(B)$ consists of all $f \in H(B)$ such that

$$\sup_{z\in B} \mu(z)|f(z)| < \infty,$$

where $\mu(z) = \mu(|z|)$ and μ is normal on the interval $[0, 1)$ (see, for example, [4]).

Let $u \in H(B)$ and φ be a holomorphic self-map of B. For $f \in H(B)$ weighted composition operator is defined by $(uC_\varphi f)(z) = u(z)f(\varphi(z))$. In the last four decades experts in the area provide function theoretic characterizations when u and φ induce bounded or compact weighted composition operators on spaces of holomorphic functions. For some classical results in the topic, see, e.g., [2].

In [3], Ohno characterized the boundedness and compactness of the operator uC_φ between $H^\infty(\mathbb{D})$ and $\mathcal{B}(\mathbb{D})$ on the unit disk $\mathbb{D}$. The results from [3] are corrected, and extended in the setting of the unit polydisk in [5], and in the unit ball setting in [7]. Closely related results can be found in [1, 4, 6, 9, 10, 11, 12].

In [7], among other results, we proved the following theorem, regarding the boundedness of the operator $uC_\varphi : \mathcal{B}(B) \to H^\infty(B)$.

Theorem A. *Let φ be a holomorphic self-map of B and $u \in H(B)$. Then the operator $uC_\varphi : \mathcal{B} \to H^\infty$ is bounded if and only if $uC_\varphi : \mathcal{B}_0 \to H^\infty$ is bounded if and only if $u \in H^\infty$ and $M := \sup_{z\in B} |u(z)| \ln \frac{2}{1-|\varphi(z)|^2} < \infty$.*

Moreover, if $uC_\varphi : \mathcal{B} \to H^\infty$ is bounded, then $\|uC_\varphi\|_{\mathcal{B}\to H^\infty} \asymp M$, where the notation $a \asymp b$ means that there is a positive constant C such that $b/C \le a \le Cb$.

One of the interesting questions is to find the exact value of the norm of weighted composition operators. Our aim here is to calculate $\|uC_\varphi\|_{\mathcal{B}\to H^\infty_\mu}$.

We need the following lemma, which should be folklore and whose proof follows from the next sequence of relations (compare with Lemma 2.2 in [8]):

$$|f(z) - f(0)| = \left| \int_0^1 \langle \nabla f(tz), \bar{z} \rangle \right| \le b(f) \int_0^1 \frac{|z|dt}{1-|z|^2t^2} = b(f)\frac{1}{2}\ln\frac{1+|z|}{1-|z|}.$$

Lemma 1. *Let $f \in \mathcal{B}(B)$. Then the following inequality holds*

$$|f(z)| \le |f(0)| + b(f)\frac{1}{2}\ln\frac{1+|z|}{1-|z|}. \tag{1}$$

2. The norm of the operator $uC_\varphi : \mathcal{B} \to H^\infty_\mu$

Now we are in a position to formulate and prove the main result of this note.

Theorem 1. *Assume $u \in H(B)$, φ is a holomorphic self-map of B and $uC_\varphi : \mathcal{B} \to H^\infty_\mu$ is bounded. Then*

$$\|uC_\varphi\|_{\mathcal{B}\to H^\infty_\mu} = \|uC_\varphi\|_{\mathcal{B}_0\to H^\infty_\mu} = \max\left\{ \|u\|_{H^\infty_\mu}, \frac{1}{2}\sup_{z\in B} \mu(z)|u(z)| \ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|} \right\}. \tag{2}$$

Proof. Since $f_0(z) \equiv 1 \in \mathcal{B}_0$, we have

$$\|uC_\varphi\|_{\mathcal{B}_0\to H^\infty_\mu} = \|f_0\|_{\mathcal{B}} \|uC_\varphi\|_{\mathcal{B}_0\to H^\infty_\mu} \ge \|uC_\varphi(f_0)\|_{H^\infty_\mu} = \|u\|_{H^\infty_\mu}. \tag{3}$$

For $w \in B$, set $f_w(z) = \frac{1}{2}\ln\frac{1+\langle z,w\rangle}{1-\langle z,w\rangle}$, (with $\ln 1 = 0$). Since $f_w(0) = 0$ and

$$(1-|z|^2)|\nabla f_w(z)| = \frac{(1-|z|^2)|w|}{|1-\langle z,w\rangle^2|} \le \frac{1-|z|^2}{1-|w|^2|z|^2} \le \min\left\{1, \frac{1-|z|^2}{1-|w|^2}\right\},$$

it follows that $\sup_{w\in B} \|f_w\|_{\mathcal{B}} \le 1$, and $f_w \in \mathcal{B}_0$ for each fixed $w \in B$.

From this and the boundedness of $uC_\varphi : \mathcal{B}_0 \to H^\infty_\mu$ we have that for $\varphi(w) \ne 0$ and for every $\rho \in (0,1)$ the following inequality holds

$$\begin{aligned} \|uC_\varphi\|_{\mathcal{B}_0\to H^\infty_\mu} &\ge \|uC_\varphi f_{\rho\varphi(w)/|\varphi(w)|}\|_{H^\infty_\mu} \\ &= \sup_{z\in B} \mu(z)\left| u(z)\frac{1}{2}\ln\frac{1+\rho\langle\varphi(z),\varphi(w)/|\varphi(w)|\rangle}{1-\rho\langle\varphi(z),\varphi(w)/|\varphi(w)|\rangle}\right| \\ &\ge \frac{1}{2}\mu(w)|u(w)|\ln\frac{1+\rho|\varphi(w)|}{1-\rho|\varphi(w)|}. \end{aligned} \tag{4}$$

Note that (4) obviously holds if $\varphi(w) = 0$.

Letting $\rho \to 1$ in (4), we obtain that for each $w \in B$

$$\|uC_\varphi\|_{\mathcal{B}_0 \to H_\mu^\infty} \geq \frac{1}{2}\mu(w)|u(w)| \ln \frac{1 + |\varphi(w)|}{1 - |\varphi(w)|}.$$

From this and since w is an arbitrary element of B, it follows that

$$\|uC_\varphi\|_{\mathcal{B}_0 \to H_\mu^\infty} \geq \frac{1}{2} \sup_{z \in B} \mu(z)|u(z)| \ln \frac{1 + |\varphi(z)|}{1 - |\varphi(z)|}. \tag{5}$$

If $f \in \mathcal{B}$, then Lemma 1 and the definition of the norm $\|\cdot\|_{\mathcal{B}}$ yield

$$\begin{aligned} \|uC_\varphi f\|_{H_\mu^\infty} &= \sup_{z \in B} \mu(z)|u(z) f(\varphi(z))| \\ &\leq \sup_{z \in B} \left(\mu(z)|u(z)| \left(|f(0)| + b(f) \frac{1}{2} \ln \frac{1 + |\varphi(z)|}{1 - |\varphi(z)|} \right) \right) \\ &\leq \|f\|_{\mathcal{B}} \max \left\{ \|u\|_{H_\mu^\infty}, \frac{1}{2} \sup_{z \in B} \mu(z)|u(z)| \ln \frac{1 + |\varphi(z)|}{1 - |\varphi(z)|} \right\}. \end{aligned} \tag{6}$$

From (3)-(6) and since $\|uC_\varphi\|_{\mathcal{B} \to H_\mu^\infty} \geq \|uC_\varphi\|_{\mathcal{B}_0 \to H_\mu^\infty}$, the result follows. □

References

[1] D. Clahane and S. Stević, Norm equivalence and composition operators between Bloch/ Lipschitz spaces of the unit ball, *J. Inequal. Appl.* Vol. 2006, Article ID 61018, (2006), 11 pages.

[2] C. C. Cowen and B. D. MacCluer, Composition Operators on Spaces of Analytic Functions, *CRC Press, Boca Raton, FL,* 1995.

[3] S. Ohno, Weighted composition operators between H^∞ and the Bloch space, *Taiwanese J. Math.* 5 (2001), 555-563.

[4] S. Li and S. Stević, Weighted composition operators from Bergman-type spaces into Bloch spaces, *Proc. Indian Acad. Sci. Math. Sci.* **117** (3) (2007), 371-385.

[5] S. Li and S. Stević, Weighted composition operators from α-Bloch space to H^∞ on the polydisk, *Numer. Funct. Anal. Optimization* **28** (7) (2007), 911-925.

[6] S. Li and S. Stević, Weighted composition operators from H^∞ to the Bloch space on the polydisc, *Abstr. Appl. Anal.* Vol. 2007, Article ID 48478, (2007), 12 pages.

[7] S. Li and S. Stević, Weighted composition operators between H^∞ and α-Bloch spaces in the unit ball, *Taiwan. J. Math.* (to appear).

[8] S. Stević, On an integral operator on the unit ball in $\mathbb{C}^n$, *J. Inequal. Appl.* **1** (2005), 81-88.

[9] S. Stević, Composition operators between H^∞ and the α-Bloch spaces on the polydisc, *Z. Anal. Anwendungen* **25** (4) (2006), 457-466.

[10] S. Stević, Weighted composition operators between mixed norm spaces and H_α^∞ spaces in the unit ball, *J. Inequal. Appl.* Vol. 2007, Article ID 28629, (2007), 9 pages.

[11] X. Zhu, Generalized weighted composition operators from Bloch-type spaces to weighted Bergman spaces, *Indian J. Math.* **49** (2) (2007), 139-149.

[12] X. Zhu, Products of differentiation, composition and multiplication from Bergman type spaces to Bers spaces, *Integral Transform. Spec. Funct.* **18** (3) (2007), 223-231.