A (0, 1)–Matrix Framework for Elliptic Semiplanes

M. Abreu, *† M. Funk, *† D. Labbate, *‡ V. Napolitano, *†

† Dipartimento di Matematica, Università della Basilicata,
Viale dell'Ateneo Lucano, 85100 Potenza, Italy.
e-mail: abreu@unibas.it; funk@unibas.it; vnapolitano@unibas.it

‡ Dipartimento di Matematica, Politecnico di Bari,
Via E. Orabona, 4, 70125 Bari, Italy.
e-mail: ld487sci@unibas.it

Abstract

We present algebraic constructions yielding incidence matrices for all finite Desarguesian elliptic semiplanes of types C, D, and L. Both basic ingredients and suitable notations are derived from addition and multiplication tables of finite fields. This approach applies also to the only elliptic semiplane of type B known so far. In particular, the constructions provide intrinsic tactical decompositions and partitions for these elliptic semiplanes into elliptic semiplanes of smaller order.

1 Introduction: Finite Elliptic Semiplanes

A *partial plane* is an incidence structure $S = (X, L, |)$ such that any two distinct *points* in X are incident with at most one *line* in L. If $p|l$, in abuse of language we say that p *lies on* l and that l *goes through* p. In particular, S is called *non-degenerate* if both X and L contain at least three elements. Dembowski [7] defines a *semiplane* to be a non-degenerate partial plane satisfying the following axiom of parallels:

Given a non-incident point line pair (p_0, l_0), there exists at most one line l_1 through p_0 and "parallel" to l_0 (i.e. there is no point incident with

*This research was carried out within the activity of INdAM-GNSAGA and supported by the Italian Ministry MIUR

both l_0 and l_1) and, dually, at most one point p_1 on l_0 and "parallel" to p_0 (i.e. there is no line incident with both p_0 and p_1).

A semiplane is *elliptic* of order n if there are $n+1$ points on each line and $n+1$ lines through each point.

A *Baer-subset* of a finite projective plane $\mathcal{P}$ (of order $n+1$, say) is either a Baer-subplane of $\mathcal{P}$ or, for any point line pair (p, l) of $\mathcal{P}$, incident or not, the partial sub-plane $\mathcal{B}(p, l)$ made up by p, l, and all points lying on l as well as all lines going through p. It was known already to Dembowski [7] that $\mathcal{P}\backslash\mathcal{B}$ is an elliptic semiplane (of order n). (We call any such elliptic semiplane *Desarguesian* if $\mathcal{P}$ is so.) Dembowski proved the following partial converse:

Theorem 1.1 *If $\mathcal{S} = (X, L, |)$ is an elliptic semiplane of order n, then all parallel classes in X and L have the same size, say m. Moreover, m divides $n(n+1)$, the total number of points (lines) is $n(n+1)+m$, and exactly one of the following cases holds true:*

(0) $m = 1$ and $\mathcal{S} = \mathcal{P}$ is a projective plane of order n;

(C) $m = n$ and $\mathcal{S} = \mathcal{P} - \mathcal{B}(p, l)$ for some incident point line pair $p|l$, see also Cronheim [6];

(L) $m = n+1$ and $\mathcal{S} = \mathcal{P} - \mathcal{B}(p, l)$ for some non-incident point line pair (p, l), see also Lüneburg [14];

(D) $m = n+1-\sqrt{n+1}$ and $\mathcal{S} = \mathcal{P} - \mathcal{B}$ for some Bear subplane $\mathcal{B}$ of $\mathcal{P}$;

(B) $m < n+1-\sqrt{n+1}$ and there exists a symmetric balanced incomplete block design with parameters

$$v = b = (n(n+1)+m)m^{-1}, \quad k = r = n(n+1-m)m^{-1},$$

$$\textit{and} \quad \lambda = (n-m)(n+1-m)m^{-1}.$$

Dembowski left the existence of elliptic semiplanes of type B as an open problem. In 1977 Baker [1, 2] found such an elliptic semiplane, which has 45 points, order $n = 6$, and parallel class size $m = 3$; the corresponding BIBD with parameters $(v, k, \lambda) = (15, 7, 3)$ is isomorphic to the point plane structure of the projective space $PG(3, 2)$.

2 Constructions and Algebraic Criteria for Linearity of $(0, 1)$ block matrices

A partial plane $\mathcal{L} = (X, L, |)$ gives rise to a $(0,1)$–matrix, the *incidence matrix*: fix some labelings $X = \{p_0, \ldots, p_r\}$ and $L = \{l_0, \ldots, l_s\}$, and define $M = (m_{i,j})$ with $m_{i,j} = 1$ or 0 whether or not one has $p_i|l_j$. The incidence matrix is unique up to re-ordering of rows and columns since relabeling the points (lines) of $\mathcal{L}$ results in a permutation of the rows (columns) of M.

The forbidden substructure characterizing partial planes is a *di-gon*, made up by two distinct points p_1, p_2 and two distinct lines l_1, l_2 such that for all $i, j \in \{1, 2\}$ one has $p_i | l_j$. Its incidence matrix would read $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. This implies the following criterion.

Lemma 2.1 *A* $(0,1)$*–matrix is the incidence matrix of some partial plane if, and only if, it does not contain any* 2×2 *submatrix all of whose entries are* 1.

We call such a $(0,1)$–matrix **linear**.

Big $(0,1)$–matrices are difficult to handle, in particular when checking linearity. In favorable situations, however, the $(0,1)$–matrix M under consideration reveals an appropriate block matrix structure with square blocks. Our approach consists in constructing $1-1$ correspondences between square $(0,1)$–blocks of M and elements of a finite field $GF(q)$ in such a way, that checking linearity in M can be translated into inspecting algebraic equations over $GF(q)$.

Additive correspondence

Let $q \geq 3$ be a prime power and consider the additive group $(GF(q), +)$ and fix a labeling for its elements, say $g_0, g_1, \dots, g_{q-1}$ such that $g_0 = 0$. Let $(a_{i,j})$ be the $GF(q)$–matrix of order q defined by

$$a_{i,j} := (-g_i) + g_j \quad \text{for } i, j = 0, \dots, q-1 \,.$$

Note that $(a_{i,j})$ is an addition table for $GF(q)$ where the elements

$$g_0 = 0, -g_1, -g_2, \dots, -g_{q-1} \quad \text{and} \quad g_0, g_1, g_2, \dots, g_{q-1}$$

correspond to the $1^{st}, 2^{nd}, \dots, q^{th}$ rows and the $1^{st}, 2^{nd}, \dots, q^{th}$ columns, respectively. In particular, all the entries in the main diagonal are equal to 0. For each $g \in GF(q)$, let P_g be the $(0,1)$–matrix of order q whose entry in position (i, j) is defined by

$$(P_g)_{i,j} \quad := \quad \begin{cases} 1 & \text{if } a_{i,j} = g \\ 0 & \text{otherwise} \,. \end{cases}$$

Since the element g appears in each row and column of the addition table $(a_{i,j})$ precisely once, P_g is a *permutation matrix* of order q. In particular, P_0 is the unit matrix of order q.

Definition 2.2 *Let* $B = (b_{i,j})$ *be an* $r \times s$ *matrix with entries in* $GF(q)$*. Then we "blow up"* B *to a* $(0,1)$*–matrix* $\overline{B}$ *with* qr *rows and* qs *columns in the following way:* $\overline{B}$ *is the* $r \times s$ *block matrix having* $q \times q$ *blocks* $\overline{B}_{i,j}$ *such that for all* $i = 0, \dots, r-1$ *and* $j = 0, \dots, s-1$ *one has*

$$\overline{B}_{i,j} = P_g \quad \textit{if, and only if} \quad b_{i,j} = g \,.$$

Proposition 2.3 *(Criterion 1) The* $(0,1)$*-matrix* $\overline{B}$ *is linear if, and only if, for each* 2×2 *submatrix* S *of* B*, say*

$$S = \begin{pmatrix} a & b \\ d & c \end{pmatrix}, \quad (a,b,c,d \in GF(q)),$$

one has

$$a - b + c - d \neq 0.$$

Proof. To prove linearity, assume that $\overline{B}$ had an ordinary submatrix of order 2 all of whose entries were 1. Clearly, these four entries occurred in four distinct blocks of $\overline{B}$. Since the entries lied two by two in the same row and the same column, we found entry 1 in positions

$$(i,j) \text{ in } P_a, \quad (i,k) \text{ in } P_b, \quad (l,k) \text{ in } P_c, \quad \text{and } (l,j) \text{ in } P_d,$$

for some $i,j,k,l \in \{0,\ldots,q-1\}$. By construction, this respectively implied

$$-g_i + g_j = a, \quad -g_i + g_k = b, \quad -g_l + g_k = c, \quad \text{and} \quad -g_l + g_j = d.$$

Subtracting the second and fourth equations from the sum of the first and third, we obtained

$$0 = a - b + c - d,$$

a contradiction.

The converse is easily seen to be true. □

Corollary 2.4 *Let* B *be an* $r \times s$ *matrix over* $GF(q)$*. If* B *meets the requirements of Criterion* (1)*, then so does each matrix* B' *obtained from* B *by adding a fixed element* $a \in GF(q)$ *to every entry in some row or column of* B*.*

We refer to B' as an **additive shift** of B. Reiterated additive shifts induce an equivalence relation in the set of all $r \times s$ matrices over $GF(q)$ satisfying Criterion (1).

An immediate application this Criterion has already been pointed out in [9], Proposition 7.3:

Proposition 2.5 *If* M *is the full multiplication table of* $GF(q)$*, then* $\overline{M}$ *is a linear* $(0,1)$*-matrix of order* q^2*.*

Proof. With each element $g \in GF(q)$, we associate the $q \times q$ permutation matrix P_g. We blow up the full multiplication table M of $GF(q)$ to a $q^2 \times q^2$ matrix $\overline{M}$. Apply Criterion 1: consider $M = (m_{i,j})$ and take four elements pairwise in the same row and the same column, say $m_{i,j}, m_{i,k}, m_{l,j}, m_{l,k}$. Since

M is a multiplication table, there exist elements $x_i, x_l, y_j, y_k \in GF(q)$ with $x_i \neq x_l$ and $y_j \neq y_k$ such that

$$m_{i,j} = x_i y_j,\ m_{i,k} = x_i y_k,\ m_{l,j} = x_l y_j,\ m_{l,k} = x_l y_k\,;$$

thus

$$m_{i,j} - m_{i,k} + m_{l,k} - m_{l,j} = (x_i - x_l)(y_j - y_k) \neq 0\,.$$

Hence, $\overline{M}$ is linear. □

Multiplicative correspondence

For a prime power $q \geq 3$, consider the multiplicative group $GF(q)^*$ and fix a labeling for its elements, say $g_1, \ldots, g_{q-1}$ such that $g_1 = 1$. Let $(\alpha_{i,j})$ be the $GF(q)^*$–matrix of order $q-1$ defined by

$$\alpha_{i,j} := g_i^{-1} g_j \quad \text{for } i,j = 1, \ldots, q-1\,.$$

The matrix $(\alpha_{i,j})$ is a multiplication table for $GF(q)^*$ where

$$g_1 = 1, g_2^{-1}, g_3^{-1}, \ldots, g_{q-1}^{-1} \quad \text{and} \quad g_1, g_2, \ldots, g_{q-1}$$

correspond to the the $1^{st}, 2^{nd}, \ldots, (q-1)^{st}$ rows and the the $1^{st}, 2^{nd}, \ldots, (q-1)^{st}$ columns, respectively. In particular, all the entries in the main diagonal are equal to 1. For each $g \in GF(q)$, let Q_g be the $(0,1)$–matrix of order $q-1$ whose entry in position (i,j) is defined by

$$(Q_g)_{i,j} \quad := \quad \begin{cases} 1 & \text{if } \alpha_{i,j} = g \\ 0 & \text{otherwise}. \end{cases}$$

Since the element g appears in each row and column of the multiplication table $(\alpha_{i,j})$ precisely once, G_g is again a permutation matrix, but of order $q-1$. In particular, Q_1 is the unit matrix of order $q-1$.

Definition 2.6 *Let $B = (b_{i,j})$ be an $r \times s$ matrix with entries in $GF(q)^*$. Then we "blow up" B to a $(0,1)$–matrix $\tilde{B}$ with $(q-1)r$ rows and $(q-1)s$ columns in the following way: $\tilde{B}$ is the $r \times s$ block matrix having $q-1 \times q-1$ blocks $\tilde{B}_{i,j}$ such that for all $i = 1, \ldots, r$ and $j = 1, \ldots, s$ one has*

$$\tilde{B}_{i,j} = Q_g \quad \textit{if, and only if} \quad b_{i,j} = g\,.$$

Proposition 2.7 *(Criterion 2) The $(0,1)$–matrix $\tilde{B}$ is linear if, and only if, for each 2×2 submatrix S of B, say*

$$S \quad = \quad \begin{pmatrix} a & b \\ d & c \end{pmatrix}, \quad (a,b,c,d \in GF(q)^*)\,,$$

one has

$$ab^{-1}cd^{-1} \neq 1\,.$$

Proof. Translate the additive pattern proving Criterion 1 into multiplicative terms. □

Corollary 2.8 *Let B be an $r \times s$ matrix over $GF(q)^*$. If B meets the requirements of Criterion* (2), *then so does each matrix B" obtained from B by multiplying every entry in some row or column of B by a fixed element $a \in GF(q)^*$.*

B" is said to be a **multiplicative shift** of B. Reiterated multiplicative shifts induce an equivalence relation in the set of all $r \times s$ matrices over $GF(q)^*$ satisfying Criterion (2).

An immediate application of Criterion 2 reads:

Proposition 2.9 *If S is a submatrix of the addition table of $GF(q)$ such that* 0 *does not occur as entry of S, then $\tilde{S}$ is a linear* $(0,1)$*–matrix.*

Proof. With each element $g \in GF(q)^*$, we associate the $q-1 \times q-1$ permutation matrix Q_g. We blow up S to $\tilde{S}$. To apply Criterion 2, consider $S = (\sigma_{i,j})$ and take four elements pairwise in the same row and the same column, say $\sigma_{i,j}, \sigma_{i,k}, \sigma_{l,j}, \sigma_{l,k}$. Since S comes from an addition table, there exist elements $x_i, x_l, y_j, y_k \in GF(q)$ with $x_i \neq x_l$ and $y_j \neq y_k$ such that

$$\sigma_{i,j} = x_i + y_j,\ \sigma_{i,k} = x_i + y_k,\ \sigma_{l,j} = x_l + y_j,\ \sigma_{l,k} = x_l + y_k\,.$$

Thus

$$\sigma_{i,j}\sigma_{i,k}^{-1}\sigma_{l,k}\sigma_{l,j}^{-1} = \frac{x_ix_l + x_iy_k + x_ly_j + y_jy_k}{x_ix_l + x_iy_j + x_ly_k + y_jy_k} \neq 1$$

if, and only if

$$x_iy_k + x_ly_j \neq x_iy_j + x_ly_k\,.$$

This, in turn, holds true if, and only if,

$$(x_i - x_l)(y_j - y_k) \neq 0\,.$$

Hence, $\tilde{S}$ is linear. □

3 Elliptic Semiplanes of Type C

Let's begin with the Desarguesian elliptic semiplane $\mathcal{S}^C(q-1) := \mathcal{P} \backslash \mathcal{B}(p,l)$, obtained by deleting the Baer subset $\mathcal{B}(p,l)$ with $p|l$ from a finite Desarguesian projective plane $\mathcal{P}$ of order q.

Theorem 3.1 *The* $(0,1)$*–matrix $\overline{M}$ constructed in Proposition 2.5 is an incidence matrix of the Desarguesian elliptic semiplane $\mathcal{S}^C(q-1)$, and vice versa.*

Proof. Fix an incident point line pair $p|l$ in a finite Desargesian projective plane $\mathcal{P}$ of order q and introduce non-homogeneous co-ordinates in $\mathcal{P}$, see e.g. [10] or [12]: choose l to be the line at infinity and p to be the point with co-ordinate (∞). Then the points and lines of $\mathcal{S} = \mathcal{P}\backslash\mathcal{B}(p,l)$ are exactly those with two co-ordinates, i.e. the point set and the line set can be identified with the sets

$$\{(a,b) \ : \ a,b \in GF(q)\} \quad \text{and} \quad \{[\alpha,\beta] \ : \ \alpha,\beta \in GF(q)\},$$

respectively. Incidence is given by the rule

$$(a,b) \mid [\alpha,\beta] \quad \text{if, and only if} \quad \alpha a + \beta = b\,.$$

The labeling $g_0 = 0, g_1, \ldots, g_q$ chosen when establishing the additive correspondence induces a linear order in $GF(q)$. If we adopt the canonical lexicographic order for the co-ordinates to establish labelings for both the points and lines of $\mathcal{S}$, we yield $\overline{M}$ as an incidence matrix of $\mathcal{S}$. □

Remark 3.2 *The distinction in the first place of this lexicographic order has the following geometric interpretation: The point and lines of S are grouped together in one and the same parallel class if, and only if, their first co-ordinate coincide. The* $1^{st}, 2^{nd}, \ldots, q^{th}$ *point classes correspond in* $\mathcal{P}$ *to the deleted (vertical) lines with equations*

$$x = g_0, \ \ x = g_1, \ \ldots, \ \ x = g_{q-1}\,,$$

while the $1^{st}, 2^{nd}, \ldots, q^{th}$ *line classes are characterized in* $\mathcal{P}$ *by their improper points*

$$(g_0), (g_1), \ldots, (g_{q-1})\,,$$

respectively.

The advantage of this characterization becomes clear when compared with other explicit constructions (cf e.g. [8]).

Example 1 For $GF(4) = \{0, 1, x, \overline{x}\}$ we obtain

$$\begin{pmatrix} 0 & 1 & x & \overline{x} \\ 1 & 0 & \overline{x} & x \\ x & \overline{x} & 0 & 1 \\ \overline{x} & x & 1 & 0 \end{pmatrix}$$

as an addition table already with 0 entries in the main diagonal,

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$P_x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad P_{\overline{x}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

as permutation matrices, and

$$M \quad = \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & x & \overline{x} \\ 0 & x & \overline{x} & 1 \\ 0 & \overline{x} & 1 & x \end{pmatrix}$$

as the full multiplication table. We blow up M to the following $(0,1)$–matrix $\overline{M}$, where, for convenience, the entries 0 have been omitted. In two additional rows and columns (above and to the left) there are written down the non-homogeneous point and line co-ordinates in lexicographic order; they illustrate how $\overline{M}$ works as an incidence matrix for $\mathcal{S}^C(3)$.

		0	0	0	0	1	1	1	1	x	x	x	x	$\overline{x}$	$\overline{x}$	$\overline{x}$	$\overline{x}$
		0	1	x	$\overline{x}$	0	1	x	$\overline{x}$	0	1	x	$\overline{x}$	0	1	x	$\overline{x}$
0	0	1				1				1				1			
0	1		1				1				1				1		
0	x			1				1				1				1	
0	$\overline{x}$				1				1				1				1
1	0	1					1					1					1
1	1		1			1							1			1	
1	x			1					1	1					1		
1	$\overline{x}$				1			1			1			1			
x	0	1						1					1		1		
x	1		1						1			1		1			
x	x			1		1					1						1
x	$\overline{x}$				1		1			1						1	
$\overline{x}$	0	1							1		1					1	
$\overline{x}$	1		1					1		1							1
$\overline{x}$	x			1			1						1	1			
$\overline{x}$	$\overline{x}$				1	1						1			1		

4 Elliptic Semiplanes of Type L

Next we deal with the Desarguesian elliptic semiplane $\mathcal{S}^L(q-1) := \mathcal{P} \backslash \mathcal{B}(p,l)$, obtained by deleting the Baer subset $\mathcal{B}(p,l)$ from a finite Desargesian projective plane $\mathcal{P}$ of order q where (p,l) is a non-incident point line pair in $\mathcal{P}$.

Let $A = (a_{i,j})$ be the addition table of $GF(q)$ constructed above for establishing the additive correspondence. Then, for all $i, j = 0, \dots, q-1$, one has

$$a_{i,j} + a_{j,i} = (-g_i) + g_j + (-g_j) + g_i = 0\,,$$

whence 0 shows up in the main diagonal of A. For $i, j = 0, \dots, q$, let $B = (b_{i,j})$ be the following $q+1 \times q+1$ matrix whose entries are either blank or elements in $GF(q)^*$:

$$b_{i,j} \quad := \quad \begin{cases} \text{blank} & \text{if} \quad i = j\,; \\ a_{i,j} & \text{if} \quad i, j \in \{0, \dots, q-1\} \text{ with } i \neq j\,; \\ 1 & \text{if} \quad \text{either } i = q \text{ or } j = q\,. \end{cases}$$

Finally, blow up B to $\tilde{B}$, where the blanks are substituted by copies of the $q-1 \times q-1$ matrix all of whose entries are 0.

In the light of the multiplicative shift equivalence, the choices $b_{q,j} = 1$ and $b_{i,q} = 1$ could be replaced by $b_{q,j} = a\,,\ j = 0, \dots, q-1$ and $b_{i,q} = b\,,\ i = 0, \dots, q-1$ for some $a, b \in GF(q)^*$.

Lemma 4.1 *B meets the requirements of Criterion* 2.

Proof. For entries coming from the principal minor of order q, the statement follows from Proposition 2.9. Hence the only remaining 2×2 submatrices to be examined are of types

$$\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & 1 \\ c & 1 \end{pmatrix}$$

for some $a, b, c \in GF(q)^*$. Since a, b and a, c respectively appear in the same row and column of an addition table, they can't be equal. Thus $ab^{-1} \neq 1$ and $ac^{-1} \neq 1$. □

Theorem 4.2 *The* $(0,1)$*–matrix $\tilde{B}$ is an incidence matrix of the Desarguesian elliptic semiplane $\mathcal{S}^L(q-1)$, and vice versa.*

Proof. Fix a non-incident point line pair (p, l) in a finite Desargesian projective plane $\mathcal{P}$ of order q and introduce homogeneous co-ordinates in $\mathcal{P}$: choose $l \equiv [0 : 0 : 1]$ to be the line at infinity and let $p \equiv (0 : 0 : 1)$ be the origin. Then the points of $\mathcal{S} = \mathcal{P} \backslash \mathcal{B}(p, l)$ are exactly the affine points of $\mathcal{P}$, other than the origin. Normalizing their third co-ordinate to be 1, we obtain

$$\{\,(a, b, 1)\ : a, b \in GF(q), (a, b) \neq (0, 0\}$$

as point set of $\mathcal{S}$. The lines of $\mathcal{S}$ arise from those affine lines in $\mathcal{P}$ whose affine equations read either $y = \alpha' x + \beta'$ with $\beta' \neq 0$ or $x = \mu'$ with $\mu' \neq 0$. Translating into homogeneous co-ordinates, this becomes either $[\alpha' : -1 : \beta']$ or $[-1 : 0 : \mu']$. (It's worth mentioning that we could also normalize their third co-ordinate to be 1; hence

$$\{\,[\alpha, \beta, 1] \,:\, \alpha, \beta \in GF(q), (\alpha, \beta) \neq (0,0)\}$$

would be the line set of $\mathcal{S}$ and incidence be given by the rule

$$(a, b, 1) \mid [\alpha, \beta, 1] \quad \text{if, and only if,} \quad \alpha a + \beta b \;=\; -1.)$$

The labeling $g_1 = 1, \ldots, g_q$ chosen when establishing the multiplicative correspondence induces a linear order, say $<$, in $GF(q)^*$. We extend this order by saying that all the elements in $GF(q)^*$ precede the symbol ∞ (standing for quotients with denominator zero). Again we adopt some kind of lexicographic orders (denoted by $<$ as well) for the co-ordinates to establish labelings for points and lines of $\mathcal{S}$:

$$(a_1, b_1, 1) < (a_2, b_2, 1) \text{ if } \begin{cases} a_1^{-1}b_1 < a_2^{-1}b_2 & ; \\ a_1 < a_2 & \text{if } \; a_1^{-1}b_1 = a_2^{-1}b_2 \,; \\ b_1 < b_2 & \text{if } \; a_1^{-1}b_1 = a_2^{-1}b_2 \text{ and } a_1 = a_2 \,. \end{cases}$$

$$[\alpha_1, -1, \beta_1] < [\alpha_2, -1, \beta_2] \quad \text{if} \quad \begin{cases} \alpha_1 < \alpha_2 & ; \\ \beta_1 < \beta_2 & \text{if} \quad \alpha_1 = \alpha_2 \,. \end{cases}$$

$$[\alpha, -1, \beta] < [-1, 0, \mu] \quad \text{in any case.}$$

$$[-1, 0, \mu_1] < [-1, 0, \mu_2] \quad \text{if} \quad \mu_1 < \mu_2 \,.$$

With respect to these labelings, $\tilde{B}$ is the incidence matrix of $\mathcal{S}$. □

Remark 4.3 *The point and lines of* S *are grouped together according to the parallel classes they make up such that the* $1^{st}, 2^{nd}, \ldots, (q+1)^{st}$ *point classes correspond in* $\mathcal{P}$ *to the deleted lines going through the origin with equations*

$$y = g_0 x, \;\; y = g_1 x, \;\ldots,\; y = g_{q-1} x, \;\; \textit{and} \;\; x = 0\,,$$

while the $1^{st}, 2^{nd}, \ldots, (q+1)^{st}$ *line classes are characterized in* $\mathcal{P}$ *by their improper points*

$$(g_0), (g_1), \ldots, (g_{q-1}) \;\; \textit{and} \;\; (\infty)\,,$$

respectively.

Example 2 Let $GF(5)^* = \{1, 2, 3, 4\}$ and write down the multiplication table with respect to the order

$$1, \;\; 2^{-1} = 3, \;\; 3^{-1} = 2, \;\; 4^{-1} = 4 \quad \text{and} \quad 1, 2, 3, 4$$

for the first and second factors, respectively:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \\ 2 & 4 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Extract the permutation matrices

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad Q_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

With these data, one has

$$B = \begin{pmatrix} & 4 & 3 & 2 & 1 & 1 \\ 1 & & 4 & 3 & 2 & 1 \\ 2 & 1 & & 4 & 3 & 1 \\ 3 & 2 & 1 & & 4 & 1 \\ 4 & 3 & 2 & 1 & & 1 \\ 1 & 1 & 1 & 1 & 1 & \end{pmatrix}.$$

We blow up B to the following $(0,1)$–matrix $\tilde{B}$, where, for convenience, the entries 0 will again been omitted. In three additional rows and columns (above and to the left) there are written down the homogeneous point and line co-ordinates in the order defined above. The entry $-1(\equiv 4)$ in the line co-ordinates has been replaced by 4. This shows that $\tilde{B}$ is an incidence matrix for $\mathcal{S}^L(4)$. (An analogous incidence matrix for $\mathcal{S}^L(3)$ can be found in [5], Figure 6.)

	0 4 1	0 4 2	0 4 3	0 4 4	1 4 1	1 4 2	1 4 3	1 4 4	2 4 1	2 4 2	2 4 3	2 4 4	3 4 1	3 4 2	3 4 3	3 4 4	4 4 1	4 4 2	4 4 3	4 4 4	4 0 1	4 0 2	4 0 3	4 0 4
1 0 1								1			1			1			1				1			
2 0 1							1		1							1		1				1		
3 0 1						1						1	1						1				1	
4 0 1					1					1					1					1				1
1 1 1	1											1			1			1			1			
2 2 1		1									1		1							1		1		
3 3 1			1							1						1	1						1	
4 4 1				1					1					1					1					1
1 2 1		1			1											1			1		1			
2 4 1				1		1									1		1					1		
3 1 1	1						1							1						1			1	
4 3 1			1					1					1					1						1
1 3 1			1			1			1											1	1			
2 1 1	1							1		1									1			1		
3 4 1				1	1						1							1					1	
4 2 1		1					1					1					1							1
1 4 1				1			1			1			1								1			
2 3 1			1		1							1		1								1		
3 2 1		1						1	1						1								1	
4 1 1	1					1					1					1								1
0 1 1	1				1				1				1				1							
0 2 1		1				1				1				1				1						
0 3 1			1				1				1				1				1					
0 4 1				1				1				1				1				1				

5 Elliptic Semiplanes of Type D

Explicit constructions of incidence matrices for elliptic semiplanes of type D have already been presented in [9]. Therefore we may restrict ourselves to a short survey.

To this end we need a slight generalization of our notation. For each $r \in \mathbb{N}$, we understand a *finite difference set modulo r* to be a subset $S = \{s_0, \ldots, s_{\kappa-1}\} \subseteq \mathbb{Z}_r$ such that the $\kappa^2 - \kappa$ differences

$$\delta_{i,j} :\equiv s_i - s_j \pmod r$$

are pairwise distinct for $i, j = 0, \ldots, \kappa - 1$ with $i \neq j$. If $r = \kappa^2 - \kappa + 1$, then S is called *perfect* [4].

Lemma 5.1 [13] *Let* $C = < c_0, \ldots, c_{r-1} >$ *be a circulant* $(0,1)$*–matrix,* $S := \{i \in \{0, \ldots, r-1\} \mid c_i = 1\}$, *and* $|S| = \kappa$. *Then C is linear if, and only if, S is a difference set modulo r.*

Instances of perfect difference sets are $\{0,1,3\}$ modulo 7 and $\{0,1,4,6\}$ modulo 13, which represent incidence matrices for $PG(2,2)$ and $PG(2,3)$, respectively.

Now, for a prime r, we allow difference sets modulo r as entries in B once in every row and every column (typically in the main diagonal). For convenience, we will mention (r) as an index above and to the right of every such generalized matrix over $GF(r)$. Criterion (1) still holds true if applied for every choice of some element out of each difference set (cf. [9], Theorem 5.5).

Finite projective planes $\mathcal{P} = PG(2,q^2)$ can be partitioned into Baer subplanes $\mathcal{B} = PG(2,q)$ and Singer cycle results guarantee circulant $(0,1)$–matrices C as incidence matrices for $\mathcal{B}$, for details see e.g. [11]. Deleting just one copy of C in a suitable incidence matrix for $\mathcal{P}$, we obtain incidence matrices for Desargesian elliptic semiplanes of type D and order q^2-1, denoted by $\mathcal{S}^D(q^2-1)$.

For instance,

$$\begin{pmatrix} 0,1,3 & 6 \\ 6 & 0,1,3 \end{pmatrix}^{(7)}$$

represents an incidence matrix of $\mathcal{S}^D(3)$, and $\mathcal{S}^D(8)$ has an incidence matrix induced by:

$$\begin{pmatrix} 0,1,4,6 & 12 & 8 & 11 & 11 & 8 \\ 12 & 0,1,4,6 & 12 & 8 & 11 & 11 \\ 8 & 12 & 0,1,4,6 & 12 & 8 & 11 \\ 11 & 8 & 12 & 0,1,4,6 & 12 & 8 \\ 11 & 11 & 8 & 12 & 0,1,4,6 & 12 \\ 8 & 11 & 11 & 8 & 12 & 0,1,4,6 \end{pmatrix}^{(13)}$$

Note that the linearity of these incidence matrices can easily been verified using Criterion 1.

6 The Elliptic Semiplane of Type B

Baker [1, 2] presented his elliptic semiplane, denoted here by $\mathcal{S}^B$, in terms of an incidence matrix, which is a 15×15 block matrix having $(0,1)$ square blocks of order 3. Adopting suitable additive shift operations, his incidence matrix can be transformed into the following 15×15 block square matrix which can be written down in terms of our additive correspondence for $GF(3)$:

			0	5	10	1	6	11	2	7	12	3	8	13	4	9	14
			0	0	0	1	1	0	1	0	1	1	1	0	1	1	0
			0	1	1	0	0	0	1	1	0	1	0	1	1	1	0
			0	1	1	0	1	1	0	0	0	1	1	0	1	0	1
			1	1	0	0	1	1	0	1	1	0	0	0	1	1	0
0	1000		2	2	1			0		0				0			0
5	0110		2	0	0	1					1	1			1		
10	1110		1	0	1		2		2				2			2	
1	0100		0			0	2	0			1		0				2
6	0011			1		1	1	0	2					1	0		
11	0111				2	1	2	2		0		2				1	
2	0010		0			1			0	2	0			0		2	
7	1101			1			2		1	1	0	1					0
12	1111				2			0	1	2	2		2		1		
3	0001				0	0			0			2	2	1			0
8	1010		1				1			1		2	0	0	1		
13	1011			2				2			2	1	0	1		2	
4	1100		0					2	2			0			1	0	0
9	0101			1		0				0			1		1	2	1
14	1001				2		1				1			2	2	2	0

Let's denote this matrix by B_{15}.

The above table also reveals a deeper correlation between Baker's elliptic semiplane and the corresponding BIBD, i.e. the point plane structure of $PG(3,2)$, which will be discussed in the sequel.

Let

$$\langle C \rangle = \langle \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \rangle$$

generate a Singer cycle by its transitive action on the points of $PG(3,2)$: if p_0 is the point having homogeneous coordinates $(1:0:0:0)$, we define

$$p_{i+1} := p_i C \, , \quad i = 0, \ldots, 14 \, .$$

Let π_0 be the plane containing the line $l_0 = p_0 p_5$ and the point $p_1 \equiv (0:1:0:0)$: in this way, π_0 gets homogeneous coordinates $[0:0:0:1]$. Define

$$\pi_{i+1} := \pi_i (C^{-1})^T \, , \quad i = 0, \ldots, 14 \, .$$

Turning all blank and non-blank entries of B_{15} into 0 and 1, respectively, we obtain a point plane incidence matrix of $PG(3,2)$. Obviously, this corresponds to the process of identifying parallel classes of points and lines in Baker's elliptic semiplane. In the above table, additional rows and columns (above and to the left) indicate the Singer cycle indices $i \in \{0, \ldots, 14\}$ as well as the homogeneous co-ordinates of p_i and π_j, $i, j \in \{0, \ldots, 14\}$.

It's easy but somewhat lengthy to check the following statement:

Proposition 6.1 *The 5^{th} power of the Singer cycle C induces an automorphism of order 3 in Baker's elliptic semiplane. The effect of the induced automorphism is to add 1 to each non-blank entry of B_{15}.*

7 Epilogue: Tactical Decompositions and Partitions

The reader will have noticed that throughout the paper another question has already been solved without explicitly being mentioned so far: do Desarguesian elliptic semiplanes admit significant tactical decompositions or partitions into elliptic semiplanes of smaller order?

Recall that a *tactical decomposition* of $\mathcal{S} = (X, L, |)$ is a partition of X into s pairwise disjoint point sets $X_1, \cdots, X_s$ together with a partition of L into t pairwise disjoint line sets $L_1, \cdots, L_t$, such that, for each choice of i and j, $i = 1, \ldots, s$, $j = 1, \ldots, t$, the partial subplanes $(X_i, L_j, |)$ satisfy the following condition (cf e.g. [5]):

The numbers

$$\alpha_{i,j} := |\{l \in L_j \ : \ p|l\}| \quad \textit{for some fixed} \ p \in X_i$$

$$\beta_{i,j} := |\{p \in X_i \ : \ p|l\}| \quad \textit{for some fixed} \ l \in L_j$$

do not depend on the choice of $p \in X_i$ and $l \in L_j$, respectively.

A *partition of an elliptic semiplane $\mathcal{S} = (X, L, |)$ into elliptic semiplanes of smaller order* is a tactical decomposition with $s = t$ such that, for all $i = 1, \ldots, s$, the partial plane $(X_i, L_i, |)$ makes up a non-degenerate elliptic semiplane again.

Clearly, the block structures occurring in the $(0, 1)$ block matrices constructed here meet the requirements of tactical decompositions, see e.g. [9]. More in detail, we have the following result.

Proposition 7.1 *Let $\mathcal{S}$ be a Desarguesian elliptic semiplane of order n.*

(i) If $\mathcal{S}$ is of type C, then $n = q - 1$ and $\mathcal{S}$ admits a tactical decomposition into partial subplanes each of which is a union of q disjoint flags, i.e. incident point line pairs.

(ii) If $\mathcal{S}$ is of type L, then $n = q - 1$ and $\mathcal{S}$ admits a tactical decomposition into partial subplanes each one being a union of $q - 1$ disjoint flags.

(*iii*) *If S is of type D, then $n = q^2 - 1$ and S admits a partition into $q^2 - q$ Baer subplanes of order q (which can also be seen as elliptic semiplanes).*

(*iv*) *Baker's elliptic semiplane S^B admits a partition into 5 copies of the Desargesian elliptic semiplane $S^C(2)$.*

Proof. In cases (i), (ii), and (iii), the statement is immediately clear. In case (iv), both

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}$$

are additive shifts of the full multiplication table

$$(GF(3), \times) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix},$$

while

$$\begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

turns out to be an additive shift of the full multiplication table of $GF(3)$ where all entries have been multiplied by $-1 \equiv 2$. Note that $(-GF(3), \times)$ is not additively shift equivalent to $(GF(3), \times)$, but induces an incidence matrix for $S^C(2)$ as well . □

Final Remark 7.2 (*i*) *According to the* 35 *lines in $PG(3,2)$, there is a total number of* 35 *elliptic subplanes isomorphic to $S^C(2)$ in S^B. The partition of S^B into* 5 *such copies of $S^C(2)$ corresponds to a line spread of $PG(3,2)$, namely the* 5 *lines*

$$l_i := \{p_i, p_{i+5}, p_{i+10}\},$$

indices taken modulo 15.

(*ii*) *Since $S^C(2)$ is isomorphic to Pappus' famous configuration, we may also say that Baker's elliptic semiplane S^B admits a partition into* 5 *disjoint Pappus configurations and that S^B contains a total number of* 35 *of them.*

Acknowledgment

The authors would like to thank Prof. G. Ebert for the valuable suggestions provided during his visit to Potenza in 2004.

References

[1] R. D. Baker, An Elliptic Semiplane, J. Comb. Th. A 25 (1978), 193–195.

[2] R. D. Baker, Elliptic semi-planes I : Existence and classification, in: Proc. 8^{th} Conf. Combinatorics, Graph Theory, and Computing, Baton Rouge 1977, Congressus Numerantium 19, pp. 61–73.

[3] R. D. Baker, G. L. Ebert, Elliptic semiplanes II: Structure theory, Ars Combinatoria 12 (1981), 147–164.

[4] L. D. Baumert, *Cyclic difference sets*, Springer–Verlag, Berlin – Heidelberg – New York 1971.

[5] A. Betten and D. Betten, Tactical Decompositions and some Configurations v_4, J. Geom. 66 (1999), 27–41.

[6] A. Cronheim, T-groups and their geometry, Illinois J. Math. 9 (1965), 1–30.

[7] P. Dembowski, *Finite Geometries*. Springer, New York 1968, reprint 1997.

[8] M. Funk, On Configurations of Type n_κ with Constant Degree of Irreducibility, J. Comb. Th. A 65 (1994), 173–201.

[9] M. Funk, D. Labbate, V. Napolitano, Tactical decompositions of symmetric configurations, submitted to Discrete Math.

[10] M. Hall, Projective planes, Trans. Amer. Math. Soc. 54 (1943), 229–277.

[11] J. W. P. Hirschfeld, Projective Geometries over Finite Fields (2^{nd} edition), Clarendon Press, Oxford 1998.

[12] D. Hughes and F. C. Piper, *Projective Planes*. Springer, New York 1973.

[13] M. J. Lipman, The Existence of small tactical configurations, in: *Graphs and Combinatorics*, Springer Lecture Notes in Mathematics 406 (1974), 319–324.

[14] H. Lüneburg, Charakterisierungen der endlichen desarguesschen projektiven Ebenen, Math. Z. 85 (1964), 419–450.