The uniqueness of the generalized quadrangle of order 5 with an axis of symmetry

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Abstract

We show that every generalized quadrangle of order $(4,6)$ with a spread of symmetry is isomorphic to the Ahrens-Szekeres generalized quadrangle $AS(5)$. It then easily follows that every generalized quadrangle of order 5 with an axis of symmetry is isomorphic to the classical generalized quadrangle $Q(4,5)$.

Keywords: generalized quadrangle, spread of symmetry, axis of symmetry
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1 Introduction

A *generalized quadrangle* of order (s,t), $s,t \in \mathbb{N} \setminus \{0\}$, or shortly a GQ$(s,t)$, is a point-line incidence structure Q which satisfies the following properties:

(a) each point is incident with $t+1$ lines and two distinct points are incident with at most one line;

(b) each line is incident with $s+1$ points and two distinct lines are incident with at most one point;

(c) for every line L and every point p not incident with L, there exists a unique line through p meeting L.

If $s=t$, then we also say that Q has order s. The point-line dual of a GQ(s,t) is a GQ(t,s).

Let Q be a generalized quadrangle of order (s,t). For every point x of Q, let $x^\perp$ denote the set of all points collinear with x (so $x \in x^\perp$). If

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X is a nonempty set of points of Q, then we define $X^\perp := \bigcap_{x \in X} x^\perp$ and $X^{\perp\perp} := (X^\perp)^\perp$. If x and y are different points, then $|\{x,y\}^\perp|$ is equal to either $s+1$ or $t+1$ depending on whether x and y are collinear or not. The set $\{x,y\}^{\perp\perp}$ is called the *span* of the pair (x,y). If x and y are collinear, then $\{x,y\}^{\perp\perp}$ coincides with the set of points of the line xy. If x and y are not collinear, then $\{x,y\}^{\perp\perp}$ is also called the *hyperbolic line* through x and y; since $\{x,y\}^\perp$ contains two noncollinear points, this hyperbolic line contains at most $t+1$ points. If the hyperbolic line through two noncollinear points x and y contains precisely $t+1$ points, then the pair (x,y) is called *regular*. A point x is called *regular* if the pair (x,y) is regular for every point y not collinear with x.

If x is a regular point of a generalized quadrangle Q of order s with $s \neq 1$, then a new generalized quadrangle $P(Q,x)$ can be derived from it, see [5] or [6]. The points of $P(Q,x)$ are the points of Q not collinear with x and the lines of $P(Q,x)$ are on the one hand the lines of Q not containing x and on the other hand the hyperbolic lines of Q through x (natural incidence). The generalized quadrangle $P(Q,x)$ has order $(s-1,s+1)$.

The generalized quadrangle $W(q)$, q prime power, is the GQ of the points and totally isotropic lines of a symplectic polarity in $PG(3,q)$. Its point-line dual is the generalized quadrangle $Q(4,q)$ whose points and lines are the points and lines lying on a nonsingular parabolic quadric in $PG(4,q)$. If q is even, then $W(q) \cong Q(4,q)$, i.e. $W(q)$ is self-dual. Every point of $W(q)$ is regular. So, we can construct a generalized quadrangle $P(W(q),x)$ of order $(q-1,q+1)$ for every point x of $W(q)$. Since the automorphism group of $W(q)$ acts transitively on the point set, essentially one GQ of order $(q-1,q+1)$ arises this way. If q is odd, then $P(W(q),x)$ is isomorphic to the so-called Ahrens-Szekeres generalized quadrangle $AS(q)$, see [1] or [6].

A *spread* of a GQ is a set of lines partitioning the point set. If S is a spread in a GQ(s,t) with $t \neq 1$, then there are at most $s+1$ automorphisms of the GQ which fix each line of S, see [2]. If there are precisely $s+1$ such automorphisms, then S is called a *spread of symmetry*.

If x is a point of a GQ(s,t) with $s \neq 1$, then there are at most t automorphisms of the GQ which fix every point of $x^\perp$, see [6]. If there are precisely t such automorphisms, then x is called a *center of symmetry*. An *axis of symmetry* is the dual notion of a center of symmetry. Every point of $W(q)$ is a center of symmetry. Dually, every line of $Q(4,q)$ is an axis of symmetry.

Let Q_1 and Q_2 denote two GQ's. If x_i, $i \in \{1,2\}$, is a point of Q_i, then we say that (Q_1,x_1) is equivalent with (Q_2,x_2) if there exists an isomorphism from Q_1 to Q_2 mapping x_1 to x_2. If S_i, $i \in \{1,2\}$, is a spread of Q_i, then we say that (Q_1,S_1) is equivalent with (Q_2,S_2) if there exists an isomorphism from Q_1 to Q_2 mapping S_1 to S_2.

Now, suppose that Q is a generalized quadrangle of order $s \geq 2$ with a regular point x. Then the hyperbolic lines through x define a spread $S(Q,x)$ of $P(Q,x)$. If x is a center of symmetry, then the s automorphisms of Q which fix each point of $x^\perp$ induce s automorphisms of $P(Q,x)$ fixing each line of $S(Q,x)$. Hence, $S(Q,x)$ is a spread of symmetry of $P(Q,x)$. Conversely, if S^* is a spread of symmetry in a generalized quadrangle Q^* of order $(s-1, s+1)$, $s \geq 2$, then by [4], there exists, up to equivalence, a unique pair (Q,x), with Q a generalized quadrangle of order s and x a regular point of Q, such that $(P(Q,x), S(Q,x))$ is equivalent with (Q^*, S^*).

All finite generalized quadrangles of order $(1,t)$, $(2,t)$ and $(3,t)$ have been classified. If Q is a generalized quadrangle of order $(4,t)$, then $t \in \{1,2,4,6,8,11,12,16\}$ and unique examples exist in the cases $t=1$, $t=2$ and $t=4$. We refer to [6] for more details. So, the "smallest order" for which not all GQ's have been determined is the order $(4,6)$. A unique GQ is known with these parameters, namely $AS(5)$. In this paper, we will prove the following result.

Theorem 1 *If a* GQ *of order* $(4,6)$ *has a spread of symmetry, then it is isomorphic to* $AS(5)$.

Since $AS(5)$ has only one spread of symmetry, see [2], there exists up to isomorphism only one GQ of order 5 with a center of symmetry. So, we also have:

Corollary 1 *Every* GQ *of order* 5 *with a center of symmetry is isomorphic to* $W(5)$*, or dually, every* GQ *of order* 5 *with an axis of symmetry is isomorphic to* $Q(4,5)$.

To prove these results we will make use of so-called admissible triples. These are objects which were introduced in [2].

2 Admissible triples

Definition. An *admissible triple* is a triple $T = (\mathcal{L}, G, \Delta)$, where:

- G is a nontrivial group. We put $s := |G| - 1 \geq 1$.
- $\mathcal{L}$ is a linear space, different from a point, in which each line is incident with exactly $s+1$ points. We denote the point set of $\mathcal{L}$ by P. Then $\mathcal{L}$ has order $(s, t-1)$, where $t := \frac{|P|-1}{s}$.

- Δ is a map from $P \times P$ to G such that the following holds for any points x, y and z of $\mathcal{L}$:

(AT) x, y and z are collinear $\Leftrightarrow \Delta(x,y) + \Delta(y,z) = \Delta(x,z)$.

If T is an admissible triple, then $\Delta(x,x) = 0$ and $\Delta(y,x) = -\Delta(x,y)$ for all points x and y of $\mathcal{L}$. Notice that we have used the additive notation for the group G.

Theorem 2 ([2]) *Suppose that $T = (\mathcal{L}, G, \Delta)$ is an admissible triple and let P denote the point set of $\mathcal{L}$. Let Γ be the graph with vertex set $G \times P$, two vertices (g_1, x_1) and (g_2, x_2) being adjacent whenever*

- *$x_1 = x_2$ and $g_1 \neq g_2$, or*
- *$x_1 \neq x_2$ and $g_2 = g_1 + \Delta(x_1, x_2)$.*

Then Γ is the collinearity graph of a generalized quadrangle Q of order (s,t). Moreover, the set $L_x := \{(g,x) \mid g \in G\}$ is a line of Q for every point x in P and the lines L_x, $x \in P$, form a spread of symmetry S in Q.

Example. Let $\mathcal{L}$ be the Desarguesian affine plane AG$(2,q)$ coordinatized in the natural way by the finite field $\mathbb{F}_q$. Let G be the additive group of $\mathbb{F}_q$. For all points (x_1, y_1) and (x_2, y_2) of $\mathcal{L}$, we define $\Delta[(x_1,y_1),(x_2,y_2)] := \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$. Now, three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) of $\mathcal{L}$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

or equivalently, if and only if $\Delta[(x_1,y_1),(x_3,y_3)] = \Delta[(x_1,y_1),(x_2,y_2)] + \Delta[(x_2,y_2),(x_3,y_3)]$. It was shown in [2] that the generalized quadrangle associated with the admissible triple $(\mathcal{L}, G, \Delta)$ is isomorphic to $P(W(q), x)$, where x is any point of $W(q)$.

For every admissible triple T, we put $\Omega(T) := (Q, S)$ where Q and S are as in Theorem 2. The following theorem is one of the key results which we will use during our proof of Theorem 1.

Theorem 3 ([2]) *If S is a spread of symmetry of a generalized quadrangle Q, then there exists an admissible triple T such that $\Omega(T)$ is equivalent with (Q, S).*

Theorem 4 ([2],[3]) *Suppose that $T = (\mathcal{L}, G, \Delta)$ is an admissible triple and let P denote the point set of $\mathcal{L}$. Let $\mathcal{L}'$ be a linear space isomorphic to $\mathcal{L}$ and let G' denote a group isomorphic to G. Let α denote an isomorphism from $\mathcal{L}$ to $\mathcal{L}'$, let θ denote an isomorphism from G to G' and let f denote an arbitrary map from P to G. For all points x and y of $\mathcal{L}$, we define*

$$\Delta'(\alpha(x), \alpha(y)) := [f(x) + \Delta(x, y) - f(y)]^{\theta}.$$

Then $T' := (\mathcal{L}', G', \Delta')$ is an admissible triple and $\Omega(T')$ is equivalent with $\Omega(T)$.

Definition. Let $T_1 = (\mathcal{L}_1, G_1, \Delta_1)$ and $T_2 = (\mathcal{L}_2, G_2, \Delta_2)$ be two admissible triples. Let P_i, $i \in \{1, 2\}$, denote the point set of $\mathcal{L}_i$.

- If $\mathcal{L}_1$ and $\mathcal{L}_2$ are lines, then we say that T_1 and T_2 are equivalent if $\mathcal{L}_1$ and $\mathcal{L}_2$ contain the same number of points.
- If $\mathcal{L}_1$ or $\mathcal{L}_2$ is not a line, then we say that T_1 and T_2 are equivalent if

  (i) there exists an isomorphism α from $\mathcal{L}_1$ to $\mathcal{L}_2$,

  (ii) there exists an isomorphism θ from G_1 to G_2,

  (iii) there exists a map $f : P_1 \rightarrow G_1$

  such that

  $$\Delta_2(\alpha(x), \alpha(y)) = [f(x) + \Delta_1(x, y) - f(y)]^{\theta}$$

  for all points x and y of $\mathcal{L}_1$.

Theorem 5 ([3]) *Two admissible triples T_1 and T_2 are equivalent if and only if $\Omega(T_1)$ and $\Omega(T_2)$ are equivalent.*

Lemma 1 *If $T = (\mathcal{L}, G, \Delta)$ is an admissible triple and if o is an arbitrary point of $\mathcal{L}$, then there exists an admissible triple $T' = (\mathcal{L}, G, \Delta')$ equivalent with T such that $\Delta'(o, x) = 0$ for every point x of $\mathcal{L}$.*

Proof. In the above definition, we put α equal to the trivial automorphism of $\mathcal{L}$, θ equal to the trivial automorphism of G and we define $f(x) := \Delta(o, x)$ for every point x of $\mathcal{L}$. We then have $\Delta'(o, x) = \Delta(o, o) + \Delta(o, x) - \Delta(o, x) = 0$ for every point x of $\mathcal{L}$. □

Lemma 2 *If an admissible triple $T = (\mathcal{L}, G, \Delta)$ gives rise to generalized quadrangle Q of order $(s-1, s+1)$, $s \geq 2$, then G is a group of order s and $\mathcal{L}$ is an affine plane of order s.*

Proof. Clearly, G has order $(s-1)+1=s$. The linear space $\mathcal{L}$ has $1+(s-1)(s+1)=s^2$ points and s points on each line. So, $\mathcal{L}$ is an affine plane of order s. □

In particular, if Q is a generalized quadrangle of order $(4,6)$, then we may suppose that $\mathcal{L}$ is the Desarguesian affine plane $\mathrm{AG}(2,5)$ coordinatized by the finite field $\mathbb{F}_5$ and that G is the additive group of $\mathbb{F}_5$. In the following section, we will show that $\mathrm{AG}(2,5)$ admits only one admissible triple up to equivalence.

3 Determination of all admissible triples arising from $\mathrm{AG}(2,5)$

We will determine all admissible triples $(AG(2,5),\mathbb{F}_5,\Delta)$. By Lemma 1, we may choose an arbitrary point o in $\mathrm{AG}(2,5)$ and suppose that $\Delta(o,x)=0$ for every point of $\mathrm{AG}(2,5)$. If x and y are two different points of $\mathrm{AG}(2,5)$, then by Property (AT) we have that $o\in xy$ if and only if $\Delta(x,y)=0$.

Lemma 3 *(a) If $L=\{x_1,x_2,x_3,x_4,x_5\}$ is a line of $\mathrm{AG}(2,5)$ not passing through o, then $\mathbb{F}_5=\{\Delta(x_1,x_i)\,|\,i\in\{1,\ldots,5\}\}$.*

(b) If L is a line not passing through o, then we can label the points of L by the elements of the set $\{x_i\,|\,i\in\mathbb{F}_5\}$ such that $\Delta(x_i,x_j)=j-i$.

(c) For every $f\in\mathbb{F}_5$ and for every point x different from o, there are precisely five points y such that $\Delta(x,y)=f$.

(d) If L is a line through the origin and if x is a point not contained in L, then $\mathbb{F}_5=\{\Delta(x,l)\,|\,l\in L\}$.

Proof.

(a) For all $i,j\in\{1,\ldots,5\}$ with $i\neq j$, we have $\Delta(x_i,x_j)\neq 0$. Hence, $\Delta(x_1,x_j)=\Delta(x_1,x_i)+\Delta(x_i,x_j)\neq\Delta(x_1,x_i)$.

(b) Choose a point of L and label it x_1. Then for every $f\in\mathbb{F}_5$, let x_f denote the unique point of L such that $\Delta(x_1,x_f)=f-1$. Then $\Delta(x_i,x_j)=\Delta(x_i,x_1)+\Delta(x_1,x_j)=(1-i)+(j-1)=j-i$ for all $i,j\in\mathbb{F}_5$.

(c) If $f=0$, then the required points are precisely the five points on the line ox. If $f\neq 0$, then by (a) there exists a point y with $\Delta(x,y)=f$ on each of the five lines through x different from ox.

(d) Let l_1 and l_2 denote two different points of L. Then $\Delta(l_1, l_2) = 0$. By Property (AT) it follows that $\Delta(x, l_2) \neq \Delta(x, l_1) + \Delta(l_1, l_2) = \Delta(x, l_1)$. □

Now, consider an arbitrary line L not passing through o and an arbitrary line M through o not parallel with L. By Lemma 3 we may label the points of L by the elements of the set $\{x_f \mid f \in \mathbb{F}_5\}$ in such a way that $L \cap M = \{x_1\}$ and $\Delta(x_i, x_j) = j - i$ for all $i, j \in \mathbb{F}_5$. Also, by Lemma 3, $\mathbb{F}_5 = \{\Delta(x_2, m) \mid m \in M\}$. Now, $\Delta(x_2, 0) = 0$ and $\Delta(x_2, x_1) = 4$. Let p, q and r denote the points of M such that $\Delta(x_2, p) = 3$, $\Delta(x_2, q) = 2$ and $\Delta(x_2, r) = 1$.

- For every $i \in \{3, 4, 5\}$, we have $\Delta(p, x_i) \notin \{\Delta(p, o)+\Delta(o, x_i), \Delta(p, x_1)+\Delta(x_1, x_i), \Delta(p, x_2)+\Delta(x_2, x_i)\}$ or $\Delta(p, x_i) \notin \{0, i-1, i\}$. So, $\Delta(p, x_3) \in \{1, 4\}$, $\Delta(p, x_4) \in \{1, 2\}$ and $\Delta(p, x_5) \in \{1, 2, 3\}$. This leads to 12 possibilities for the triple $(\Delta(p, x_3), \Delta(p, x_4), \Delta(p, x_5))$, but since we must also have that $\Delta(p, x_5) - \Delta(p, x_4) \neq 1$, $\Delta(p, x_5) - \Delta(p, x_3) \neq 2$ and $\Delta(p, x_4) - \Delta(p, x_3) \neq 1$, only three possibilities remain, namely $(1, 1, 1)$, $(4, 1, 3)$ and $(4,2,2)$.
- For every $i \in \{3, 4, 5\}$, we have $\Delta(q, x_i) \notin \{\Delta(q, o)+\Delta(o, x_i), \Delta(q, x_1)+\Delta(x_1, x_i), \Delta(q, x_2) + \Delta(x_2, x_i)\}$ or $\Delta(q, x_i) \notin \{0, i-1, i+1\}$. So, $\Delta(q, x_3) \in \{1, 3\}$, $\Delta(q, x_4) \in \{1, 2, 4\}$ and $\Delta(q, x_5) \in \{2, 3\}$. Since $\Delta(q, x_5) - \Delta(q, x_4) \neq 1$, $\Delta(q, x_5) - \Delta(q, x_3) \neq 2$ and $\Delta(q, x_4) - \Delta(q, x_3) \neq 1$, we again have only three possibilities for $(\Delta(q, x_3), \Delta(q, x_4), \Delta(q, x_5))$, namely the possibilities (1,4,2), (3,1,3) and (3,2,2).
- For every $i \in \{3, 4, 5\}$, we have $\Delta(r, x_i) \notin \{\Delta(r, o)+\Delta(o, x_i), \Delta(r, x_1)+\Delta(x_1, x_i), \Delta(r, x_2) + \Delta(x_2, x_i)\}$ or $\Delta(r, x_i) \notin \{0, i-1, i+2\}$. So, $\Delta(r, x_3) \in \{1, 3, 4\}$, $\Delta(r, x_4) \in \{2, 4\}$ and $\Delta(r, x_5) \in \{1, 3\}$. Since $\Delta(r, x_5) - \Delta(r, x_4) \neq 1$, $\Delta(r, x_5) - \Delta(r, x_3) \neq 2$ and $\Delta(r, x_4) - \Delta(r, x_3) \neq 1$, we again have only three possibilities for $(\Delta(r, x_3), \Delta(r, x_4), \Delta(r, x_5))$, namely the possibilities (1,4,1), (3,2,1) and (4,4,3).

Now, for every point x of AG(2, 5), we define

$$A_x := (\Delta(x, x_1), \Delta(x, x_2), \Delta(x, x_3), \Delta(x, x_4), \Delta(x, x_5)).$$

By the above reasoning we know that there are at most 27 possibilities for (A_p, A_q, A_r). By Lemma 3, we have that the elements $\Delta(x_i, p)$, $\Delta(x_i, q)$ and $\Delta(x_i, r)$ are mutually different for every $i \in \{3, 4, 5\}$. Using this fact, only four cases remain:

(I) $A_p = (0, 2, 1, 1, 1)$, $A_q = (0, 3, 3, 2, 2)$ and $A_r = (0, 4, 4, 4, 3)$;

(II) $A_p = (0,2,4,1,3)$, $A_q = (0,3,1,4,2)$ and $A_r = (0,4,3,2,1)$;

(III) $A_p = (0,2,4,1,3)$, $A_q = (0,3,3,2,2)$ and $A_r = (0,4,1,4,1)$;

(IV) $A_p = (0,2,4,2,2)$, $A_q = (0,3,3,1,3)$ and $A_r = (0,4,1,4,1)$.

Now, consider the line M' through the points o and x_2 and let p', q' and r' those points on M' such that $\Delta(x_3,p') = 3$, $\Delta(x_3,q') = 2$ and $\Delta(x_3,r') = 1$. By symmetry, we have the following possibilities for $A_{p'}$ $A_{q'}$ and $A_{r'}$:

(I') $A_{p'} = (1,0,2,1,1)$, $A_{q'} = (2,0,3,3,2)$ and $A_{r'} = (3,0,4,4,4)$;

(II') $A_{p'} = (3,0,2,4,1)$, $A_{q'} = (2,0,3,1,4)$ and $A_{r'} = (1,0,4,3,2)$;

(III') $A_{p'} = (3,0,2,4,1)$, $A_{q'} = (2,0,3,3,2)$ and $A_{r'} = (1,0,4,1,4)$;

(IV') $A_{p'} = (2,0,2,4,2)$, $A_{q'} = (3,0,3,3,1)$ and $A_{r'} = (1,0,4,1,4)$.

We will now show that only the case (II) + (II') can occur by deriving a contradiction for each of the 15 remaining cases. We will use the following observation which holds for any two different points x and y of AG(2,5) not contained in L:

- if the line xy is parallel with L, then no entry of the vector $A_x - A_y$ is equal to $\Delta(x,y)$;
- if the line xy intersects L in the point x_i, then the i-th entry of $A_x - A_y$ is equal to $\Delta(x,y)$ and the remaining entries are different from $\Delta(x,y)$.

The case (I) + (I')

In the following table, we list the possibilities for $\Delta(p,p')$, $\Delta(p,q')$ and $\Delta(p,r')$ using the above-mentioned observations. One should interpret the table in the following way: if the line pq' is parallel with L, then $\Delta(p,q') \in \{1\}$ (notice that $\Delta(p,q') = 0$ is not possible), if the line pq' is not parallel with L, then $\Delta(p,q') \in \{4\}$.

line	difference	$\parallel$	$\nparallel$
pp'	$A_p - A_{p'} = (4,2,4,0,0)$	$\{1,3\}$	—
pq'	$A_p - A_{q'} = (3,2,3,3,4)$	$\{1\}$	$\{4\}$
pr'	$A_p - A_{r'} = (2,2,2,2,2)$	$\{1,3,4\}$	—

From the table it follows that pp' and pr' are two lines through p parallel with L, a contradiction.

The case (I) + (II')

line	difference	$\parallel$	$\nparallel$
qp'	$A_q - A_{p'} = (2,3,1,3,1)$	$\{4\}$	—
qq'	$A_q - A_{q'} = (3,3,0,1,3)$	$\{2,4\}$	$\{1\}$
qr'	$A_q - A_{r'} = (4,3,4,4,0)$	$\{1,2\}$	—

From the table it follows that qp' and qr' are two lines through q parallel with L, a contradiction.

The case (I) + (III')

line	difference	$\parallel$	$\nparallel$
pp'	$A_p - A_{p'} = (2,2,4,2,0)$	$\{1,3\}$	$\{4\}$
pq'	$A_p - A_{q'} = (3,2,3,3,4)$	$\{1\}$	$\{4\}$
pr'	$A_p - A_{r'} = (4,2,2,0,2)$	$\{1,3\}$	—

The line pr' is the unique line through p parallel with L. So, $\Delta(p,p') = 4$ and $\Delta(p,q') = 4$, contradicting the fact that $\Delta(p,p') \neq \Delta(p,q')$.

The case (I) + (IV')

line	difference	$\parallel$	$\nparallel$
pp'	$A_p - A_{p'} = (3,2,4,2,4)$	$\{1\}$	—
pq'	$A_p - A_{q'} = (2,2,3,3,0)$	$\{1,4\}$	—
pr'	$A_p - A_{r'} = (4,2,2,0,2)$	$\{1,3\}$	—

Each of the lines pp', pq' and pr' would be parallel with L, a contradiction.

The case (II) + (I')

line	difference	$\parallel$	$\nparallel$
pp'	$A_p - A_{p'} = (4,2,2,0,2)$	$\{1,3\}$	—
pq'	$A_p - A_{q'} = (3,2,1,3,1)$	$\{4\}$	—
pr'	$A_p - A_{r'} = (2,2,0,2,4)$	$\{1,3\}$	$\{4\}$

Each of the lines pp' and pq' would be parallel with L, a contradiction.

The case (II) + (III')

line	difference	$\parallel$	$\nparallel$
pp'	$A_p - A_{p'} = (2,2,2,2,2)$	$\{1,3,4\}$	—
pq'	$A_p - A_{q'} = (3,2,1,3,1)$	$\{4\}$	—
pr'	$A_p - A_{r'} = (4,2,0,0,4)$	$\{1,3\}$	—

Each of the lines pp', pq' and pr' would be parallel with L, a contradiction.

The case (II) + (IV')

line	difference	$\parallel$	$\nparallel$
pp'	$A_p - A_{p'} = (3,2,2,2,1)$	$\{4\}$	$\{1\}$
pq'	$A_p - A_{q'} = (2,2,1,3,2)$	$\{4\}$	$\{1,3\}$
pr'	$A_p - A_{r'} = (4,2,0,0,4)$	$\{1,3\}$	—

The line pr' is the unique line through p parallel with L. But then each of the elements $\Delta(p,p')$, $\Delta(p,q')$ and $\Delta(p,r')$ would belong to $\{1,3\}$, contradicting the fact that they are mutually different.

The case (III) + (I')

line	difference	$\parallel$	$\nparallel$
pp'	$A_p - A_{p'} = (4,2,2,0,2)$	$\{1,3\}$	—
pq'	$A_p - A_{q'} = (3,2,1,3,1)$	$\{4\}$	—
pr'	$A_p - A_{r'} = (2,2,0,2,4)$	$\{1,3\}$	$\{4\}$

Each of the lines pp' and pq' would be parallel with L, a contradiction.

The case (III) + (II')

line	difference	$\parallel$	$\nparallel$
qp'	$A_q - A_{p'} = (2,3,1,3,1)$	$\{4\}$	—
qq'	$A_q - A_{q'} = (3,3,0,1,3)$	$\{2,4\}$	$\{1\}$
qr'	$A_q - A_{r'} = (4,3,4,4,0)$	$\{1,2\}$	—

Each of the lines qp' and qr' would be parallel with L, a contradiction.

The case (III) + (III')

line	difference	$\parallel$	$\nparallel$
pp'	$A_p - A_{p'} = (2,2,2,2,2)$	$\{1,3,4\}$	—
pq'	$A_p - A_{q'} = (3,2,1,3,1)$	$\{4\}$	—
pr'	$A_p - A_{r'} = (4,2,0,0,4)$	$\{1,3\}$	—

Each of the lines pp', pq' and pr' would be parallel with L, a contradiction.

The case (III) + (IV')

line	difference	$\parallel$	$\nparallel$
pp'	$A_p - A_{p'} = (3,2,2,2,1)$	$\{4\}$	$\{1\}$
pq'	$A_p - A_{q'} = (2,2,1,3,2)$	$\{4\}$	$\{1,3\}$
pr'	$A_p - A_{r'} = (4,2,0,0,4)$	$\{1,3\}$	—

The line pr' is the unique line through p parallel with L. But then each of the elements $\Delta(p,p')$, $\Delta(p,q')$ and $\Delta(p,r')$ would belong to $\{1,3\}$, contradicting the fact that they are mutually different.

The case (IV) + (I')

line	difference	$\parallel$	$\nparallel$
pp'	$A_p - A_{p'} = (4,2,2,1,1)$	$\{3\}$	—
pq'	$A_p - A_{q'} = (3,2,1,4,0)$	—	$\{1,4\}$
pr'	$A_p - A_{r'} = (2,2,0,3,3)$	$\{1,4\}$	—

Each of the lines pp' and pr' would be parallel with L, a contradiction.

The case (IV) + (II')

line	difference	$\parallel$	$\nparallel$
pp'	$A_p - A_{p'} = (2,2,2,3,1)$	$\{4\}$	$\{1,3\}$
pq'	$A_p - A_{q'} = (3,2,1,1,3)$	$\{4\}$	—
pr'	$A_p - A_{r'} = (4,2,0,4,0)$	$\{1,3\}$	—

Each of the lines pq' and pr' would be parallel with L, a contradiction.

The case (IV) + (III')

line	difference	$\parallel$	$\nparallel$
qp'	$A_q - A_{p'} = (2,3,1,2,2)$	$\{4\}$	$\{1\}$
qq'	$A_q - A_{q'} = (3,3,0,3,1)$	$\{2,4\}$	$\{1\}$
qr'	$A_q - A_{r'} = (4,3,4,0,4)$	$\{1,2\}$	—

The line qr' is the unique line through q parallel with L. But then $\Delta(q,p') = 1$ and $\Delta(q,q') = 1$, contradicting the fact that $\Delta(q,p') \neq \Delta(q,q')$.

The case (IV) + (IV')

line	difference	$\parallel$	$\nparallel$
qp'	$A_q - A_{p'} = (3,3,1,2,1)$	$\{4\}$	$\{2\}$
qq'	$A_q - A_{q'} = (2,3,0,3,2)$	$\{1,4\}$	—
qr'	$A_q - A_{r'} = (4,3,4,0,4)$	$\{1,2\}$	—

Each of the lines qq' and qr' would be parallel with L, a contradiction.

Since only the case (II) + (II') can occur, we have shown the following (see also Lemma 3, (a)+(d)).

Lemma 4 *If x is a point of* AG(2, 5) *different from o and if L is a line of* AG(2, 5) *not parallel with ox, then $\mathbb{F}_5 = \{\Delta(x,l) \mid l \in L\}$.*

Definition. For every point x of AG(2, 5) different from o and for every $f \in \mathbb{F}_5$, let L_x^f denote the set of points of y such that $\Delta(x,y) = f$.

Lemma 5 *For every point x of* AG(2, 5) *different from o and for every $f \in \mathbb{F}_5$, L_x^f is a line parallel with ox.*

Proof. By Lemma 3 (c), L_x^f contains precisely 5 points. If y_1 and y_2 are two different points of L_x^f, then by Lemma 4, y_1y_2 must be parallel with ox. The lemma now easily follows. □

Another consequence of the fact that only the case (II) +(II') can occur is the following lemma.

Lemma 6 *Let L be a line of* AG(2, 5) *not containing o, let x be a point of* AG(2, 5) *different from o such that ox is not parallel with L and put $L \cap ox = \{u\}$. Let $\kappa \in \mathbb{F}_5$ and $v, w \in L$ such that $\Delta(u,w) = \kappa \cdot \Delta(u,v)$. Then $\Delta(x,w) = \kappa \cdot \Delta(x,v)$.*

We are now ready to determine Δ. In AG(2, 5) there exist points x^* and y^* such that $\Delta(x^*, y^*) = 1$. Since the points o, x^* and y^* are not collinear we can choose our reference system in such a way that $o = (0,0)$, $x^* = (1,0)$ and $y^* = (0,1)$. By Lemma 3 (d), there exists a permutation λ of $\mathbb{F}_5$ such that $\Delta[(1,0),(0,\lambda(i))] = i$ for every $i \in \mathbb{F}_5$. Obviously, $\lambda(0) = 0$ and $\lambda(1) = 1$. Similarly, there exists a permutation μ of $\mathbb{F}_5$ such that $\Delta[(\mu(i),0),(0,1)] = i$ for every $i \in \mathbb{F}_5$. Again $\mu(0) = 0$ and $\mu(1) = 1$.

Lemma 7 *For all $i, j \in \mathbb{F}_5$, $\Delta[(\mu(i),0),(0,\lambda(j))] = ij$.*

Proof. Obviously, this property holds if $i \in \{0,1\}$ or $j \in \{0,1\}$. So, suppose that $i, j \in \{2,3,4\}$. By Lemma 5, we have $\Delta[(1,0),(1,\lambda(j))] = \Delta[(1,0),(0,\lambda(j))] = j = j \cdot \Delta[(1,0),(0,1)] = j \cdot \Delta[(1,0),(1,1)]$. By Lemmas 5 and 6, we then have $\Delta[(\mu(i),0),(0,\lambda(j))] = \Delta[(\mu(i),0),(1,\lambda(j))] = j \cdot \Delta[(\mu(i),0),(1,1)] = j \cdot \Delta[(\mu(i),0),(0,1)] = ij$. □

Now, choose points (x_1, y_1) and (x_2, y_2) with $x_1 \neq 0 \neq y_1$. The line through (x_2, y_2) parallel with the line through $(0,0)$ and (x_1, y_1) contains the points $(\frac{x_2y_1 - y_2x_1}{y_1}, 0)$ and $(0, \frac{y_2x_1 - x_2y_1}{x_1})$. By Lemma 5, $\Delta[(x_1,y_1),(x_2,y_2)] = \Delta[(x_1,y_1)$ $(\frac{x_2y_1 - x_1y_2}{y_1}, 0)] = \Delta[(0,y_1),(\frac{x_2y_1 - x_1y_2}{y_1}, 0)] = -\mu^{-1}(\frac{x_2y_1 - x_1y_2}{y_1}) \cdot \lambda^{-1}(y_1)$.

On the other hand, $\Delta[(x_1,y_1),(x_2,y_2)] = \Delta[(x_1,y_1),(0,\frac{y_2x_1-x_2y_1}{x_1})] = \Delta[(x_1,0),(0,\frac{y_2x_1-x_2y_1}{x_1})] = \mu^{-1}(x_1)\cdot\lambda^{-1}(\frac{y_2x_1-x_2y_1}{x_1})$. As a consequence,

$$-\mu^{-1}(\frac{x_2y_1-x_1y_2}{y_1})\cdot\lambda^{-1}(y_1) = \mu^{-1}(x_1)\cdot\lambda^{-1}(\frac{y_2x_1-x_2y_1}{x_1})$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{F}_5$ with $x_1 \neq 0 \neq y_1$. Putting $x_1 = y_1 \neq 0$ and $x_2 \neq y_2$, we obtain

$$\frac{\lambda^{-1}(x_1)}{\mu^{-1}(x_1)} = -\frac{\lambda^{-1}(y_2-x_2)}{\mu^{-1}(x_2-y_2)}.$$

Hence, $\lambda = \mu$ and $\lambda^{-1}(a) = -\lambda^{-1}(-a)$ for all $a \in \mathbb{F}_5$. (Recall that $\lambda(1) = \mu(1) = 1$ and $\lambda(0) = \mu(0) = 0$.) In particular, $\lambda^{-1}(4) = -\lambda^{-1}(1) = 4$. Now, for all $x_1, y_1, x_2, y_2 \in \mathbb{F}_5$ with $x_1 \neq 0 \neq y_1$, we have

$$\Delta[(x_1,y_1),(x_2,y_2)] = \lambda^{-1}(x_1)\lambda^{-1}(\frac{x_1y_2-x_2y_1}{x_1}).$$

Since the points $(1,1)$, $(1,2)$ and $(1,3)$ are collinear we have $\Delta[(1,1),(1,2)] + \Delta[(1,2),(1,3)] = \Delta[(1,1),(1,3)]$ or $\lambda^{-1}(2) = 2$. Now, $\lambda^{-1}(3) = -\lambda^{-1}(2) = 3$. So, we have shown that λ and μ are trivial permutations of $\mathbb{F}_5$ and that $\Delta[(x_1,y_1),(x_2,y_2)] = x_1y_2 - x_2y_1$ for all x_1, y_1, x_2, y_2 with $x_1 \neq 0 \neq y_1$. One easily verifies that this formula remains valid if $x_1 = 0$ or $y_1 = 0$. So,

$$\Delta[(x_1,y_1),(x_2,y_2)] = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

for all points (x_1,y_1) and (x_2,y_2) of AG(2,5). We have shown earlier that this Δ indeed gives rise to an admissible triple.

Conclusion. We have shown that AG(2,5) admits, up to equivalence, only one admissible triple. As a consequence, there exists, up to equivalence, only one pair (Q,S) where Q is a generalized quadrangle of order $(4,6)$ and where S is a spread of symmetry of Q. This also implies that there exists, up to equivalence, only one pair (Q,x) where Q is a generalized quadrangle of order 5 and where x is a center of symmetry of Q.

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