Bounds on Powerful Alliance Numbers

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Abstract

Let $G = (V, E)$ be a graph. Then $S \subseteq V$ is an excess-t global powerful alliance if $|N[v] \cap S| \geq |N[v] \cap (V - S)| + t$ for every $v \in V$. If $t = 0$ this definition reduces to that of a global powerful alliance. Here we determine bounds on the cardinalities of such sets S.

Keywords: Domination, defensive alliances, offensive alliances, powerful alliances, global alliances, extremal graphs

1 Introduction

The concept of alliances in graphs has been developed in the past few years in order to model situations in which entities such as nations or businesses unite for mutual benefit in thwarting common enemies. Several different types of alliances have been introduced to achieve varying goals. Two, defensive and offensive, are mentioned briefly here and this paper concentrates on a combination of these called powerful alliances. We employ the following notation. Let $G = (V, E)$ be a graph. For any $v \in V$, $N(v) = \{w \in V : vw \in E\}$ is its *open neighborhood* and $N[v] = \{v\} \cup N(v)$ is its *closed neighborhood.* The *closed neighborhood of* $S \subseteq V$ is $N[S] = (\cup_{v \in S} N[v])$. The *boundary* of S, denoted ∂S, is the set $N[S] \cap (V - S)$. The subgraph induced by S is denoted $\langle S \rangle$.

Set $S \subseteq V$ is a *defensive alliance* if $|N[v] \cap S| \geq |N[v] \cap (V - S)|$ for all $v \in S$ and is an *offensive alliance* if $|N[v] \cap S| \geq |N[v] \cap (V - S)|$ for all $v \in \partial S$. A defensive alliance S can successfully defend every member from an attack by the vertices of ∂S and an offensive alliance can prevail in an attack on any vertex of ∂S. An alliance S is *minimal* if $S - \{v\}$ is not an alliance of the same type for every $v \in S$, and is *critical* if no proper subset of S is an alliance of the same type. It is possible for an alliance to be minimal but not critical [10]. Finally, S is a *global* alliance if it also is a dominating set. Defensive and offensive alliances are discussed in [4, 5, 7, 8, 9, 10, 12, 13, 14].

Set $S \subseteq V$ is a *powerful alliance* if it is both defensive and offensive, that is, if $|N[v] \cap S| \geq |N[v] \cap (V - S)|$ for all $v \in S \cup \partial S = N[S]$. For a given graph G, the smallest cardinality of a powerful alliance is the *powerful alliance number* which is denoted by $a_p(G)$, and the smallest cardinality of a global powerful alliance is the *global powerful alliance number* which is denoted by $\gamma_{a_p}(G)$. Powerful alliances of these cardinalities are called *a_p-sets* and *γ_{a_p}-sets*, respectively, and similar definitions will apply to other parameters. The abbreviation *gpa* will be employed for the phrase *global powerful alliance.* Notice that a_p-sets and γ_{a_p}-sets are both minimal and critical and that the concepts of minimal and critical are the same for gpa's. Powerful alliances have been studied in Brigham, Dutton, Haynes and Hedetniemi [1].

Powerful alliances can be generalized as follows. Set $S \subseteq V$ is an *excess-t powerful alliance* if $|N[v] \cap S| \geq |N[v] \cap (V - S)| + t$ for every $v \in S \cup \partial S$. This definition makes sense if $-\delta(G) \leq t \leq \delta(G)$ where $\delta(G)$ is the minimum degree of G. Values of t in this interval are termed *feasible.* We mainly shall be concerned with this concept when G is d-regular in which case $-d \leq t \leq d$. Since $S = V$ satisfies the definition for an excess-t gpa, we have that a minimum one exists for each t such that $-d \leq t \leq d$. Note that $t = d$ implies $V - S$ must be empty. This special case is of little interest and, unless otherwise noted, we therefore assume $t < d$. It is straightforward to show that $S \subseteq V$ is an excess-t gpa of d-regular graph G if and only if every vertex of $V - S$ has at least $\lceil \frac{d+1+t}{2} \rceil$ neighbors in S and every vertex of S has at most $\lfloor \frac{d+1-t}{2} \rfloor$ neighbors in $V - S$.

We will employ the notation $a_p(G, t)$ for the smallest cardinality of an excess-t powerful alliance of arbitrary graph G, and $a_p(G, d, t)$ if G is d-regular. The corresponding notations for gpa's are $\gamma_{a_p}(G, t)$ and $\gamma_{a_p}(G, d, t)$, respectively. Thus, for arbitrary graphs, $a_p(G, 0) = a_p(G)$ and $\gamma_{a_p}(G, 0) = \gamma_{a_p}(G)$ while, for d-regular graphs, $a_p(G, 0) = a_p(G, d, 0) = a_p(G)$ and $\gamma_{a_p}(G, 0) = \gamma_{a_p}(G, d, 0) = \gamma_{a_p}(G)$. In addition to minimum (global) powerful alliances, we also shall be concerned with minimal (global) powerful alliances of largest cardinality, and, in a straightforward extension of the notation, this cardinality is written $\Gamma_{a_p}(G, t)$ or $\Gamma_{a_p}(G, d, t)$. For given values of n and t (respectively n, d, and t), it is of interest to find the smallest value of $\gamma_{a_p}(G, t)$ (resp. $\gamma_{a_p}(G, d, t)$) and the largest value of $\Gamma_{a_p}(G, t)$ (resp. $\Gamma_{a_p}(G, d, t)$) taken over all graphs (resp. d-regular graphs) having n vertices. We denote these values in an obvious manner by $\tau(n, t)$, $\tau(n, d, t)$, $T(n, t)$, and $T(n, d, t)$. Note that an earlier comment implies $\gamma_{a_p}(G, d, d) = \tau(n, d, d) = \Gamma_{a_p}(G, d, d) = T(n, d, d) = n$ where G is any d-

regular graph on n vertices. We call a triple (n, d, t) *feasible* if it is possible to find an n vertex d-regular graph having an excess-t gpa. Suppose S is a minimal powerful alliance and $v \in S$. Then removing v from S destroys the degree requirements for at least one vertex w of G. Vertex w may be v itself, a neighbor of v in S, or a neighbor of v in $V - S$. We shall refer to w as a *critical* vertex.

We present bounds, including Nordhaus-Gaddum types, for the above parameters, and determine exact values for $\tau(n, d, t)$ and $T(n, d, t)$ for all feasible triples (n, d, t). Chellali and Haynes [3] have found a sharp bound on $\gamma_{a_p}(G)$ for trees.

2 Lower Bounds for Global Powerful Alliance Numbers

Lower bounds for gpa numbers are developed in this section. The symbol $\Delta(G)$ represents the maximum degree of graph G. The argument G often is omitted from this invariant, and others, if the graph in question is clear. We employ $deg(x)$ for the degree of vertex x and, for $M \subseteq V$, $deg_M(x)$ represents $|N(x) \cap M|$.

Observation 1 *For any graph G and feasible integer t, $\gamma_{a_p}(G, t) \geq \tau(n, t)$ $\geq \lceil \frac{\Delta+1+t}{2} \rceil$ $(\gamma_{a_p}(G) \geq \tau(n, 0) \geq \lceil \frac{\Delta+1}{2} \rceil)$.*

Proof: Let x be a vertex of degree Δ. If $x \in S$, the requirements for an excess-t powerful alliance dictate $|S| \geq deg_S(x) + 1 \geq deg(x) - deg_S(x) + t$ so $2deg_S(x) \geq \Delta + t - 1$ and the result follows. On the other hand, if $x \in V - S$, $|S| \geq deg_S(x) \geq deg(x) - deg_S(x) + t + 1$ and again the result is obtained. □

Observation 2 *Let G be a d-regular graph of order n and (n, d, t) be a feasible triple. Then $\gamma_{a_p}(G, d, t) \geq \tau(n, d, t) \geq \left\lceil \frac{n \lceil \frac{d+1+t}{2} \rceil}{d+1} \right\rceil$ $\Big(\gamma_{a_p}(G) \geq \tau(n, d, 0) \geq \lceil \frac{n}{2} \rceil$ if d is odd and $\gamma_{a_p}(G) \geq \tau(n, d, 0) \geq \left\lceil \frac{n(d+2)}{2(d+1)} \right\rceil$ if d is even$\Big)$.*

Proof: The result is trivially true if $(d, t) = (0, 0)$, $(1, -1)$, $(1, 0)$, and $(1, 1)$ so we assume $d \geq 2$. Let S be a $\gamma_{a_p}(G, d, t)$-set of d-regular graph G, and let e be the number of edges of G with exactly one end point in S. Then $(n - \gamma_{a_p}(G, d, t)) \lceil \frac{d+1+t}{2} \rceil \leq e \leq \gamma_{a_p}(G, d, t) \lfloor \frac{d+1-t}{2} \rfloor$. Simplifying yields $\gamma_{a_p}(G, d, t) \geq \left\lceil \frac{n \lceil \frac{d+1+t}{2} \rceil}{d+1} \right\rceil$. □

We will show later that the bound of Observation 2 is best possible. The observation sometimes aids in determining the value of $\gamma_{a_p}(G)$ for certain graphs. For example, with $t = 0$, consider the 4-regular graph shown in Figure 1. The circled seven vertices form a gpa. Observation 2 yields $\gamma_{a_p}(G) \geq \frac{11(6)}{2(5)}$ which implies $\gamma_{a_p}(G) \geq 7$, and hence $\gamma_{a_p}(G) = 7$.

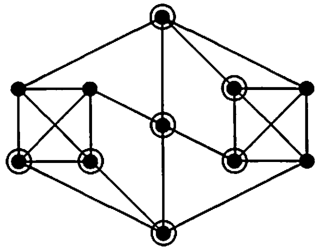

Figure 1: Graph G having $\gamma_{a_p}(G) = 7$

The final lower bound employs the following theorem that, with its corollary, is of independent interest. For the remainder of this section we assume $t = 0$.

Theorem 3 *Let $G = (V, E)$ be a graph. If $f \in E$, then $\gamma_{a_p}(G) - 1 \leq \gamma_{a_p}(G - f) \leq \gamma_{a_p}(G) + 2$.*

Proof: To show the first inequality let S be a γ_{a_p}-set of $G - f$ where $f = xy \in E$. The possible locations of the end vertices of f are now considered.

1. $\{x, y\} \subseteq S$. Then S also is a gpa of G so $\gamma_{a_p}(G) \leq \gamma_{a_p}(G - f)$.

2. $x \in S$ and $y \in V - S$. Here $S \cup \{y\}$ is a gpa of G implying $\gamma_{a_p}(G) \leq \gamma_{a_p}(G - f) + 1$.

3. $\{x, y\} \subseteq V - S$. Here either of $S \cup \{x\}$ or $S \cup \{y\}$ is a gpa of G so again we have $\gamma_{a_p}(G) \leq \gamma_{a_p}(G - f) + 1$.

These cases exhaust all possibilities and thus establish the result.

A similar approach establishes the second inequality. Let S be a γ_{a_p}-set of G where $f = xy \in E$. Again we consider the locations of the end vertices of f,

1. $\{x, y\} \subseteq S$. Let x and y have neighbors $\hat{x}$ and $\hat{y}$, respectively, in $V - S$. Then $S \cup \{\hat{x}, \hat{y}\}$ is a gpa of $G - f$. If $\hat{x} = \hat{y}$, $S \cup \hat{x}$ is a gpa of $G - f$. If x (resepectively y) has no neighbor in $V - S$, there is no need to add $\hat{x}$ (respectively $\hat{y}$) into S. In any event, $\gamma_{a_p}(G - f) \leq \gamma_{a_p}(G) + 2$.

2. $x \in S$ and $y \in V - S$. Now $S \cup \{y\}$ is a gpa of $G - f$ so $\gamma_{a_p}(G - f) \leq \gamma_{a_p}(G) + 1$.

3. $\{x, y\} \subseteq V-S$. Set S remains a gpa of $G-f$ and $\gamma_{a_p}(G-f) \leq \gamma_{a_p}(G)$.

Again all possibilities have been examined and the proof is complete. □

Corollary 4 *Let $G = (V, E)$ be a graph. If $f \notin E$, then $\gamma_{a_p}(G) - 2 \leq \gamma_{a_p}(G+f) \leq \gamma_{a_p}(G) + 1$.*

Let us now consider all graphs on a fixed number of vertices n for which γ_{a_p} is as small as possible. Let $G = (V, E)$ be such a graph that has the minimum number of edges. Thus $f \in E$ implies $\gamma_{a_p}(G-f) > \gamma_{a_p}(G)$. We have the following observation.

Observation 5 *Let $G = (V, E)$ be a graph on n vertices for which γ_{a_p} is as small as possible and which, among all such graphs, has the minimum number of edges. Let S be a γ_{a_p}-set. Then*

1. *$V - S$ is an independent set.*
2. *For $x \in V - S$, $deg(x) = 1$.*
3. *If $x \in S$, $|N(x) \cap S| = \left\lfloor \frac{deg(x)}{2} \right\rfloor$.*
4. *V contains at most one vertex of even degree.*

Proof:

1. Suppose edge f has both end vertices in $V - S$. Then S also is a gpa of $G - f$ implying $\gamma_{a_p}(G-f) \leq \gamma_{a_p}(G)$, a contradiction.

2. Suppose $x \in V - S$, $deg(x) > 1$, and f is an edge having x as an end vertex. Then, since $V - S$ is independent, S remains a gpa of $G - f$ and again we obtain the contradiction $\gamma_{a_p}(G-f) \leq \gamma_{a_p}(G)$.

3. For any vertex $x \in S$, let a_x be the number of neighbors it has in S and b_x the number in $V - S$. Suppose $a_x > \left\lfloor \frac{deg(x)}{2} \right\rfloor$ so $b_x = deg(x) - a_x < \left\lceil \frac{deg(x)}{2} \right\rceil$. Let y be a neighbor of x in $\langle S \rangle$. Since S is a gpa, $a_x + 1 \geq b_x = deg(x) - a_x$ and $a_y + 1 \geq b_y = deg(y) - a_y$. The latter inequality implies $a_y \geq \left\lfloor \frac{deg(y)}{2} \right\rfloor$. If edge $f = xy$ is removed, we have $|N[x] \cap S| = a_x \geq \left\lfloor \frac{deg(x)}{2} \right\rfloor + 1 \geq b_x = |N[x] - S|$. Since S no longer is a gpa, we must have $a_y < b_y = deg(y) - a_y$ so $2a_y < deg(y)$ which implies $a_y \leq \left\lfloor \frac{deg(y)}{2} \right\rfloor$ which in turn implies $a_y = \left\lfloor \frac{deg(y)}{2} \right\rfloor$. This means y has a neighbor z in $V - S$. Construct a new graph G' from $G - f$ by removing edge yz and adding edge xz. In G' vertex

x has $a_x - 1$ neighbors in S and $b_x + 1$ in $V - S$. Furthermore, in G', $|N[x] \cap S| = a_x \geq \left\lceil \frac{deg(x)}{2} \right\rceil \geq b_x + 1 = |N[x] - S|$. Therefore S is also a gpa of G' so $\gamma_{a_p}(G') \leq \gamma_{a_p}(G)$, a contradiction that shows x has exactly $\left\lfloor \frac{deg(x)}{2} \right\rfloor$ neighbors in S.

4. All vertices of $V - S$ have odd degree since they are monovalent. The vertices in S of even degree form an independent set since, if two were adjacent, the edge joining them could be removed and S would remain a gpa, a contradiction. Suppose $x \in S$ is a vertex of even degree, $y \in S$ is a neighbor of odd degree, z is a degree one neighbor of y, and $w \in S$ is a second vertex of even degree. Then S remains a γ_{a_p}-set for the graph obtained from G by removing edge yz and adding edge wz. Now S has adjacent even degree vertices x and y, a contradiction that yields the result. □

Observation 5 can be employed to find a sharp lower bound on $\gamma_{a_p}(G)$. Before proceeding, it is necessary to describe a family of graphs of even order n that will be employed in showing sharpness both here and later. These graphs, denoted G_s where s is a parameter, have a gpa number that is small as a fraction of n, and the gpa number of its complement, $\overline{G_s}$, has a value of $\frac{n}{2}$. Start with a K_s with vertices $v_1, v_2, \ldots, v_s$. Append s vertices $w_{i1}, w_{i2}, \ldots, w_{is}$ of degree one to vertex v_i for $1 \leq i \leq s$. Then $n = s^2 + s$.

Proposition 6 $\gamma_{a_p}(G_s) = s = \frac{n}{s+1}$.

Proof: The maximum degree of G_s is $\Delta = 2s - 1$ so, by Observation 1, $\gamma_{a_p}(G_s) \geq s$. Let $S = \{v_1, v_2, \ldots, v_s\}$. Then $x \in V - S$ has degree 1 and $|N[x] \cap S| = 1$. Also $x \in S$ has degree $2s - 1$ and $|N[x] \cap S| = s$. Thus S is a dominating powerful alliance. □

Now consider $\overline{G_s}$. It is constructed from a K_{s^2} with vertices $\{w_{ij} : 1 \leq i, j \leq s\}$, an independent set $\{v_1, v_2, \ldots, v_s\}$, and edges from each v_i to all the vertices of the complete graph except for a set $A_i = \{w_{i1}, w_{i2}, \ldots, w_{is}\}$ of cardinality s. Note the sets A_i for $1 \leq i \leq s$ partition the set $\{w_{ij} : 1 \leq i, j \leq s\}$. Each v_i has degree $s^2 - s$ and each w_{ij} has degree $s^2 + s - 2$.

Proposition 7 $\gamma_{a_p}(\overline{G_s}) = \frac{s^2+s}{2} = \frac{n}{2}$.

Proof: Since $\Delta = s^2 + s - 2$, Observation 1 implies $\gamma_{a_p}(\overline{G_s}) \geq \frac{s^2+s}{2}$. Let S be any set of $\frac{s^2+s}{2}$ vertices taken from the w_{ij}'s, with the restriction that at least one vertex of each A_i is not included in S. This is possible if $s \geq 3$. Then any w_{ij} has all of these $\frac{s^2+s}{2}$ vertices of S in its closed neighborhood. Furthermore, any v_i has at least $\frac{s^2+s}{2} - (s-1) = \frac{s^2-s}{2} + 1$ vertices of S in its closed neighborhood. Thus S is a dominating powerful alliance and the result follows. □

Now we can proceed to the lower bound.

Theorem 8 *For any graph G on n vertices,*

$$\gamma_{a_p}(G) \geq \tau(n,0) \geq \begin{cases} \lceil \sqrt{n+.25} - .5 \rceil & \text{if } n \text{ is even} \\ \lceil \sqrt{n} \rceil & \text{if } n \text{ is odd} \end{cases}$$

and these bounds are sharp.

Proof: Let $t_n = \tau(n,0) = \min\{\gamma_{a_p}(G) : G \text{ has } n \text{ vertices}\}$. Select G so it has n vertices, $\gamma_{a_p}(G) = t_n$, and the minimum number of edges. From Observation 5, G has $n - t_n$ degree one vertices. Let S be the remaining t_n vertices which form a γ_{a_p}-set of G. Each vertex of S has at most $t_n - 1$ neighbors in S and hence at most t_n neighbors in $V - S$. Thus $n - t_n \leq t_n^2$. Solving this inequality yields $t_n \geq \lceil \sqrt{n+.25} - .5 \rceil$ which is valid for all n. A slight improvement can be made when n is odd. Again using Observation 5, the G in this case must have exactly one vertex of even degree. To minimize t_n, we must maximize $n - t_n$ which is accomplished by maximizing the number of edges of $\langle S \rangle$. This in turn results when the even degree vertex has $t_n - 1$ neighbors in S and all other vertices of S have $t_n - 2$ neighbors in S. Thus each of the t_n vertices of S has at most $t_n - 1$ neighbors in $V - S$. It follows that $n - t_n \leq t_n(t_n - 1)$ or $t_n \geq \sqrt{n}$.

When n is even the bound is achieved by the graph G_s described above. When n is odd, S is formed by joining a vertex to all vertices of a K_s minus a one factor, where s is even, and then appending s monovalent vertices to each of the $s+1$ vertices of S. □

3 Upper Bounds

We begin with some straightforward results. Recall that, for general graphs, t is an integer such that $-\delta \leq t \leq \delta$.

Observation 9 *Let G be a graph. Then $a_p(G,t) \leq \gamma_{a_p}(G,t) \leq n - \lceil \frac{\delta-t}{2} \rceil$ $(a_p(G) \leq \gamma_{a_p}(G) \leq n - \lceil \frac{\delta}{2} \rceil)$.*

Proof: Let S be a subset of $n - \lceil \frac{\delta-t}{2} \rceil$ vertices. Then any vertex of S has at most $\lceil \frac{\delta-t}{2} \rceil$ vertices of its closed neighborhood in $V - S$ and at least $\lfloor \frac{\delta+t}{2} \rfloor + 1$ in S. Furthermore, any vertex of $V - S$ also has at most $\lceil \frac{\delta-t}{2} \rceil$ vertices of $V - S$ in its closed neighborhood and at least $\lfloor \frac{\delta+t}{2} \rfloor + 1$ in S. It is straightforward to show $\lfloor \frac{\delta+t}{2} \rfloor + 1 - \lceil \frac{\delta-t}{2} \rceil \geq t$. Thus S is an excess-t

powerful alliance. It also dominates since each vertex of $V - S$ has at least $\lfloor \frac{\delta-t}{2} \rfloor + 1$ neighbors in S. □

The bound of Observation 9 is achieved for a_p and γ_{a_p} (when $t = 0$) by C_4, C_5, and K_n. When equality exists in Observation 9, other points can be made as the next two results show.

Observation 10 *Let G be a graph. If $\gamma_{a_p}(G,t) = n - \lceil \frac{\delta-t}{2} \rceil$, then every subset of $\lceil \frac{\delta-t}{2} \rceil + 1$ vertices is dominated in G by a single vertex.*

Proof: Let $V - S$ be an arbitrary set of $\lceil \frac{\delta-t}{2} \rceil + 1$ vertices. Then S, the set of other vertices, is not an excess-t gpa. Thus there is a vertex x such that $|N[x] \cap (V - S)| \geq |N[x] \cap S| - t + 1$. If $x \in S$, this means $deg_{V-S}(x) \geq deg(x) + 1 - deg_{V-S}(x) - t + 1$ or $deg_{V-S}(x) \geq \lceil \frac{\delta-t}{2} \rceil + 1$ so x dominates $V - S$. A similar analysis shows, if $x \in V - S$, that $deg_{V-S}(x) \geq \lceil \frac{\delta-t}{2} \rceil$ so once again x dominates $V - S$. □

The domination number of graph G is denoted $\gamma(G)$.

Corollary 11 *Let G be a graph. If $\gamma_{a_p}(G) = n - \lceil \frac{\delta-t}{2} \rceil$, then $\gamma(G) \leq \left\lceil \frac{n}{\lceil \frac{\delta-t}{2} \rceil + 1} \right\rceil$.*

An open question is if the bound of Corollary 11 holds in general, not just in this special case.

Observation 12 *Let G be a graph. If $a_p(G,t) < \gamma_{a_p}(G,t)$, then $a_p(G,t) \leq n - \delta - 1$.*

Proof: Since a minimum excess-t powerful alliance doesn't dominate the graph, the undominated vertex and all its neighbors, of which there are at least δ, are not in the alliance. □

Let $\overline{N[x]}$ be the closed neighborhood of vertex x in the complement $\overline{G}$ of graph G. We explore relationships between $\gamma_{a_p}(G)$ and $\gamma_{a_p}(\overline{G})$.

Theorem 13 *If $\gamma_{a_p}(G) \leq \lceil \frac{n}{2} \rceil - 1$, then $\gamma_{a_p}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Proof: Let S be an arbitrary gpa of G for which $\gamma_{a_p}(G) \leq s = |S| \leq \lceil \frac{n}{2} \rceil - 1$. Then, for any vertex $x \in V$, with a_x defined by $a_x = |N(x) \cap S|$, we have $a_x + 1 \geq deg(x) - a_x$ when $x \in S$, and $a_x \geq deg(x) + 1 - a_x$ otherwise. Therefore, $2a_x - deg(x) + \lambda_x \geq 0$, where $\lambda_x = +1$ if $x \in S$ and $\lambda_x = -1$ otherwise. In $\overline{G}$, for every $x \in V$, $|\overline{N[x]} - (V - S)| = s - a_x$, and $|\overline{N[x]} \cap (V - S)| = n - s - (deg(x) - a_x)$. Since $deg(x) - a_x \leq a_x + \lambda_x \leq s \leq \lceil \frac{n}{2} \rceil - 1$, we have that $n - s - (deg(x) - a_x) \geq n - s - (\lceil \frac{n}{2} \rceil - 1) = \lfloor \frac{n}{2} \rfloor + 1 - s \geq \lceil \frac{n}{2} \rceil - s$. That is, $\lceil \frac{n}{2} \rceil - s \geq 1$ vertices of the closed neighborhood of any vertex lie in $V - S$. Hence, a set obtained by removing any $\lceil \frac{n}{2} \rceil - s - 1$ vertices from $V - S$ is still a dominating set of $\overline{G}$, a fact we employ below.

Using the identity $s = n-s-n+2s$ and the fact $a_x \geq deg(x) - a_x - \lambda_x$ we have $s - a_x \leq n - s - (deg(x) - a_x) + 2s - n + \lambda_x$. That is, $|\overline{N[x]} - (V-S)| \leq |\overline{N[x]} \cap (V-S)| + 2s - n + \lambda_x$. Since $2s - n + \lambda_x \leq 0$, and since $V-S$ was shown above to be a dominating set of $\overline{G}$, $V-S$ is a gpa of $\overline{G}$. Thus the result holds when $\gamma_{a_p}(G) = \lceil \frac{n}{2} \rceil - 1$, since in this case $|V-S| = n - (\lceil \frac{n}{2} \rceil - 1) = \lfloor \frac{n}{2} \rfloor + 1$.

Let Q be a largest set of vertices from $V-S$ that is not a gpa of $\overline{G}$. That is, there exists $x \in V$ such that $|\overline{N[x]} \cap Q| < |\overline{N[x]} - Q|$. Let $B = (V-S) - Q$ and $r = |B| \geq 1$. With respect to $\overline{G}$, let s_x, b_x, and q_x be the number of neighbors of vertex x in sets S, B, and Q, respectively. We now bound r depending upon the location of the vertex x.

1. $x \in S$. Then $b_x \leq r$, $\lambda_x = +1$, and, since $B \cup Q = V - S$ is a gpa of $\overline{G}$, $s_x + 1 \leq b_x + q_x + 2s - n + 1$. Since $|\overline{N[x]} \cap Q| < |\overline{N[x]} - Q|$ implies $q_x < (s_x + 1) + b_x$, $q_x < 2b_x + q_x + 2s - n + 1$, or $r \geq b_x \geq \lceil \frac{n}{2} \rceil - s$.

2. $x \in B$. In this case $b_x \leq r - 1$ and $\lambda_x = -1$. As in the previous case, since Q is not a gpa of $\overline{G}$, $q_x < s_x + 1 + b_x$. Since $B \cup Q$ is a gpa of $\overline{G}$ and, here, since $x \notin S$, we have $s_x \leq b_x + 1 + q_x + 2s - n - 1$. Then $r \geq b_x + 1 \geq \lceil \frac{n}{2} \rceil - s + 1$.

3. $x \in Q$. In this case $b_x \leq r$ and $\lambda_x = -1$. Also $s_x \leq b_x + q_x + 1 + 2s - n - 1$, and $q_x + 1 < s_x + b_x$. Together these imply $r \geq b_x \geq \lceil \frac{n}{2} \rceil - s + 1$.

Hence we must remove at least the minimum of the three cases, $\lceil \frac{n}{2} \rceil - s$ vertices, to obtain a subset of $V - S$ which is not a gpa of $\overline{G}$. From the above, a set obtained by removing any set of $\lceil \frac{n}{2} \rceil - s - 1$ vertices from $V - S$ is a dominating set of $\overline{G}$. Therefore $\gamma_{a_p}(\overline{G}) \leq n - s - (\lceil \frac{n}{2} \rceil - s - 1) = \lfloor \frac{n}{2} \rfloor + 1$.
□

Corollary 14 *For any graph G either* $\max\{\gamma_{a_p}(G), \gamma_{a_p}(\bar{G})\} \leq \lceil \frac{n+1}{2} \rceil$ *or* $\lfloor \frac{n+1}{2} \rfloor \leq \min\{\gamma_{a_p}(G), \gamma_{a_p}(\bar{G})\}$.

Surprisingly, determining useful upper bounds for $a_p(G)$ and $\gamma_{a_p}(G)$ appears to be difficult, even if consideration is restricted to regular graphs. We have been able to show such a bound for cubic graphs only, and we close this section with that result. Let S be a γ_{a_p}-set of a 3-regular graph G. Let A_i be the set of vertices in S having i neighbors in S, $1 \leq i \leq 3$, and $a_i = |A_i|$. Similarly let B_i be the set of vertices in $V - S$ having i neighbors in S, $2 \leq i \leq 3$, and $b_i = |B_i|$. Let $x \in A_3$. Then, since S is minimal, x must have a critical vertex neighbor $y \in A_1$, and the sum of the degrees of x and y in $\langle S \rangle$ is four, or an average of two per vertex. After finding such a pair for all vertices in A_3, all remaining vertices have degree at most two in

$\langle S \rangle$. Thus $\sum_{x \in S} deg_{\langle S \rangle}(x) \leq 2|S|$. Since $\sum_{x \in S} deg_G(x) = 3|S|$, it follows that the number of edges e between S and $V - S$ is bounded as $e \geq |S|$.

Next consider vertices $v \in B_3$. Every neighbor of such a v must either be a critical vertex or be adjacent to a critical vertex, and the adjacent critical vertex is not v since $v \in B_3$. There are two situations in which we can interchange two vertices to create a new gpa S' and which moves v from B_3 to B_2 with respect to S'. The two situations are:

1. v has a neighbor x which is a critical vertex, the remaining neighbor of x that is in $V - S$ is w, and the neighbor u of x that is in S is not a critical vertex. Then create $S' = (S - \{x\}) \cup \{w\}$. Observe that $|S'| = |S|$ and the degree rules for a gpa are true for S'. Thus S' is also a γ_{a_p}-set. It is possible for u to be a second neighbor of v. The transformation is illustrated in Part (1) of Figure 2 where vertices in S (S') appear at the bottom of the figure. Directed edges mean that the terminal vertex may be in either S or $V - S$.

2. v has a neighbor x which is not a critical vertex, exactly one of its two neighbors in S, say s, is a critical vertex with neighbors c and d in $V - S$, and its other neighbor u in S, of course, is not a critical vertex. As in the first case, the new γ_{a_p}-set S' is given by $S' = (S - \{x\}) \cup \{c\}$. Again it is possible for u to be a second neighbor of v. This transformation is shown in Part (2) of Figure 2.

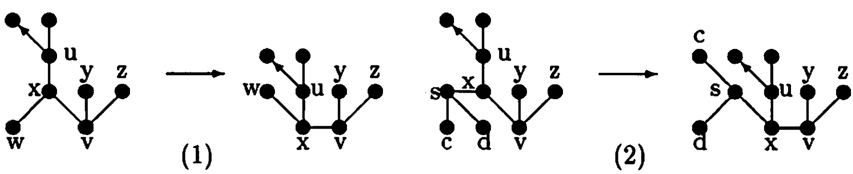

Figure 2: Transformations

Apply the above two transformations until it is no longer possible. For convenience we call the γ_{a_p}-set which results S. All comments and notation will now be with respect to this set, including the unique association of vertices in A_1 with vertices in A_3.

The next goal is to associate with each vertex remaining in B_3 a unique vertex of A_1 not associated with a vertex of A_3. Since the transformations can no longer be carried out, no neighbor of $v \in B_3$ satisfies the conditions of either of the two cases. Thus a neighbor x which is a critical vertex must have its neighbor in S be a critical vertex. On the other hand, if x is not a critical vertex, then its two neighbors in S must both be critical vertices. Consider the latter situation and let the two neighbors in S be a and b. We select one arbitrarily, say a, and let that be the unique vertex in A_1 that is associated with v. Since $x \in A_2$, a cannot already be associated with a vertex of A_3. Furthermore, no neighbor of a can be in B_3 since then the first transformation would have been applied. Thus vertex a can not be required for another association.

The only remaining case is if each neighbor of v is a critical vertex with its neighbor in S also a critical vertex. If x is such a neighbor of v we see it has not previously been associated with any vertex of $A_3 \cup B_3$. We consider the three neighbors of v to be candidates for the desired association. Construct a bipartite graph $B = (X, Y, \mathcal{E})$ where X is the set of vertices of B_3 which have not yet received an associated vertex of A_1, $Y = \cup_{v \in X} N(v)$ and $vx \in \mathcal{E}$ if and only if $x \in N(v)$. Let T be any subset of X having k vertices, $1 \leq k \leq |X|$. There are $3k$ edges between T and $N(T)$. The vertices of $N(T)$ can be endpoints of at most two of the edges. Thus $|N(T)| \geq \frac{3k}{2} > k = |T|$. By Hall's theorem (see [6]) there is a system of distinct representatives for X and we take them as the unique associated vertices from A_1.

Since we have found a unique association between vertices of A_1 with vertices of $A_3 \cup B_3$, $a_1 \geq a_3 + b_3$. Observe that $|S| = a_1 + a_2 + a_3$, $|V - S| = b_2 + b_3$, $e = 2b_2 + 3b_3 = 2(n - |S|) + b_3$, and $e = 2a_1 + a_2 = |S| + a_1 - a_3$. It follows that $|S| = 2n - 2|S| + b_3 - (a_1 - a_3) \leq 2n - 2|S| + b_3 - (a_3 + b_3 - a_3) = 2n - 2|S|$ which gives the following theorem.

Theorem 15 *For any 3-regular graph G, $\gamma_{a_p}(G) \leq \frac{2n}{3}$.*

An unsatisfactory demonstration of sharpness is illustrated by the graph shown in Figure 3. The four top vertices form a γ_{a_p}-set. However, we have been unable to find other examples of sharpness, and we wonder if this is unique.

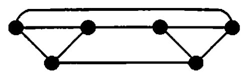

Figure 3: A graph for which $\gamma_{a_p} = \frac{2n}{3}$

4 Nordhaus-Gaddum Results for $\gamma_{a_p}(G)$

All results in this section correspond to $t = 0$. Some lower bounds are given first.

Theorem 16 *Let G be a graph. Then $\gamma_{a_p}(G) + \gamma_{a_p}(\overline{G}) \geq \left\lceil \frac{n + \Delta(G) - \delta(G) + 1}{2} \right\rceil$.*

Proof: From Observation 1, $\gamma_{a_p}(G) \geq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \geq \frac{\Delta(G)+1}{2}$ and $\gamma_{a_p}(\overline{G}) \geq \left\lceil \frac{\Delta(\overline{G})+1}{2} \right\rceil \geq \frac{\Delta(\overline{G})+1}{2} = \frac{n-1-\delta(G)+1}{2}$. Thus $\gamma_{a_p}(G)+\gamma_{a_p}(\overline{G}) \geq \frac{\Delta(G)+1+n-\delta(G)}{2}$ and the result follows. □

The graph G_s defined earlier shows sharpness of the bound for Theorem 16. For it, $\gamma_{a_p}(G_s) + \gamma_{a_p}(\overline{G_s}) = s + \frac{s^2+s}{2} = \frac{s^2+3s}{2}$. Furthermore, since $n = s^2 + s$, $\Delta(G_s) = 2s - 1$ and $\delta(G_s) = 1$, we have $\left\lceil \frac{n+\Delta(G_s)-\delta(G_s)+1}{2} \right\rceil = \left\lceil \frac{s^2+s+2s-1-1+1}{2} \right\rceil = \left\lceil \frac{s^2+3s-1}{2} \right\rceil$. Since $s^2 + 3s$ is even, this is the same as $\left\lceil \frac{s^2+3s}{2} \right\rceil = \frac{s^2+3s}{2}$.

The following theorem is better than Theorem 16 for some graphs.

Theorem 17 *Let G be a graph. Then* $\gamma_{a_p}(G) + \gamma_{a_p}(\overline{G}) \geq \gamma(G) + \gamma(\overline{G}) + \left\lceil \frac{n-\Delta(G)+\delta(G)-3}{2} \right\rceil$.

Proof: Let S be a smallest gpa of G and X any $\left\lfloor \frac{\delta(G)}{2} \right\rfloor$ vertices of S. Let $v \in V(G)$. If $v \notin S$, it has at least $\left\lceil \frac{deg(v)+1}{2} \right\rceil > \left\lfloor \frac{\delta(G)}{2} \right\rfloor$ neighbors in S and hence at least one in $S - X$. If $v \in X$, it has $\left\lceil \frac{deg(v)-1}{2} \right\rceil \geq \left\lfloor \frac{\delta(G)}{2} \right\rfloor$ neighbors in S and hence at least one in $S - X$. It follows that $S - X$ is a dominating set of G and $\gamma(G) \leq |S - X| \leq \gamma_{a_p}(G) - \left\lfloor \frac{\delta(G)}{2} \right\rfloor$. Applying the argument to $\overline{G}$ and combining, we have $\gamma(G) + \gamma(\overline{G}) \leq \gamma_{a_p}(G) + \gamma_{a_p}(\overline{G}) - \left(\left\lfloor \frac{\delta(G)}{2} \right\rfloor + \left\lfloor \frac{\delta(\overline{G})}{2} \right\rfloor\right) = \gamma_{a_p}(G)+\gamma_{a_p}(\overline{G})-\left(\left\lfloor \frac{\delta(G)}{2} \right\rfloor + \left\lfloor \frac{n-\Delta(G)-1}{2} \right\rfloor\right) \leq \gamma_{a_p}(G)+\gamma_{a_p}(\overline{G}) - \left\lceil \frac{n-\Delta(G)+\delta(G)-3}{2} \right\rceil$ and the result follows. □

Notice that Theorem 16 gives $\gamma_{a_p}(C_4) + \gamma_{a_p}(\overline{C_4}) \geq 3$ and $\gamma_{a_p}(C_5) + \gamma_{a_p}(\overline{C_5}) \geq 3$. The corresponding bounds from Theorem 17 are both 5 while the actual values are 5 and 8, respectively. The bound of Theorem 17 can be rewritten as $\gamma(G) + \gamma(\overline{G}) + \left\lceil \frac{n-\Delta(G)+\delta(G)-3}{2} \right\rceil = \left\lceil \frac{n+\Delta(G)-\delta(G)+1}{2} \right\rceil + \gamma(G)+\gamma(\overline{G})-(\Delta(G)-\delta(G)+2)$ where the first term on the right hand side is the bound of Theorem 16. Thus Theorem 16 represents an improvement over Theorem 17 whenever $\gamma(G) + \gamma(\overline{G}) - (\Delta(G) - \delta(G) + 2) < 0$. This occurs, for example, in $G = K_{1,n}$ when $n \geq 3$. In this case $\gamma(G)+\gamma(\overline{G}) = 3$ and $\Delta(G) - \delta(G) + 2 = n + 1$.

The bound of Theorem 16 for regular graphs is $\lceil \frac{n+1}{2} \rceil$. The next theorem shows a greatly improved bound for such graphs.

Theorem 18 *Let G be a d-regular graph of order n, $d \geq 1$. If n is odd, let $m = min\{d + 1, n - d\}$ and, if n is even, let m be whichever of $d + 1$*

and $n-d$ is odd. Then $\gamma_{a_p}(G)+\gamma_{a_p}(\overline{G}) \geq n+\lceil \frac{n}{2m} \rceil + \epsilon$ where $\epsilon = 1$ if n is odd and $\epsilon = 0$ if n is even.

Proof: Observation 2 shows that $\gamma_{a_p}(G) \geq \left\lceil \frac{n\lceil \frac{d+1}{2} \rceil}{d+1} \right\rceil \geq \lceil \frac{n}{2} \rceil$. Applying the same reasoning to the $(n-d-1)$-regular graph $\overline{G}$, we have $\gamma_{a_p}(\overline{G}) \geq \left\lceil \frac{n\lceil \frac{n-d}{2} \rceil}{n-d} \right\rceil \geq \lceil \frac{n}{2} \rceil$. Consider first that $n = 2k$ is even which implies exactly one of $d+1$ and $n-d$ is odd. Without loss of generality assume $d+1$ is odd. Then $\gamma_{a_p}(G)+\gamma_{a_p}(\overline{G}) \geq \left\lceil \frac{2k\lceil \frac{d+1}{2} \rceil}{d+1} \right\rceil + k = \left\lceil \frac{2k\frac{d+2}{2}}{d+1} \right\rceil + k = \left\lceil k + \frac{k}{d+1} \right\rceil + k = 2k + \left\lceil \frac{k}{d+1} \right\rceil = n + \lceil \frac{n}{2m} \rceil$. If $n = 2k+1$ is odd, both $d+1$ and $n-d$ are odd. Again without loss of generality assume $d+1 \leq n-d$. It follows that $\gamma_{a_p}(G)+\gamma_{a_p}(\overline{G}) \geq \left\lceil \frac{(2k+1)\lceil \frac{d+1}{2} \rceil}{d+1} \right\rceil + \left\lceil \frac{(2k+1)\lceil \frac{n-d}{2} \rceil}{n-d} \right\rceil = \left\lceil \frac{(2k+1)\frac{d+2}{2}}{d+1} \right\rceil + \left\lceil \frac{(2k+1)\frac{n-d+1}{2}}{n-d} \right\rceil = \left\lceil k + \frac{1}{2} + \frac{2k+1}{2(d+1)} \right\rceil + \left\lceil k + \frac{1}{2} + \frac{2k+1}{2(n-d)} \right\rceil \geq k + \left\lceil \frac{2k+1}{2(d+1)} \right\rceil + k + \left\lceil \frac{1}{2} + \frac{2k+1}{2(n-d)} \right\rceil = 2k + \lceil \frac{n}{2m} \rceil + \left\lceil \frac{1}{2} + \frac{n}{2(n-d)} \right\rceil$. It is easy to see that $\frac{n+1}{2} \leq n-d \leq n-1$ which implies $\frac{1}{2} < \frac{n}{2(n-d)} < 1$ so $\left\lceil \frac{1}{2} + \frac{n}{2(n-d)} \right\rceil = 2$. Therefore $\gamma_{a_p}(G)+\gamma_{a_p}(\overline{G}) \geq 2k + \lceil \frac{n}{2m} \rceil + 2 = n + \lceil \frac{n}{2m} \rceil + 1$. □

The bound of Theorem 18 is sharp when n is even. To see this, consider C_n where $n = 12k$ for some positive integer k. Label the vertices in order by $0, 1, \ldots, n-1$. Since $d = 2$, $m = 3$ and the theorem yields $\gamma_{a_p}(C_n) + \gamma_{a_p}(\overline{C_n}) \geq n + \lceil \frac{n}{6} \rceil = \frac{7n}{6}$. It is known that $\gamma_{a_p}(C_n) = \frac{2n}{3}$ (see [1]) and, from the proof to the theorem, $\gamma_{a_p}(\overline{C_n}) \geq \frac{n}{2}$. We will show equality holds in this latter case. Let $S = \{0, 1, 4, 5, 8, 9, \ldots, n-4, n-3\}$. Notice that $|S| = \frac{n}{2}$ and for any vertex v, $|\overline{N[v]} \cap S| = \frac{n}{2} - 1$ so S dominates. The degree of every vertex of $\overline{C_n}$ is $n-d-1 = n-3$ so S will be a gpa if every vertex has its closed neighborhood contain at least $\lceil \frac{n-d}{2} \rceil = \frac{n}{2} - 1$ vertices of S. We have just seen that this is the case. Thus $\gamma_{a_p}(C_n) + \gamma_{a_p}(\overline{C_n}) = \frac{2n}{3} + \frac{n}{2} = \frac{7n}{6}$ and the theorem's bound is seen to be sharp. Similarly, $C_n = C_{12k+1}$ can be used to show sharpness when n is odd. We know $\gamma_{a_p}(C_n) = \lceil \frac{2n}{3} \rceil = \lceil \frac{24k+2}{3} \rceil = 8k+1$. Label the vertices on the cycle as before in order from 0 to $n-1$ and let $S = \{0, 1, 4, 5, 8, 9, \ldots, n-5, n-4, n-2, n-1\}$. It is easy to see that $|S| = 6k+2$ and S is a gpa. Thus $\gamma_{a_p}(\overline{C_{12k+1}}) \leq 6k+2$. Hence $\gamma_{a_p}(C_n) + \gamma_{a_p}(\overline{C_n}) \leq 8k+1+6k+2 = 14k+3$. Using the result of the Theorem 18 we calculate $\gamma_{a_p}(C_n) + \gamma_{a_p}(\overline{C_n}) \geq 12k+2+\lceil \frac{12k+1}{6} \rceil = 12k+2+2k+1 = 14k+3$. Therefore the set S must be a minimum gpa for $\overline{C_{12k+1}}$ and again we see the theorem is sharp.

We close this section with two simple upper bounds.

Observation 19 *Let G be a graph. Then $a_p(G) + a_p(\overline{G}) \leq \gamma_{a_p}(G) + \gamma_{a_p}(\overline{G}) \leq \left\lfloor \frac{3n+\Delta(G)-\delta(G)+1}{2} \right\rfloor$.*

Proof: From Observation 9 we have $\gamma_{a_p}(G) \leq n - \left\lceil \frac{\delta(G)}{2} \right\rceil$ and $\gamma_{a_p}(\overline{G}) \leq n - \left\lceil \frac{\delta(\overline{G})}{2} \right\rceil = n - \left\lceil \frac{n-\Delta(G)-1}{2} \right\rceil$. Thus $\gamma_{a_p}(G) + \gamma_{a_p}(\overline{G}) \leq 2n - \frac{\delta(G)}{2} - \frac{n-\Delta(G)-1}{2} = \frac{3n+\Delta(G)-\delta(G)+1}{2}$. □

Corollary 20 *If G is a regular graph, $a_p(G)+a_p(\overline{G}) \leq \gamma_{a_p}(G)+\gamma_{a_p}(\overline{G}) \leq \lfloor \frac{3n+1}{2} \rfloor$.*

The bound of Corollary 20 is achieved for C_5.

5 Properties of Excess-t Global Powerful Alliances for Regular Graphs

Throughout this section all graphs are regular, with degree d, unless otherwise stated. Let S be any minimal excess-t gpa of graph G. It follows from a comment in the Introduction that $\delta(\langle S\rangle) \geq \lceil \frac{d-1+t}{2} \rceil$ and $\Delta(\langle V-S\rangle) \leq d - \lceil \frac{d+1+t}{2} \rceil = \lfloor \frac{d-1-t}{2} \rfloor$. This means that each vertex of $V-S$ has at least $\lceil \frac{d+1+t}{2} \rceil$ neighbors in S, so $|S| \geq \lceil \frac{d+1+t}{2} \rceil$. Furthermore, since S has at least one critical vertex, $\delta(\langle S\rangle) = \lceil \frac{d-1+t}{2} \rceil$. Such a vertex has exactly $\lfloor \frac{d+1-t}{2} \rfloor$ neighbors in $V-S$ which implies $|V-S| \geq \lfloor \frac{d+1-t}{2} \rfloor$ and $|S| \leq n - \lfloor \frac{d+1-t}{2} \rfloor$.

The following observation and its corollary show that $\gamma_{a_p}(G,d,t)$ is monotonic in t.

Observation 21 *For any d-regular graph G and $-d \leq t \leq d$, S is an excess-i gpa for any i such that $-d \leq i \leq t$.*

Proof: This is immediate if $t = -d$, so assume $t > -d$. Observe that $\delta(\langle S\rangle) = \lceil \frac{d-1+t}{2} \rceil \geq \lceil \frac{d-1+i}{2} \rceil$ and $\Delta(\langle V-S\rangle) \leq \lfloor \frac{d-1-t}{2} \rfloor \leq \lfloor \frac{d-1-i}{2} \rfloor$. Thus S is an excess-i gpa. □

Corollary 22 *For any d-regular graph G and $-d < t \leq d$, $\gamma_{a_p}(G,d,t-1) \leq \gamma_{a_p}(G,d,t)$.*

We have noted that the definition of an excess-t gpa reduces to that of a standard gpa when $t = 0$, so $\gamma_{a_p}(G,d,0) = \gamma_{a_p}(G)$. More generally, if $m = \lceil \frac{d+1+t}{2} \rceil$, $\gamma_{a_p}(G,d,t) \geq \gamma_m(G)$ where γ_m is the *m-domination number*, since every vertex of $V-S$ has at least m neighbors in S. It follows that

$\gamma_{a_p}(G,d,-d) \geq \gamma_1(G) = \gamma(G)$, the domination number of G. Recall that $\gamma_{a_p}(G,d,d) = n$ so we usually assume $t < d$.

When $t = d-1$ or $t = d-2$, $m = \lceil \frac{d+1+t}{2} \rceil = d$ which implies $V-S$ is an independent set. Thus, using Corollary 22, $\gamma_{a_p}(G,d,d-1) \geq \gamma_{a_p}(G,d,d-2) \geq \gamma_d(G) = \alpha_0(G)$, the *vertex cover number* of G. This suggests a relationship to $p(G)$, the *packing number* of G, which is the maximum number of vertices that are pairwise at least distance three apart. See [11] for more information on these parameters.

Theorem 23 *For any d-regular graph G,* $\gamma_{a_p}(G,d,d-1) = \gamma_{a_p}(G,d,d-2) = n - p(G) \geq \alpha_0(G)$.

Proof: Let $\hat{S}$ be a $\gamma_{a_p}(G,d,d-2)$-set. The paragraph preceding the theorem shows $V-\hat{S}$ is independent. Furthermore, $\delta(\langle \hat{S} \rangle) = d-1$ so no vertex of $\hat{S}$ has two neighbors in $V-\hat{S}$. It follows that each pair of vertices in $V-\hat{S}$ is at least distance three apart and hence $\gamma_{a_p}(G,d,d-1) \geq \gamma_{a_p}(G,d,d-2) \geq n-p(G)$. Also $V-X$ for any maximum packing X is an excess-$(d-1)$ gpa and, using Corollary 22, $\gamma_{a_p}(G,d,d-2) \leq \gamma_{a_p}(G,d,d-1) \leq n-p(G)$, establishing the result. □

Corollary 24 *For any d-regular graph G,* $\gamma_{a_p}(G) + p(G) \leq n$. *If* $d \leq 2$, $\gamma_{a_p}(G) + p(G) = n$.

Proof: From Theorem 23, if $d \geq 3$, $\gamma_{a_p}(G) = \gamma_{a_p}(G,d,0) \leq \gamma_{a_p}(G,d,d-2) = n - p(G)$. When $d = 1$, $\gamma_{a_p}(G) = p(G) = \frac{n}{2}$. When $d = 2$, $\gamma_{a_p}(G) = \gamma_{a_p}(G,d,0) = \gamma_{a_p}(G,d,d-2) = n - p(G)$. □

Lemma 25 *Let G be a d-regular graph. If* $-d+1 \leq t \leq d$, *then* $\gamma_{a_p}(G,d,t-1) = \gamma_{a_p}(G,d,t)$ *if and only if* $d+t$ *is odd.*

Proof: Suppose $d+t$ is even. Let S be a $\gamma_{a_p}(G,d,t)$-set and $x \in S$. Then $\delta(\langle S \rangle - x) \geq \delta(\langle S \rangle) - 1 = \lceil \frac{d-1+t}{2} \rceil - 1 = \lceil \frac{d-1+t-2}{2} \rceil = \lceil \frac{d-1+t-1}{2} \rceil$ where the last equality follows since $d+t$ is even. Similarly, $\Delta(\langle V-S \rangle + x) \leq \lfloor \frac{d-1-t}{2} \rfloor + 1 = \lfloor \frac{d-1-t+1}{2} \rfloor$. Therefore, $S - \{x\}$ is an excess-$(t-1)$ gpa so $\gamma_{a_p}(G,d,t-1) < \gamma_{a_p}(G,d,t)$.

Assume next that $d+t$ is odd. Let $\hat{S}$ be a $\gamma_{a_p}(G,d,t-1)$-set. Since $d+t$ is odd, $\delta(\langle \hat{S} \rangle) = \left\lceil \frac{d-1+(t-1)}{2} \right\rceil = \lceil \frac{d-1+t}{2} \rceil$ and $\Delta(\langle V-\hat{S} \rangle) \leq \left\lfloor \frac{d-1-(t-1)}{2} \right\rfloor = \lfloor \frac{d-1-t}{2} \rfloor$. These imply $\hat{S}$ is an excess-t gpa from which it follows that $\gamma_{a_p}(G,d,t) \leq |\hat{S}| = \gamma_{a_p}(G,d,t-1)$. □

Observation 21, Theorem 23, and Lemma 25 lead immediately to the following result.

Observation 26 *Let G be a d-regular graph and i an integer such that $0 \leq i \leq d-1$. Then $\gamma_{a_p}(G,d,2i-d) = \gamma_{a_p}(G,d,2i-d+1) < \gamma_{a_p}(G,d,2i-d+2)$. Furthermore, $\gamma_{a_p}(G,d,-d) = \gamma(G)$ and $\gamma_{a_p}(G,d,d-1) = n - p(G) < \gamma_{a_p}(G,d,d) = n$.*

Corollary 22, Lemma 25, and Observation 26 have similar counterparts for $\tau(n,d,t)$ and $T(n,d,t)$. The results for $\tau(n,d,t)$ are virtually identical.

Observation 27 *Let (n,d,t) be a feasible triple.*

1. *For t such that $-d < t \leq d$, $\tau(n,d,t-1) \leq \tau(n,d,t)$.*
2. *If $-d+1 \leq t \leq d$, then $\tau(n,d,t-1) = \tau(n,d,t)$ if and only if $d+t$ is odd.*
3. *Let i be an integer such that $0 \leq i \leq d-1$. Then $\tau(n,d,2i-d) = \tau(n,d,2i-d+1) < \tau(n,d,2i-d+2)$. Furthermore, $\tau(n,d,-d) = \min\{\gamma(G) : G$ is d-regular with n vertices$\}$, and $\tau(n,d,d-1) = n - \max\{p(G) : G$ is d-regular with n vertices$\} < \tau(n,d,d) = n$.*

Proof: The proofs for 1, 2, and the first part of 3 follow directly from those of Observation 21, Corollary 22, Lemma 25, and Observation 26, respectively, where one now argues in tems of minimum values over all d-regular graphs on n vertices. For the latter part of 3 we have to select the smallest value of $\gamma(G)$ and the largest value of $p(G)$ in order to minimize the value of γ_{a_p} over all d-regular graphs G on n vertices. □

A complete parallel is not possible for $T(n,d,t)$. The *upper domination number* $\Gamma(G)$ (see [11]) of graph G is the cardinality of a largest minimal dominating set of G.

Observation 28 *Let (n,d,t) be a feasible triple.*

1. *For t such that $-d < t \leq d$, $T(n,d,t-1) \leq T(n,d,t)$.*
2. *If $-d+1 \leq t \leq d$, then $T(n,d,t-1) = T(n,d,t)$ if $d+t$ is odd.*
3. *$T(n,d,-d) = \max\{\Gamma(G) : G$ is d-regular with n vertices$\}$, and $T(n,d,d-1) = n - \min\{p(G) : G$ is d-regular with n vertices$\} < T(n,d,d) = n$.*

Proof: Again the proofs for 1 and 2 follow directly from those of Observation 21, Corollary 22, and Lemma 25, respectively, where the largest values of minimal gpa's are employed. For 3 we recognize $\Gamma(G)$ is an excess-$(-d)$ gpa. Therefore we select the largest value of $\Gamma(G)$ in order to maximize the value of Γ_{a_p} over all d-regular graphs G on n vertices. Similarly, we must minimize the value of $p(G)$ in order to maximize Γ_{a_p}. □

The following corollary is an immediate consequence of Lemma 25, Observation 26, and the facts that $\gamma(G) = \gamma_{a_p}(n, d, -d)$, $\gamma_{a_p}(G) = \gamma_{a_p}(n, d, 0)$, and $n - p(G) = \gamma_{a_p}(n, d, d-1) < \gamma_{a_p}(n, d, d)$.

Corollary 29 *Let G be a d-regular graph with $d \geq 2$. Then $\gamma(G) + \lfloor \frac{d}{2} \rfloor \leq \gamma_{a_p}(G) \leq n - p(G) - \lceil \frac{d}{2} \rceil - 1$.*

6 Extremal Graphs for $\tau(n, d, t)$

This section determines the value for $\tau(n, d, t)$ and constructs for each feasible triple (n, d, t) a d-regular graph G such that $\gamma_{a_p}(G, d, t) = \tau(n, d, t)$. From Observation 2 we have $\tau(n, d, t) \geq \left\lceil \frac{n \lceil \frac{d+1+t}{2} \rceil}{d+1} \right\rceil$ for any d-regular graph G.

We define a graph with n vertices and m edges to be *nearly regular* if the degree of each vertex is either $\lfloor \frac{2e}{n} \rfloor$ or $\lceil \frac{2e}{n} \rceil$. The next lemma indicates a construction technique for forming nearly regular graphs containing a stated number of edges. It is based on the fact that K_n has an edge decomposition into $\frac{n-1}{2}$ Hamiltonian cycles if n is odd and into $\frac{n-2}{2}$ Hamiltonian cycles and a 1-factor if n is even (see [2], pp 203, 206).

Lemma 30 *Let n and d be positive integers such that nd is even and $d \leq n-1$, and e a nonnegative integer such that $2e \leq nd$. Then a graph G having n vertices and e edges can be constructed in such a way that $\delta = \lfloor \frac{2e}{n} \rfloor$ and $\Delta = \lceil \frac{2e}{n} \rceil$.*

Proof: Let $e = mn + r$ where $0 \leq r \leq n - 1$. It is easy to see that $\lfloor \frac{2e}{n} \rfloor = \begin{cases} 2m & \text{if } 2r < n \\ 2m+1 & \text{if } 2r \geq n \end{cases}$. Construct G as follows:

1. Create a $2m$-regular graph on the vertices of G by incorporating m of the Hamiltonian cycles in the decomposition.

2. If $r \leq \lfloor \frac{n}{2} \rfloor$, add r independent edges taken from a remaining Hamiltonian cycle or 1-factor. Every vertex has degree $2m$ or $2m + 1$ (only $2m + 1$ if $r = \frac{n}{2}$).

3. If $r > \lfloor \frac{n}{2} \rfloor$, there must be another Hamiltonian cycle in the decomposition. Add in $\lfloor \frac{n}{2} \rfloor$ independent edges taken from that cycle. At this point, every vertex has degree $2m + 1$ except possibly one of degree $2m$ if n is odd. Now there are $r - \lfloor \frac{n}{2} \rfloor \leq n - 1 - \lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil - 1$ edges still to be included. These can be selected from the missing edges in this last Hamiltonian cycle, taking care to have one, and only one, of them incident to the vertex of degree $2m$ if it exists. Every vertex has degree $2m + 1$ or $2m + 2$. □

References in this and the following section to creating nearly regular graphs refer to employing the technique of Lemma 30. We now construct graphs which show that the lower bound for $\tau(n,d,t)$ given above can be achieved except for one special case for which that bound must be increased by one. Suppose S is an excess-t gpa of $G = (V,E)$. Let $s = |S|$; e_S and e_{V-S} be the number of edges in $\langle S\rangle$ and $\langle V-S\rangle$, respectively; and e be the number of edges joining a vertex of S and a vertex of $V-S$. It is immediate that $|E| = \frac{nd}{2} = e_S + e_{V-S} + e$, $s\delta(\langle S\rangle) \leq 2e_S$, and $2e_{V-S} \leq (n-s)\Delta(\langle V-S\rangle)$. Several preliminary results are given next.

Observation 31 *Let S be an excess-t gpa of G.*

1. $0 \leq \Delta(\langle V-S\rangle) \leq \lfloor \frac{d-1-t}{2} \rfloor$.
2. $\lceil \frac{d-1+t}{2} \rceil \leq \delta(\langle S\rangle) < d$.
3. $(n-s)(d-\Delta(\langle V-S\rangle)) \leq (n-s)d - 2e_{V-S} = e = sd - 2e_S \leq s(d-\delta(\langle S\rangle))$.
4. $s\delta(\langle S\rangle) \leq 2e_S = 2e_{V-S} + 2sd - nd \leq (n-s)\Delta(\langle V-S\rangle) + 2sd - nd$.
5. $s \geq \left\lceil \frac{n(d-\Delta(\langle V-S\rangle))}{2d-\Delta(\langle V-S\rangle)-\delta(\langle S\rangle)} \right\rceil + \epsilon$ *where* $\epsilon = 1$ *when both of* $\delta(\langle S\rangle)$ *and* $\frac{n(d-\Delta(\langle V-S\rangle))}{2d-\Delta(\langle V-S\rangle)-\delta(\langle S\rangle)}$ *are odd integers and* $\epsilon = 0$ *otherwise.*

Proof: Parts 1 and 2 are requirements any excess-t gpa must satisfy. The equalities of Part 3 reflect the fact that both S and $V-S$ must have the same number e of end points of the edges between them while the inequalities provide obvious lower and upper bounds on this number. The first inequality and the equality of Part 4 follow immediately from Part 3 and the final inequality arises since $2e_{V-S} \leq (n-s)\Delta(\langle V-S\rangle)$. For Part 5 we solve for s from the inequality between the first and last terms of Part 4 to produce the given lower bound, except for the ϵ. When there is equality throughout Part 4, both $s\delta(\langle S\rangle)$ and $(n-s)\Delta(\langle V-S\rangle)$ must be even. Furthermore, $s = \frac{n(d-\Delta(\langle V-S\rangle))}{2d-\Delta(\langle V-S\rangle)-\delta(\langle S\rangle)}$. Alternatively, if $\frac{n(d-\Delta(\langle V-S\rangle))}{2d-\Delta(\langle V-S\rangle)-\delta(\langle S\rangle)}$ is an integer, s will be equal to it and we have equality throughout Part 4. But if $\frac{n(d-\Delta(\langle V-S\rangle))}{2d-\Delta(\langle V-S\rangle)-\delta(\langle S\rangle)}$ and $\delta(\langle S\rangle)$ are both odd, we cannot have such equality and it is impossible to construct a graph, Thus the lower bound on s must be at least one larger in this case which accounts for the need of ϵ. □

We now can show the value of $\tau(n,d,t)$.

Theorem 32 *For feasible triples (n,d,t), $\tau(n,d,t) = \left\lceil \frac{n\lceil \frac{d+1+t}{2}\rceil}{d+1} \right\rceil + \epsilon$ where $\epsilon = 1$ when both $\lceil \frac{d-1+t}{2} \rceil$ and $\frac{n\lceil \frac{d+1+t}{2}\rceil}{d+1}$ are odd integers and $\epsilon = 0$ otherwise.*

Proof: The quantity $\left\lceil \frac{n(d-\Delta(\langle V-S\rangle))}{2d-\Delta(\langle V-S\rangle)-\delta(\langle S\rangle)} \right\rceil + \epsilon$ is easily checked to be a decreasing function as either $\Delta(\langle V-S\rangle)$ increases or $\delta(\langle S\rangle)$ decreases. In particular, then, the inequality of Observation 31 Part 5 must hold when $\Delta(\langle V-S\rangle)$ has its largest possible value of $\lfloor \frac{d-1-t}{2} \rfloor$ and $\delta(\langle S\rangle)$ has its smallest possible value of $\lceil \frac{d-1+t}{2} \rceil$. Substituting these limiting values into Observation 31 Part 5 shows $\tau(n,d,t) \geq \left\lceil \frac{n\lceil \frac{d+1+t}{2} \rceil}{d+1} \right\rceil + \epsilon$ where $\epsilon = 1$ when both $\lceil \frac{d-1+t}{2} \rceil$ and $\frac{n\lceil \frac{d+1+t}{2} \rceil}{d+1}$ are odd integers and $\epsilon = 0$ otherwise.

Next we show this lower bound can be achieved by demonstrating the existence of d-regular graphs on n vertices having an excess-t gpa with $\left\lceil \frac{n\lceil \frac{d+1+t}{2} \rceil}{d+1} \right\rceil + \epsilon$ vertices. Suppose first that $\epsilon = 0$ and let S be a set of $s = \left\lceil \frac{n\lceil \frac{d+1+t}{2} \rceil}{d+1} \right\rceil$ vertices. We attempt to construct a nearly regular graph on S such that $\delta(\langle S\rangle) = \lceil \frac{d-1+t}{2} \rceil$. For such a graph, e_S must satisfy

$$s\left\lceil \frac{d-1+t}{2} \right\rceil \leq 2e_S < s\left(\left\lceil \frac{d-1+t}{2} \right\rceil + 1\right). \tag{1}$$

Simultaneously we want a nearly regular graph on $V-S$ where $\Delta(\langle V-S\rangle) = \lfloor \frac{d-1-t}{2} \rfloor$ and, because of Observation 31 Part 4, $2e_{V-S} = nd - 2sd + 2e_S$. Thus we need

$$(n-s)\left(\left\lfloor \frac{d-1-t}{2} \right\rfloor - 1\right) < 2e_{V-S} \leq (n-s)\left\lfloor \frac{d-1-t}{2} \right\rfloor. \tag{2}$$

A graph can be constructed if a value for e_S can be found satisfying Equation 1 which also allows satisfaction of Equation 2. Observation 31 Part 4 shows the right inequality of Equation 2 is satisfied when $2e_S$ assumes its minimum value of $s\lceil \frac{d-1+t}{2} \rceil$. If the left inequality also holds, we will be able to construct a graph. If it doesn't, increase e_S, always maintaining the equality $2e_S = 2e_{V-S} + 2sd - nd$. Every increase of one of e_S corresponds to an increase of one in e_{V-S}. We now show that the left inequality of Equation 2 will hold when $2e_S = s\left(\lceil \frac{d-1+t}{2} \rceil + 1\right) - 1$, its maximum possible value. This means that at some point in the increasing of e_S there is a value of it in the range of Equation 1 and a corresponding value of $2e_{V-S} = 2e_S - 2sd + nd$ in the range of Equation 2.

By way of contradiction, assume $2e_{V-S} = 2e_S - 2sd + nd = s\left(\lceil \frac{d-1+t}{2} \rceil + 1\right) - 1 - 2sd + nd$ and $2e_{V-S} \leq (n-s)\left(\lfloor \frac{d-1-t}{2} \rfloor - 1\right)$. Combining these to form $s\left(\lceil \frac{d-1+t}{2} \rceil + 1\right) - 1 - 2sd + nd \leq (n-s)\left(\lfloor \frac{d-1-t}{2} \rfloor - 1\right)$ and simplifying gives $n - 1 \leq (d+1)\left(s - \frac{n\lceil \frac{d+1+t}{2} \rceil}{d+1}\right)$. Since $s = \left\lceil \frac{n\lceil \frac{d+1+t}{2} \rceil}{d+1} \right\rceil$, the multiplier of $d+1$ is less than one so it follows that the right hand side of the inequality is at most d. Thus $d \leq n-1 \leq (d+1)\left(s - \frac{n\lceil \frac{d+1+t}{2} \rceil}{d+1}\right) \leq d$. It follows that $d = n-1$ which means that $n \lceil \frac{d+1+t}{2} \rceil$ is divisible by $d+1 = n$. Therefore, $s = \frac{n\lceil \frac{d+1+t}{2} \rceil}{d+1}$ so $d = n-1 = (d+1)\left(s - \frac{n\lceil \frac{d+1+t}{2} \rceil}{d+1}\right) = 0$ which is a contradiction for nontrivial graphs. Thus there is some value of e_S satisfying Equation 1 and $2e_{V-S} = 2e_S - 2sd + nd$ which results in e_{V-S} satisfying Equation 2. Therefore an extremal graph can be constructed and the theorem holds when $\epsilon = 0$.

As indicated previously, it is not possible to construct an excess-t gpa on $\frac{n\lceil \frac{d+1+t}{2} \rceil}{d+1}$ vertices when $\epsilon = 1$. However, since $(s+1)\lceil \frac{d-1+t}{2} \rceil$ is even, it is possible in this case to construct one having $\frac{n\lceil \frac{d+1+t}{2} \rceil}{d+1} + 1$ vertices in the manner described above with s increased by one. □

As a corollary, there is an interesting relation between certain lower and upper excess-t gpa numbers.

Corollary 33 *If $t \leq d-2$ and $d+t$ is odd, $\tau(n,d,t) + \tau(n,d,-t-2) \leq n \leq \tau(n,d,t) + T(n,d,-t-2)$. If $t \leq d-3$ and $d+t$ is even, $\tau(n,d,t) + \tau(n,d,-t-3) \leq n \leq \tau(n,d,t) + T(n,d,-t-3)$.*

Proof: Interpret the set $V - S$ resulting from the construction leading to Theorem 32 as an excess-$\hat{t}$ gpa for some value of $\hat{t}$. Let us actually take $\hat{t}$ to be the largest integer for which $V - S$ is an excess-$\hat{t}$ gpa. Recall $\delta(\langle V - S \rangle) \geq \lfloor \frac{d-1-t}{2} \rfloor - 1$ and $\Delta(\langle S \rangle) \leq \lceil \frac{d-1+t}{2} \rceil + 1$. The requirements of an excess-$\hat{t}$ gpa will be most restrictive for vertices of $V - S$ and S, respectively, whose degrees are equal to these limiting values. Thus, for vertices of minimum degree of $V - S$, we must have $\lfloor \frac{d-1-t}{2} \rfloor - 1 + 1 \geq \lceil \frac{d+1+t}{2} \rceil + 1 + \hat{t}$. Similarly for vertices of maximum degree of S we must have $\lfloor \frac{d-1-t}{2} \rfloor \geq \lceil \frac{d-1+t}{2} \rceil + 1 + 1 + \hat{t}$. Suppose $d+t$ is odd. Evaluating the two inequalities shows that $\hat{t} \leq -t-2$. Thus $V - S$ is an excess-$(-t-2)$ gpa and we have $\tau(n,d,-t-2) \leq n - \tau(n,d,t) \leq T(n,d,-t-2)$ which gives the result in this case. The argument when $d+t$ is even is completely analogous. □

7 Extremal Graphs for $T(n,d,t)$

In this section we determine $T(n,d,t)$ for all feasible triples (n,d,t). Of course the smallest possible value for n is $d+1$ in which case $G = K_{d+1}$, $|S| = \lceil \frac{d+1+t}{2} \rceil$, and $|V - S| = \lfloor \frac{d+1-t}{2} \rfloor$. Suppose graph G has an excess-t gpa of cardinality $n-k$, that is, $|V - S| = k$. Since each vertex of $V - S$

must have at least $\lceil \frac{d+1+t}{2} \rceil$ neighbors in S and each vertex of S can have at most $\lfloor \frac{d+1-t}{2} \rfloor$ neighbors in $V - S$, the number of vertices in S must be at least $\left\lceil \frac{k \lceil \frac{d+1+t}{2} \rceil}{\lfloor \frac{d+1-t}{2} \rfloor} \right\rceil$. The next lemma determines an upper bound for $|S|$.

Lemma 34 *For a feasible triple (n, d, t), let $G = (V, E)$ be an arbitrary d-regular graph on n vertices and let $S \subseteq V$ be any minimal excess-t gpa of G. Then, with $k = n - |S|$, $|S| \leq \left\lfloor \frac{kd}{\lfloor \frac{d+1-t}{2} \rfloor} \right\rfloor \lceil \frac{d+1+t}{2} \rceil$.*

Proof: Suppose c is the number of vertices of $V - S$ that are critical vertices and have at least one neighbor in S that is not a critical vertex. These vertices have a total of $c \lceil \frac{d+1+t}{2} \rceil$ edges to S. Suppose further that α of these edges lead to critical vertices. Then these c vertices can act as critical vertices for at most $c \lceil \frac{d+1+t}{2} \rceil - \alpha$ vertices of S other than ones that are critical vertices. By the definition of c, each of the c vertices has at most $\lceil \frac{d-1+t}{2} \rceil$ edges terminating at critical vertices in S. Thus $\alpha \leq c \lceil \frac{d-1+t}{2} \rceil$.

It follows from the above that the number of edges from $V - S$ terminating at critical vertices in S is at most $(k-c)d+\alpha$, implying the number of critical vertices in S is at most $\left\lfloor \frac{(k-c)d+\alpha}{\lfloor \frac{d+1-t}{2} \rfloor} \right\rfloor$. These allow $\left\lfloor \frac{(k-c)d+\alpha}{\lfloor \frac{d+1-t}{2} \rfloor} \right\rfloor \lceil \frac{d+1+t}{2} \rceil$ vertices in S. Now we can compute an upper bound for $|S|$ as follows:

$$
\begin{aligned}
|S| &\leq \left\lfloor \frac{(k-c)d+\alpha}{\lfloor \frac{d+1-t}{2} \rfloor} \right\rfloor \left\lceil \frac{d+1+t}{2} \right\rceil + c \left\lceil \frac{d+1+t}{2} \right\rceil - \alpha \\
&\leq \left\lfloor \frac{(k-c)d+\alpha}{\lfloor \frac{d+1-t}{2} \rfloor} \right\rfloor \left\lceil \frac{d+1+t}{2} \right\rceil + c \left\lceil \frac{d+1+t}{2} \right\rceil \\
&= \left\lfloor \frac{(k-c)d+\alpha}{\lfloor \frac{d+1-t}{2} \rfloor} + c \right\rfloor \left\lceil \frac{d+1+t}{2} \right\rceil \\
&= \left\lfloor \frac{kd - cd + \alpha + c \lfloor \frac{d+1-t}{2} \rfloor}{\lfloor \frac{d+1-t}{2} \rfloor} \right\rfloor \left\lceil \frac{d+1+t}{2} \right\rceil \\
&= \left\lfloor \frac{kd + \alpha - c(d - \lfloor \frac{d+1-t}{2} \rfloor)}{\lfloor \frac{d+1-t}{2} \rfloor} \right\rfloor \left\lceil \frac{d+1+t}{2} \right\rceil \\
&= \left\lfloor \frac{kd + \alpha - c \lceil \frac{d-1+t}{2} \rceil}{\lfloor \frac{d+1-t}{2} \rfloor} \right\rfloor \left\lceil \frac{d+1+t}{2} \right\rceil \\
&\leq \left\lfloor \frac{kd}{\lfloor \frac{d+1-t}{2} \rfloor} \right\rfloor \left\lceil \frac{d+1+t}{2} \right\rceil .
\end{aligned}
$$

□

Lemma 34 allows determination of the value for $T(n,d,t)$.

Theorem 35 *For a feasible triple (n,d,t),*

$$T(n,d,t) = \min\left\{n - \left\lfloor\frac{d+1-t}{2}\right\rfloor, \left\lfloor\frac{md}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rfloor\left\lceil\frac{d+1+t}{2}\right\rceil\right\}$$

where m is the smallest integer for which $n \le m + \left\lfloor\frac{md}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rfloor\left\lceil\frac{d+1+t}{2}\right\rceil$.

Proof: The fact that the first expression is an upper bound follows from remarks in Section 5. The second is an upper bound because of Lemma 34. We now show that equality holds by describing the construction of a d-regular graph on n vertices possessing a minimal excess-t gpa having $q = \min\left\{n - \lfloor\frac{d+1-t}{2}\rfloor, \left\lfloor\frac{md}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rfloor\lceil\frac{d+1+t}{2}\rceil\right\}$ vertices.

Let S and $V-S$ be sets of q and $k = n-q$ vertices, respectively. We have seen that, if S is to be an excess-t gpa, at least $k\lceil\frac{d+1+t}{2}\rceil$ edges must join S and $V-S$ and at most $q\lfloor\frac{d+1-t}{2}\rfloor$ such edges can exist. This implies we must have $q \ge \left\lceil\frac{k\lceil\frac{d+1+t}{2}\rceil}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rceil$. Suppose instead, for q as determined above, that $q < \left\lceil\frac{k\lceil\frac{d+1+t}{2}\rceil}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rceil$. When $q = n-\lfloor\frac{d+1-t}{2}\rfloor$, $k = \lfloor\frac{d+1-t}{2}\rfloor$ and hence $n < \lfloor\frac{d+1-t}{2}\rfloor + \lceil\frac{d+1+t}{2}\rceil = d+1$, a contradiction. Now assume $\left\lfloor\frac{md}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rfloor\lceil\frac{d+1+t}{2}\rceil = q < \left\lceil\frac{k\lceil\frac{d+1+t}{2}\rceil}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rceil$. Since $k = n-q \le m$, we have $\left\lfloor\frac{md}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rfloor\lceil\frac{d+1+t}{2}\rceil = q \le \left\lfloor\frac{k\lceil\frac{d+1+t}{2}\rceil}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rfloor \le \left\lfloor\frac{m\lceil\frac{d+1+t}{2}\rceil}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rfloor \le \frac{m\lceil\frac{d+1+t}{2}\rceil}{\lfloor\frac{d+1-t}{2}\rfloor}$. Therefore, $\left\lfloor\frac{md}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rfloor \le \frac{m}{\lfloor\frac{d+1-t}{2}\rfloor}$ and implies $m = 0$ or $d = 1$, both of which lead to contradictions.

Partition S into S_1 and S_2 having q_1 and q_2 vertices, respectively, where $q_1 = \min\left\{q, \left\lfloor\frac{kd}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rfloor\right\}$ and $q_2 = q - q_1$. Finally let $d' = \min\{q_2, d\}$. This situation is depicted in Figure 4. The symbols e, F, and D refer to the number of edges placed between the indicated sets. The end points of such edges always will be distributed as evenly as possible among the vertices of a given set, and $|D| \le 1$ always.

We make the following definitions:

1. $e = q_1\lfloor\frac{d+1-t}{2}\rfloor$ is the number of edges between S_1 and $V-S$. This will never change if $q_2 \ge 2$ so each of the q_1 vertices of S_1 will be a critical vertex in that case. If $q_2 \le 1$, at most one vertex of S_1 may change from being a critical vertex. Suppose $q_1 = \left\lfloor\frac{kd}{\lfloor\frac{d+1-t}{2}\rfloor}\right\rfloor$.

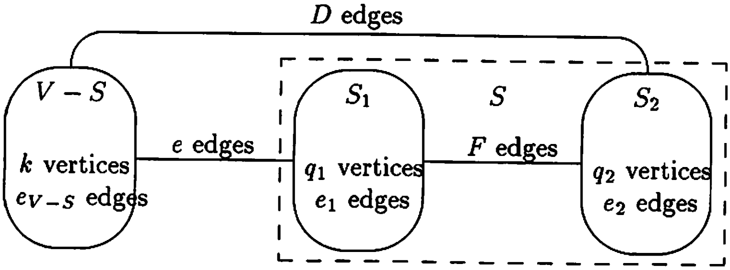

Figure 4: Notation for Theorem 35

Then $e = \left\lfloor \frac{kd}{\lfloor \frac{d+1-t}{2} \rfloor} \right\rfloor \lfloor \frac{d+1-t}{2} \rfloor$. Again letting $c = \lfloor \frac{d+1-t}{2} \rfloor$, $kd \geq e = \lfloor \frac{kd}{c} \rfloor c = \lceil \frac{kd-c+1}{c} \rceil c \geq kd - c + 1 \geq kd - k(c-1) = k(d+1-c) = k \left\lceil \frac{d+1+t)}{2} \right\rceil$ which is required. Furthermore, the inequality is strict unless $c = 1$ in which case $e = kd$ so $kd - e = 0$, an even integer. We have seen earlier in the proof that, if $q_1 = q$, then $e \geq k \left\lceil \frac{d+1+t)}{2} \right\rceil$. Furthermore, if $q = n - \lfloor \frac{d+1-t}{2} \rfloor$, $e > k \left\lceil \frac{d+1+t)}{2} \right\rceil$ unless $q = \left\lceil \frac{d+1+t)}{2} \right\rceil$ and then we again have $kd - e = 0$. If $q = \left\lfloor \frac{md}{\lfloor \frac{d+1-t}{2} \rfloor} \right\rfloor$ and $e = q \lfloor \frac{d+1-t}{2} \rfloor = k \left\lceil \frac{d+1+t)}{2} \right\rceil$, then $\frac{md}{\lfloor \frac{d+1-t}{2} \rfloor} \leq \frac{k}{\lfloor \frac{d+1-t}{2} \rfloor}$, a contradiction.

2. $e'_{V-S} = kd - e$.

3. $e'_2 = q_2(d'-1)$. Note $e'_2 = 0$ if $q_2 \leq 1$.

4. $F' = q_2 d - e'_2$.

5. $e'_1 = q_1 \lceil \frac{d-1+t}{2} \rceil - F'$.

6. $\epsilon_1 = 1$ if e'_1 is odd and 0 otherwise.

7. $\epsilon_2 = 1$ if $e'_{V-S} = kd - e$ is odd and 0 otherwise.

Now G can be constructed as follows. When edges are placed in $V - S$, S_1, and S_2, they are done so as to make the resultant subgraphs nearly regular.

1. If $q_2 \leq 1$ (S_2 either is empty or contains a single vertex having all d of its neighbors in S_1):

   (a) $e_{V-S} = \frac{e'_{V-S}+\epsilon_2}{2} = \frac{kd-e+\epsilon_2}{2} = \frac{kd-q_1\lfloor \frac{d+1-t}{2} \rfloor+\epsilon_2}{2}$ edges are placed in $V - S$.

(b) $e_1 = \frac{e_1' + \epsilon_2}{2} = \frac{q_1\left\lceil \frac{d-1+t}{2}\right\rceil - F' + \epsilon_2}{2} = \frac{q_1\left\lceil \frac{d-1+t}{2}\right\rceil - q_2 d + q_2(d'-1) + \epsilon_2}{2}$ edges are placed in $\langle S_1 \rangle$. Note that one vertex of S_1 is no longer a critical vertex if $\epsilon_2 = 1$. However, since it is adjacent to another vertex of S_1 and the vertex of S_2, if there is one, is adjacent to d vertices of S_1, there is no problem.

(c) $e_2 = 0$ edges are placed in $\langle S_2 \rangle$.

(d) $D = 0$ edges are placed between S_2 and $V - S$.

(e) $e - \epsilon_2$ edges are placed between S_1 and $V - S$. If $\epsilon_2 = 1$, the comments in part 1 of the definitions above show $e > k\left\lceil \frac{d+1+t}{2}\right\rceil$ so this reduction is acceptable.

(f) $F = F' = q_2 d - q_2(d'-1)$ edges are placed between S_2 and S_1.

2. If $q_2 \geq 2$:

(a) $e_{V-S} = \frac{e'_{V-S} - \epsilon_2}{2} = \frac{kd - e - \epsilon_2}{2} = \frac{kd - q_1\left\lfloor \frac{d+1-t}{2}\right\rfloor - \epsilon_2}{2}$ edges are placed in $V - S$.

(b) $e_1 = \frac{e_1' - \epsilon_1}{2} = \frac{q_1\left\lceil \frac{d-1+t}{2}\right\rceil - F' - \epsilon_1}{2} = \frac{q_1\left\lceil \frac{d-1+t}{2}\right\rceil - q_2 d + q_2(d'-1) - \epsilon_1}{2}$ edges are placed in $\langle S_1 \rangle$.

(c) $e_2 = \frac{e_2' - (\epsilon_1 + \epsilon_2)}{2} = \frac{q_2(d'-1) - (\epsilon_1 + \epsilon_2)}{2}$ edges are placed in $\langle S_2 \rangle$.

(d) $D = \epsilon_2$ edges are placed between S_2 and $V - S$.

(e) e edges are placed between S_1 and $V - S$.

(f) $F = F' + \epsilon_1 = q_2 d - q_2(d'-1) + \epsilon_1$ edges are placed between S_2 and S_1.

It is straightforward to verify that the e, D, and F edges can be placed so that the resultant graph is d-regular. Set S is an excess-t gpa since it dominates the graph and every vertex of S either is a critical vertex or is adjacent to a critical vertex in S. □

References

[1] R. C. Brigham, R. D. Dutton, T. W. Haynes and S. T. Hedetniemi, Powerful alliances in graphs, submitted.

[2] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, McGraw Hill, Boston, 2004.

[3] M. Chellali and T. W. Haynes, Global powerful alliances in trees, manuscript.

[4] O. Favaron, G. Fricke, W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, P. Kristiansen, R. C. Laskar and D. Skaggs, Offensive alliances in graphs, *Proceedings* 17^{th} *International Symposium Computer Information Science* (I. Cicekli, N. K. Cicekli and E. Gelenbe, Eds.), ISCIS xvii, October 28-30, 2002, Orlando, FL, CRC Press, pp. 298-302.

[5] G. H. Fricke, L. M. Lawson, T. W. Haynes, S. M. Hedetniemi and S. T. Hedetniemi, A note on defensive alliances in graphs, *Bulletin of the Institute of Combinatorics and its Applications* 38 (2003) 37-41.

[6] M. Hall, Jr., *Combinatorial Theory*, Second Edition, John Wiley & Sons, New York, 1986.

[7] T. W. Haynes, S. T. Hedetniemi and M. A. Henning, Global defensive alliances in graphs, *The Electronic Journal of Combinatorics* 10 (2003) R47.

[8] T. W. Haynes, S. T. Hedetniemi and M. A. Henning, A characterization of trees with equal domination and global strong alliance numbers, submitted.

[9] T. W. Haynes, S. T. Hedetniemi and M. A. Henning, Global defensive alliances, *Proceedings* 17^{th} *International Symposium Computer Information Science* (I. Cicekli, N. K. Cicekli and E. Gelenbe, Eds.), ISCIS xvii, October 28-30, 2002, Orlando, FL, CRC Press, pp. 303-307.

[10] S. M. Hedetniemi, S. T. Hedetniemi and P. Kristiansen, Alliances in graphs, *Journal of Combinatorics and Combinatorial Computing* 48 (2004) 157-177.

[11] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[12] P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi, Introduction to alliances in graphs, *Proceedings* 17^{th} *International Symposium Computer Information Science* (I. Cicekli, N. K. Cicekli and E. Gelenbe, Eds.), ISCIS xvii, October 28-30, 2002, Orlando, FL, CRC Press, pp. 308–312.

[13] A. McRae, W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi and P. Kristiansen, The algorithmic complexity of alliances in graphs, manuscript.

[14] K. H. Shafique and R. D. Dutton, On satisfactory partitioning of graphs, *Congressus Numerantium* 154 (2002) 183-194.