SOME CHARACTERIZATIONS OF HARMONIC BERGMAN SPACES IN THE UNIT BALL

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Abstract

Some new characterizations for harmonic Bergman space on the unit ball B in $\mathbb{R}^n$ are given in this paper. They can be described as derivative free characterizations.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper $B(a,r) = \{x \in \mathbb{R}^n \mid |x-a| < r\}$ denotes the open ball centered at $a \in \mathbb{R}^n$ of radius $r > 0$, where $|x|$ denotes the norm of $x \in \mathbb{R}^n$, B is the open unit ball in $\mathbb{R}^n$ and $\partial B = \{x \in \mathbb{R}^n \mid |x| = 1\}$ is the boundary of B. Let dV denote the Lebesgue measure on $\mathbb{R}^n$, v_n the volume of B, $V_{a,r} = V(B(a,r))$, $dV_s(x) = (1-|x|)^s dV(x)$ and $\mathcal{H}(B)$ the set of harmonic functions on B.

For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space $b^p_\alpha(B) = b^p_\alpha$ is the space of all harmonic functions u on B such that

$$\|u\|_{b^p_\alpha} = \left(\int_B |u(x)|^p (1-|x|)^\alpha dV(x)\right)^{1/p} < +\infty.$$

When $p \geq 1$, $b^p_\alpha(B)$ is a Banach space with the norm $\|\cdot\|_{b^p_\alpha}$ and when $p \in (0,1)$ a complete metric space with the metric $d_{b^p_\alpha}(u,v) = \int_B |(u-v)(x)|^p dV_\alpha(x)$. For some papers regarding weighted Bergman type spaces of harmonic functions, see, for example, [1], [2], [4], [5], [7], [8], [9], [12], [14] and the references therein.

Throughout the paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \asymp b$ means that there is a positive constant C such that $C^{-1}b \leq a \leq Cb$ and we say that the quantities a and b are comparable.

We say that a locally integrable function f on B possesses the HL-property, with a constant $C > 0$ if

$$f(a) \leq \frac{C}{r^n}\int_{B(a,r)} f(x)dV(x) \quad \text{whenever } \overline{B(a,r)} \subset B.$$

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Every subharmonic function possesses the $HL-$property when $C = 1/v_n$. In [3] Fefferman and Stein proved the following result regarding the HL-property of $|u|^p$, when $u \in \mathcal{H}(B)$.

Lemma A. *Let $u \in \mathcal{H}(B)$ and $p > 0$. Then there is a constant C depending only on n and p, such that*

$$|u(x)|^p \leq \frac{C}{r^n} \int_{B(x,r)} |u(y)|^p dV(y),$$

when $0 < r < 1 - |x|$.

The next lemma contains a known characterization for harmonic Bergman space b^p_α ([5]).

Lemma B. *Suppose $0 < p < \infty$ and $\alpha > -1$. Then the following relationship holds*

$$|u(0)|^p + \int_B |\nabla u(x)|^p (1 - |x|)^{p+\alpha} dV(x) \asymp \int_B |u(x)|^p (1 - |x|)^\alpha dV(x),$$

for every $u \in \mathcal{H}(B)$.

Motivated by Lemma B, our aim here is to give some derivative free characterizations for harmonic Bergman space b^p_α (similar to that in Lemma B but without $|\nabla u|$). The paper can be considered as a continuation of our investigations in the area, see [8, 9, 10, 11, 12, 13, 14].

2. Equivalent conditions for the Bergman harmonic space

Now we are in a position to formulate and prove the main result in this paper.

Theorem 1. *Assume that $u \in \mathcal{H}(B)$, $a, b \in [0, 1]$, $a + b = 1$ and $0 < p < \infty$. Then if $p \geq 1$ the following statements are equivalent:*

(a) $u \in b^p_\alpha(B)$;
(b) $I_1 := \int_B \left(\sup_{y \in B(x,(1-|x|)/2)} \frac{|u(x)-u(y)|}{|x-y|} (1 - |x|^2)^a (1 - |y|^2)^b \right)^p dV_\alpha(x) < \infty$;
(c) $I_2 := \int_B \left(\frac{1}{V_{x,(1-|x|)/2}} \int_{B(x,(1-|x|)/2)} \frac{|u(x)-u(y)|^p}{|x-y|^p} (1-|x|^2)^{pa} (1-|y|^2)^{pb} dV(y) \right) dV_\alpha(x) <$ ∞;
(d) $I_3 := \int_B \left(\frac{1}{V_{x,(1-|x|)/2}} \int_{B(x,(1-|x|)/2)} \frac{|u(x)-u(y)|}{|x-y|} (1-|x|^2)^a (1-|y|^2)^b dV(y) \right)^p dV_\alpha(x) <$ ∞;
(e) $I_4 := \int_B \int_{B(x,(1-|x|)/2)} \frac{|u(y)-u(x)|^p}{|x-y|^p} (1 - |y|)^{p+\alpha-n} dV(y) dV(x) < \infty$.

If $p \in (0, 1)$ then all these statements except (c), are equivalent.

Moreover, if $u \in b^p_\alpha$ and $p \geq 1$ then $I_j \asymp \|u - u(0)\|^p_{b^p_\alpha}$, $j \in \{1, 2, 3, 4\}$, and if $p \in (0, 1)$ then $I_j \asymp \|u - u(0)\|^p_{b^p_\alpha}$, $j \in \{1, 3, 4\}$.

Proof. Throughout the proof we may assume $u(0) = 0$, otherwise we can consider the function $v(x) = u(x) - u(0) \in \mathcal{H}(B)$, for which obviously holds $v(0) = 0$.

$(e) \Rightarrow (a)$. The proof of the implication follows the idea of the present author suggested to the authors of [6, p. 181]. By the Cauchy's inequality we have that there is a positive constant C independent of $u \in \mathcal{H}(B)$, such that

$$(1 - |x|)|\nabla u(x)| \le C \sup_{s \in B(x,(1-|x|)/4)} |u(s) - u(x)|. \tag{1}$$

Taking the inequality in (1) to the p-th power and applying Lemma A to the harmonic function $g(s) = u(s) - u(x)$, we have that

$$\begin{aligned}[(1 - |x|)|\nabla u(x)|]^p &\le C \sup_{s \in B(x,(1-|x|)/4)} |u(s) - u(x)|^p \\ &\le \frac{C}{(1 - |x|)^n} \int_{B(x,(1-|x|)/2)} |u(y) - u(x)|^p dV(y). \end{aligned} \tag{2}$$

From (2) and by employing the following estimates

$$\frac{1 - |y|}{|x - y|} > \frac{1 - |x|}{2|x - y|} > 1, \qquad y \in B(x, (1 - |x|)/2) \tag{3}$$

and

$$\frac{1}{2}(1 - |x|) < 1 - |y| < \frac{3}{2}(1 - |x|), \qquad y \in B(x, (1 - |x|)/2), \tag{4}$$

it follows that

$$\begin{aligned}&\frac{C}{(1 - |x|)^n} \int_{B(x,(1-|x|)/2)} |u(y) - u(x)|^p dV(y) \\ &\le C \int_{B(x,(1-|x|)/2)} |u(y) - u(x)|^p \frac{(1 - |y|^2)^{p-n}}{|x - y|^p} dV(y). \end{aligned} \tag{5}$$

Hence, from (2) and (5) we have that

$$[(1 - |x|)|\nabla u(x)|]^p \le C \int_{B(x,(1-|x|)/2)} |u(y) - u(x)|^p \frac{(1 - |y|^2)^{p-n}}{|x - y|^p} dV(y). \tag{6}$$

Multiplying (6) by $(1 - |x|)^\alpha dV(x)$, then integrating over B and employing (4), we obtain

$$\begin{aligned}&\int_B [(1 - |x|)|\nabla u(x)|]^p (1 - |x|)^\alpha dV(x) \\ &\le \int_B \int_{B(x,(1-|x|)/2)} \frac{|u(y) - u(x)|^p}{|x - y|^p} (1 - |y|^2)^{p-n} dV(y)(1 - |x|)^\alpha dV(x) \\ &\le C \int_B \int_{B(x,(1-|x|)/2)} \frac{|u(y) - u(x)|^p}{|x - y|^p} (1 - |y|)^{p+\alpha-n} dV(y) dV(x). \end{aligned} \tag{7}$$

Employing Lemma B and (7) it follows that

$$\|u\|^p_{b^p_\alpha} \le C \int_B \int_{B(x,(1-|x|)/2)} \frac{|u(y)-u(x)|^p}{|x-y|^p}(1-|y|)^{p+\alpha-n} dV(y)dV(x), \tag{8}$$

and consequently the implication.

$(d) \Rightarrow (a)$. By using (3) and the asymptotic relation $V_{x,(1-|x|)/2} \asymp (1-|x|)^n$, similar to (5) it can be proved that

$$(1-|x|)|\nabla u(x)| \le \frac{C}{V_{x,(1-|x|)/2}} \int_{B(x,(1-|x|)/2)} \frac{|u(y)-u(x)|}{|x-y|}(1-|x|)^a \, dV_b(y). \tag{9}$$

Taking (9) to the p-th power, multiplying such obtained inequality by $dV_\alpha(x)$ and integrating over B, and finally using Lemma B, it follows that

$$\|u\|^p_{b^p_\alpha} \le CI_3, \tag{10}$$

from which the implication follows.

$(a) \Rightarrow (b)$. First note that if $u \in \mathcal{H}(B)$, then using the fact that partial derivatives of u are again harmonic, Lemma A and some simple calculations, it follows that the function $|\nabla u|^p$, $p > 0$ has HL-property too.

From this, by the mean value theorem, inequalities in (4), Fubini's theorem and Lemma B, we have that

$$\begin{aligned}
&\int_B \sup_{y\in B(x,(1-|x|)/2)} \frac{|u(y)-u(x)|^p}{|x-y|^p}(1-|y|)^{bp}(1-|x|)^{ap+\alpha} dV(x) \\
\le\ & C\int_B \sup_{\zeta\in B(x,(1-|x|)/2)} |\nabla u(\zeta)|^p(1-|x|)^{p+\alpha} dV(x) \\
\le\ & C\int_B \int_{\zeta\in B(x,3(1-|x|)/4)} |\nabla u(\zeta)|^p(1-|\zeta|)^{p+\alpha-n} dV(\zeta)dV(x) \\
\le\ & C\int_B \Big((1-|\zeta|)|\nabla u(\zeta)|\Big)^p (1-|\zeta|)^{\alpha-n} \int_{x\in A(\zeta)} dV(x)dV(\zeta) \\
\le\ & C\int_B [(1-|\zeta|)|\nabla u(\zeta)|]^p(1-|\zeta|)^\alpha dV(\zeta) \le C\|u\|^p_{b^p_\alpha}.
\end{aligned} \tag{11}$$

where we have used the fact that the quantity $\int_{x\in A(\zeta)} dV(x)$ is bounded, since the set $A(\zeta)$ is contained in the ball $B(\zeta, 3(1-|\zeta|))$.

Note that the implications $(b) \Rightarrow (c)$ and $(b) \Rightarrow (d)$ are trivial since

$$\max\{I_2, I_3\} \le CI_1. \tag{12}$$

$(c) \Rightarrow (d)$. By Jensen's inequality (we only use here the condition $p \ge 1$) it is easy to see that $I_3 \le CI_2$.

$(a) \Rightarrow (e)$. By (4) and some simple calculation, it is clear that

$$\int_B \int_{B(x,(1-|x|)/2)} \frac{|u(y)-u(x)|^p}{|x-y|^p}(1-|y|)^{p+\alpha-n} dV(y)dV(x)$$
$$\leq C\int_B \Big(\sup_{y\in B(x,(1-|x|)/2)} \frac{|u(x)-u(y)|}{|x-y|}(1-|x|^2)^a(1-|y|^2)^b \Big)^p dV_\alpha(x). \quad (13)$$

From (11) and (13) we have that $I_4 \leq C\|u\|^p_{b^p_\alpha}$, from which the first part of the theorem follows.

The second part of the theorem follows from (8), (10), (11), (12), (13) and the implications $(c) \Rightarrow (d) \Rightarrow (a) \Rightarrow (e)$, applied to the function $u(x) - u(0)$. □

Remark. Note that in view of (3) it easily follows that condition *(e)* in Theorem 1 is equivalent with

$$\int_B \int_{B(x,(1-|x|)/2)} \frac{|u(y)-u(x)|^p}{|x-y|^p}(1-|y|)^s dV(y)(1-|x|)^t dV(x) < \infty,$$

when s and t are real numbers such that $s + t = p + \alpha - n$.

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