On the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$ domination number of a graph

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September 4, 2007

Abstract

Suppose a network facility location problem is modelled by means of an undirected, simple graph $G = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{v_1, \dots, v_n\}$. Let $\boldsymbol{r} = (r_1, \dots, r_n)$ and $\boldsymbol{s} = (s_1, \dots, s_n)$ be vectors of nonnegative integers and consider the combinatorial optimization problem of locating the minimum number, $\gamma\langle \boldsymbol{r}, \boldsymbol{s}, G\rangle$ (say), of commodities on the vertices of G such that at least s_j commodities are located in the vicinity of (*i.e.* in the closed neigbourhood of) vertex v_j, with no more than r_j commodities placed at vertex v_j itself, for all $j = 1, \dots, n$. In this paper we establish lower and upper bounds on the parameter $\gamma\langle \boldsymbol{r}, \boldsymbol{s}, G\rangle$ for a general graph G. We also determine this parameter exactly for certain classes of graphs, such as paths, cycles, complete graphs, complete bipartite graphs and establish good upper bounds on $\gamma\langle \boldsymbol{r}, \boldsymbol{s}, G\rangle$ for a class of grid graphs in the special case where $r_j = r$ and $s_j = s$ for all $j = 1, \dots, n$.

Keywords: Graph domination.

AMS Classification: 05C69, 05C99.

1 Introduction

Cockayne [2] recently introduced a general framework for graph domination, called $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$ domination. Let $G = (\mathcal{V}, \mathcal{E})$ be a simple graph with vertex set $\mathcal{V} = \{v_1, \dots, v_n\}$. Consider an n-vector $\boldsymbol{r} = (r_1, \dots, r_n)$ of non-negative integers r_j $(j = 1, \dots, n)$. We follow the notation in [2] and define an $\boldsymbol{r}$-*function* of G as a function $f : \mathcal{V} \mapsto \mathbb{N}_0$ satisfying $f(v_j) \leq r_j$ for all $j = 1, \dots, n$, where $\mathbb{N}_0$ denotes the set of non-negative integers. The *weight* of f is defined as $|f| = \sum_{v_j \in \mathcal{V}} f(v_j)$.

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Consider another n-vector $s = (s_1, \ldots, s_n)$ of non-negative integers s_j ($j = 1, \ldots, n$). An r-function of G is said to be *s-dominating* if

$$\sum_{u \in N[v_j]} f(u) \geq s_j \quad \text{for all } v_j \in \mathcal{V},$$

where $N[v_j]$ denotes the closed neighbourhood of v_j in G. It was noted in [2] that an s-dominating r-function of G exists if and only if

$$\sum_{v_k \in N[v_j]} r_k \geq s_j \quad \text{for all} \quad j \in \{1, \ldots, n\}. \tag{1}$$

Finally, let $\gamma\langle r, s, G\rangle$ denote the smallest weight of an s-dominating r-function of G. If the graph G is clear from the context, this parameter is denoted merely by $\gamma\langle r, s\rangle$.

2 General bounds on $\gamma\langle r, s\rangle$

Let r, s and G satisfy (1), and define, for every vertex $v_j \in \mathcal{V}$, the *constraint difference*

$$T_j = \left(\sum_{v_k \in N[v_j]} r_k \right) - s_j, \quad j = 1, \ldots, n \tag{2}$$

as well as the *constraint slackness*

$$T_j^* = \min \left\{ \min_{v_k \in N[v_j]} \{T_k\}, r_j \right\}, \quad j = 1, \ldots, n. \tag{3}$$

Then $0 \leq T_j^* \leq T_j$ and $T_j^* \leq r_j$ for all $j = 1, \ldots, n$ by (1) and (3). The constraint slackness T_j^* is the maximum difference between r_j and the value of any s-dominating r-function of G at $v_j \in \mathcal{V}$. (If $T_j^* = 0$, then necessarily $f(v_j) = r_j$.)

Recall that a packing of G is a subset $\mathcal{P} \subseteq \mathcal{V}$ with the property that the distance in G between any two vertices in $\mathcal{P}$ is at least 3. Denote the cardinality of a maximum packing by ρ. The following general bounds hold for $\gamma\langle r, s\rangle$.

Theorem 1 *Let $T_1^*, \ldots, T_m^*$ be the constraint slacknesses associated with the vertices of a packing of cardinality m for a graph G with maximum degree Δ. Then*

$$\left\lceil \frac{\sum_{i=1}^n s_i}{1 + \Delta} \right\rceil \leq \gamma\langle r, s\rangle \leq \sum_{i=1}^n r_i - \sum_{j=1}^m T_j^*$$

for all r, s satisfying (1).

Proof: Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G with minimum weight. Then

$$\sum_{v_i \in \mathcal{V}} f(v_i)(\deg v_i + 1) \geq \sum_{i=1}^{n} s_i.$$

The lower bound follows, because

$$\sum_{v_i \in \mathcal{V}} f(v_i)(\deg v_i + 1) \leq (\Delta + 1) \sum_{v_i \in \mathcal{V}} f(v_i) = (\Delta + 1)\gamma\langle \boldsymbol{r}, \boldsymbol{s} \rangle.$$

For the upper bound, let $\mathcal{P}$ be a packing of cardinality m and consider the $\boldsymbol{r}$-function

$$f'(v_j) = \begin{cases} r_j - T_j^* & \text{if } v_j \in \mathcal{P} \\ r_j & \text{otherwise.} \end{cases}$$

Note that $0 \leq f'(v_j) \leq r_j$ for all $v_j \in \mathcal{P}$ by the definition of T_j^*. If $v_j \in \mathcal{P}$, then

$$\begin{aligned} \sum_{u \in N[v_j]} f'(u) &= \sum_{u \in N(v_j)} f'(u) + f(v_j) = \sum_{v_k \in N(v_j)} r_k + (r_j - T_j^*) \\ &= \sum_{v_k \in N[v_j]} r_k - T_j^* = T_j + s_j - T_j^* \geq s_j \end{aligned}$$

by (2). However, if $v_j \notin \mathcal{P}$, then v_j has at most one neighbour, v_i (say), in $\mathcal{P}$, so that

$$\sum_{u \in N[v_j]} f'(u) = \sum_{v_k \in N[v_j]} r_k - T_i^* \geq \sum_{v_k \in N[v_j]} r_k - T_j = \sum_{v_k \in N[v_j]} r_k - \left(\sum_{v_k \in N[v_j]} r_k - s_j \right) = s_j$$

by (2). Therefore f' is an $\boldsymbol{s}$-dominating function of G and hence

$$\gamma\langle \boldsymbol{r}, \boldsymbol{s} \rangle \leq |f'| = \sum_{j=1}^{n} r_j - \sum_{k=1}^{m} T_k^*. \qquad \blacksquare$$

In the balanced special case where $r_j = r$ (say) and $s_j = s$ (say) for all $j = 1, \ldots, n$, we refer merely to an r-function instead of an $\boldsymbol{r}$-function, which is s-dominating instead of $\boldsymbol{s}$-dominating, and to the parameter $\gamma\langle r, s \rangle$ instead of $\gamma\langle \boldsymbol{r}, \boldsymbol{s} \rangle$. In this special case the existence condition (1) reduces to $s \leq (\delta + 1)r$, where δ is the minimum degree of the graph, and it follows that $T_j = r(d_j + 1) - s$ for all $j = 1, \ldots, n$, where d_j denotes the degree of vertex $v_j \in \mathcal{V}$. Therefore $T_j^* \geq \min\{r(\delta + 1) - s, r\}$ for all $j = 1, \ldots, n$. Hence the bounds in Theorem 1 reduce to the following bounds in this special case.

Corollary 1 *For any graph G,*

$$\left\lceil \frac{sn}{\Delta+1} \right\rceil \leq \gamma\langle r,s\rangle \leq rn - \rho \min\{r(\delta+1)-s, r\}$$

for all $s \leq (\delta+1)r$.

Note also that in the classical domination setting (*i.e.* when $r_j = s_j = 1$ for all $j = 1, \ldots, n$) the lower bound in Corollary 1 reduces further to the well-known bound $\lceil n/(\Delta+1) \rceil \leq \gamma(G)$ for a graph G without isolated vertices (see, for example, Theorem 2.1 in [3]), which is sharp for the infinite class of complete graphs. In the classical domination setting the upper bound in Corollary 1 reduces to the bound $\gamma(G) \leq n - \rho$ for a graph G without isolated vertices. This bound coincides with the bound $\gamma(G) \leq n - \delta\rho$ by Chellali and Volkmann [1] for graphs with end-vertices. Otherwise, it is weaker than the Chellali and Volkmann bound. However, the bound $\gamma(G) \leq n - \rho$ is sharp for the infinite class of coronas of trees.

3 Special graph classes

In this section we consider only the balanced special case where $s_i = s$ (say) and $r_i = r$ (say) for all $i = 1, \ldots, n$ and establish values for and bounds on $\gamma\langle r,s\rangle$ for a number of special graph classes, including complete (bipartite) graphs, paths, cycles and a class of grid graphs.

3.1 Complete (bipartite) graphs

Let us first consider the values of $\gamma\langle r,s\rangle$ for complete graphs.

Proposition 1 (Complete graphs)
$\gamma\langle r,s,K_n\rangle = s$ *for all $n \in \mathbb{N}$ and all $r,s \in \mathbb{N}_0$ satisfying $s \leq nr$.*

Proof: It follows by Corollary 1 that $\gamma\langle r,s,K_n\rangle \geq \frac{sn}{(n-1)+1} = s$. Suppose $m, e \in \mathbb{N}_0$ such that $s = mr + e$ with $e < r$ and $m \leq n$, and consider the r-function

$$f(v_i) = \begin{cases} r & \text{if } i = 1, \ldots, m \\ e & \text{if } i = m+1 \\ 0 & \text{otherwise} \end{cases}$$

of K_n. Then f is clearly an s-dominating function of K_n and hence $\gamma\langle r,s,K_n\rangle \leq |f| = mr + e = s$. ■

Next we turn our attention to values of $\gamma\langle r,s\rangle$ for complete bipartite graphs.

Proposition 2 (Complete bipartite graphs)
Suppose $m, n \in \mathbb{N}$ and $r, s \in \mathbb{N}_0$ satisfying $m \geq n$ and $s \leq (n+1)r$. Then

$$
\begin{aligned}
\gamma\langle r, s, K_{m,n}\rangle &= \min\{X + Y\} && (4)\\
\text{for which } \lfloor X/n \rfloor + Y &\geq s, && (5)\\
\lfloor Y/m \rfloor + X &\geq s, && (6)\\
\lceil X/n \rceil &\leq r, && (7)\\
\lceil Y/m \rceil &\leq r. && (8)
\end{aligned}
$$

Proof: Suppose the partite sets of $K_{m,n}$ are M and N with $|M| = m$ and $|N| = n$. Consider the function $f : \mathcal{V} \mapsto \mathbb{N}_0$ given by

$$f(v_i) = \begin{cases} x_i & \text{if } v_i \in N \\ y_i & \text{if } v_i \in M \end{cases}$$

and let $X = \sum_{v_i \in N} x_i$ and $Y = \sum_{v_i \in M} y_i$. It is easy to see that if f is an s-dominating r-function of $K_{m,n}$ such that $x_i - x_j = d_{ij} \geq 2$ for some $i \neq j$, then f may be replaced by the s-dominating r-function

$$f'(v) = \begin{cases} x_i - \lfloor d_{ij}/2 \rfloor & \text{if } v = v_i \\ x_j + \lfloor d_{ij}/2 \rfloor & \text{if } v = v_j \\ f(u) & \text{otherwise,} \end{cases}$$

satisfying $f'(v_i) - f'(v_j) \leq 1$ and $|f'| = |f|$. By repeating this argument we may assume, without loss of generality that $x_i = \lfloor X/n \rfloor$ or $x_i = \lceil X/n \rceil$ for all $v_i \in N$. It follows by a similar argument that we may assume that $y_j = \lfloor Y/m \rfloor$ or $y_j = \lceil Y/m \rceil$ for all $v_j \in M$. Then $|f| = X + Y$, explaining the objective function (4). The constraints (5)–(6) ensure that f is an s-dominating function of $K_{n,m}$, while the constraints (7)–(8) ensure that f is an r-function of $K_{m,n}$. ∎

The result above naturally gives rise to the following algorithm for determining the value of $\gamma\langle r, s, K_{m,n}\rangle$ exactly.

Algorithm 1 (Complete Bipartite Graphs)

Input: $r, s \in \mathbb{N}_0$, $m, n \in \mathbb{N}$ satisfying $m \geq n$ and $s \leq (n+1)r$
Output: $\gamma\langle r, s, K_{m,n}\rangle$

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1 $x \leftarrow ns(m-1)/(mn-1)$
2 $y \leftarrow ms(n-1)/(mn-1)$
3.1 if $nr \leq \lfloor x \rfloor$ then output $nr + m(s - nr)$
3.2 else if $(\lceil x \rceil, \lfloor y \rfloor)$ or $(\lfloor x \rfloor, \lceil y \rceil)$ satisfies (5)–(8) then output $\lceil x \rceil + \lfloor y \rfloor$
3.3 else output $\lceil x \rceil + \lceil y \rceil$

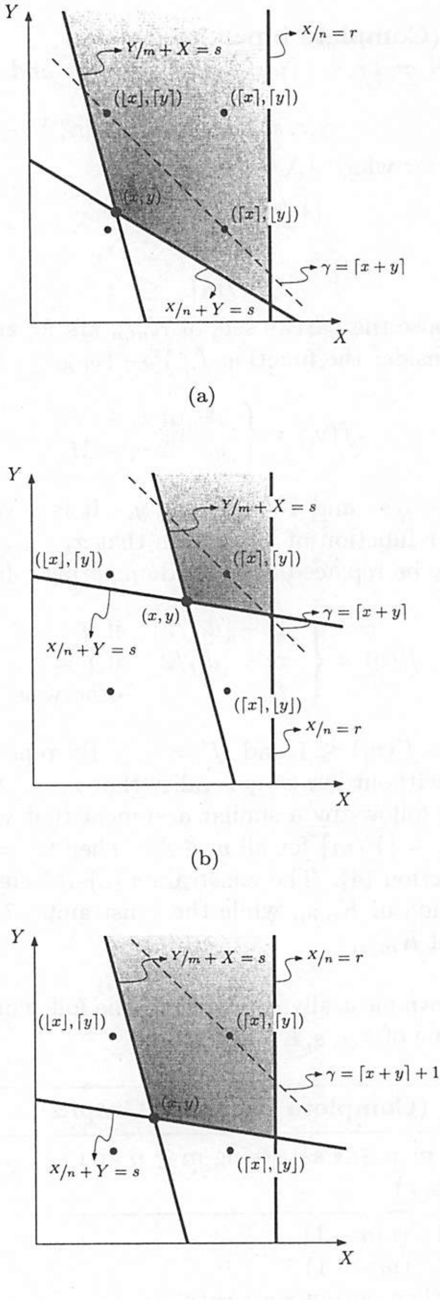


Figure 3.1: Three possibilities for the feasible domain in Proposition 2, if the constraint (7) is not binding.

The algorithm finds the smallest value of c for which $X+Y=c$, and for which X and Y are integers, by determining the lowest line with slope -1 intersecting an integer pair in the feasible domain (5)–(8) in (X,Y)-space. The values of x and y in Steps 1–2 of the algorithm are the simultaneous solutions to the system

$$\begin{cases} \frac{x}{n}+y &= s, \\ \frac{y}{m}+x &= s, \end{cases}$$

and are given by

$$x=\frac{ns(m-1)}{mn-1} \quad \text{and} \quad y=\frac{ms(n-1)}{mn-1}.$$

In Step 3.1 it is tested whether constraint (7) is binding (*i.e.* whether equality is achieved in (7)), in which case maximum values for the r-function are required at the vertices in the partite set N. Notice that we need not test whether (8) is binding, because the requirement $m \geq n$ implies $Y/m \leq X/n \leq r$. If constraint (7) is not binding and the coordinates $(\lceil x\rceil, \lfloor y\rfloor)$ and $(\lfloor x\rfloor, \lceil y\rceil)$ satisfy (5)–(8), then the situation in Figure 3.1(a) results (Step 3.2 of the algorithm). Otherwise the situations in Figure 3.1(b)–(c) result (Step 3.3 of the algorithm). If (7) is not binding, an optimal objective function value of $\lceil x+y\rceil$ is achieved in most cases (see Figures 3.1(a)–(b)), but sometimes an optimal value of $\lceil x+y\rceil+1$ results (see Figure 3.1(c)). The following result follows from these observations.

Proposition 3 (Complete bipartite graphs)
If constraint (7) is binding, then $\gamma\langle r,s,K_{m,n}\rangle = m(s-nr)+nr$. Otherwise

$$\left\lceil\frac{ns(m-1)+ms(n-1)}{nm-1}\right\rceil \leq \gamma\langle r,s,K_{m,n}\rangle \leq \left\lceil\frac{ns(m-1)+ms(n-1)}{nm-1}\right\rceil+1.$$

3.2 Paths and Cycles

The value of $\gamma\langle r,s\rangle$ for a cycle C_n of order n is established in the following proposition.

Proposition 4 (Cycles) $\gamma\langle r,s,C_n\rangle = \left\lceil\frac{sn}{3}\right\rceil$ *for all $r,s\in\mathbb{N}_0$ satisfying $s\leq 3r$.*

Proof: Let $n=3\ell+j$ for some $\ell\in\mathbb{N}_0$ and some integer $0\leq j<3$. Then it follows by Corollary 1 that

$$\gamma\langle r,s\rangle \geq \left\lceil\frac{sn}{3}\right\rceil = \left\lceil\frac{s(3\ell+j)}{3}\right\rceil = s\ell+\left\lceil\frac{sj}{3}\right\rceil \tag{9}$$

for all $s \leq 3r$. For the upper bound, suppose the vertices of C_n are labelled $v_1, \ldots, v_n$. Let $r \geq \lceil s/3 \rceil$ and let f be the r-function satisfying $\sum_{i=1}^{3} f(v_{3k+i}) = s$ and $\lceil s/3 \rceil \geq f(v_{3k+1}) \geq f(v_{3k+2}) \geq f(v_{3k+3}) \geq \lfloor s/3 \rfloor$ for all $k = 0, \ldots, \ell - 1$ and $f(v_{3\ell+i}) = f(v_i)$ for all $0 < i \leq j$. Then f is s-dominating and it is easy to verify that

$$\gamma\langle r, s\rangle \leq |f| = s\ell + \left\lceil \frac{sj}{3} \right\rceil . \tag{10}$$

The result follows by a combination of (9)–(10). ■

Next we consider the value of $\gamma\langle r, s\rangle$ for a path P_n of order n.

Proposition 5 (Paths) *Suppose $r, s \in \mathbb{N}_0$ with $s \leq 2r$. Then*

$$\gamma\langle r, s, P_n\rangle = \begin{cases} \frac{sn}{3} + s - r & \text{if } n \equiv 0 \pmod 3 \\ s\lceil \frac{n}{3} \rceil & \text{if } n \equiv 1, 2 \pmod 3). \end{cases}$$

Proof: Suppose the vertices of P_n are $v_1, \ldots, v_n$, and that v_1 and v_n are the end vertices of the path. Consider an s-dominating r-function f of P_n with minimum weight. Then $f(v_1) \geq s - f(v_2) \geq s - r$ and similarly $f(v_n) \geq s - r$. Also, for *any* three consecutive vertices u, w, v of P_n it follows that $f(u) + f(v) + f(w) \geq s$.

Consider first the case where $n = 3\ell$ for some $\ell \in \mathbb{N}$. Then

$$\gamma\langle r, s\rangle = \underbrace{f(v_1) + f(v_2)}_{\geq s} + \underbrace{f(v_3) + \cdots + f(v_{n-1})}_{\geq s(\ell-1)} + \underbrace{f(v_n)}_{\geq s-r} \geq \frac{sn}{3} + s - r. \tag{11}$$

For the upper bound, consider the r-function f' of P_n satisfying $f'(v_{3k+1}) = s - r$, $f'(v_{3k+2}) = r$ and $f'(v_{3k+3}) = 0$ for all $k = 0, \ldots, \ell - 2$, together with $f'(v_{n-2}) = s - r$, $f'(v_{n-1}) = r$ and $f'(v_n) = s - r$. Then clearly f' is an s-dominating function of P_n, and so

$$\gamma\langle r, s\rangle \leq s\ell + (s - r) = \frac{sn}{3} + s - r, \tag{12}$$

proving the proposition for the case where n is a multiple of 3 by a combination of (11)–(12).

Consider next the case where $n = 3\ell + 1$ for some $\ell \in \mathbb{N}_0$. Then

$$\gamma\langle r, s\rangle = \underbrace{f(v_1) + f(v_2)}_{\geq s} + \underbrace{f(v_3) + \cdots + f(v_{n-2})}_{\geq s(\ell-1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \geq s\left\lceil \frac{n}{3} \right\rceil . \tag{13}$$

For the upper bound, consider the r-function f'' of P_n satisfying $f''(v_{3k+1}) = \lfloor s/2 \rfloor$, $f''(v_{3k+2}) = \lceil s/2 \rceil$ and $f''(v_{3k+3}) = 0$ for all $k = 0, \dots, \ell-2$, together with $f''(v_{n-3}) = \lfloor s/2 \rfloor$, $f''(v_{n-2}) = \lceil s/2 \rceil$, $f''(v_{n-1}) = \lceil s/2 \rceil$ and $f'(v_n) = \lfloor s/2 \rfloor$. Then clearly f'' is an s-dominating function of P_n, and so

$$\gamma\langle r, s\rangle \leq s(\ell+1) = s\left\lceil \frac{n}{3} \right\rceil, \tag{14}$$

proving the proposition for the case where $n \equiv 1 \pmod 3$ by a combination of (13)–(14).

Finally, consider the case where $n = 3\ell + 2$ for some $\ell \in \mathbb{N}_0$. Then

$$\gamma\langle r, s\rangle = \underbrace{f(v_1) + \cdots + f(v_{n-2})}_{\geq s\ell} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \geq s\left\lceil \frac{n}{3} \right\rceil. \tag{15}$$

For the upper bound, consider the r-function f''' of P_n satisfying $f'''(v_{3k+1}) = \lfloor s/2 \rfloor$, $f'''(v_{3k+2}) = \lceil s/2 \rceil$ and $f'''(v_{3k+3}) = 0$ for all $k = 0, \dots, \ell-1$ together with $f'''(v_{3\ell+1}) = \lfloor s/2 \rfloor$ and $f'''(v_{3\ell+2}) = \lceil s/2 \rceil$. Then clearly f''' is an s-dominating function of P_n, and so

$$\gamma\langle r, s\rangle \leq s(\ell+1) = s\left\lceil \frac{n}{3} \right\rceil, \tag{16}$$

proving the proposition for the case where $n \equiv 2 \pmod 3$ by a combination of (15)–(16). ■

3.3 A Class of Cartesian products

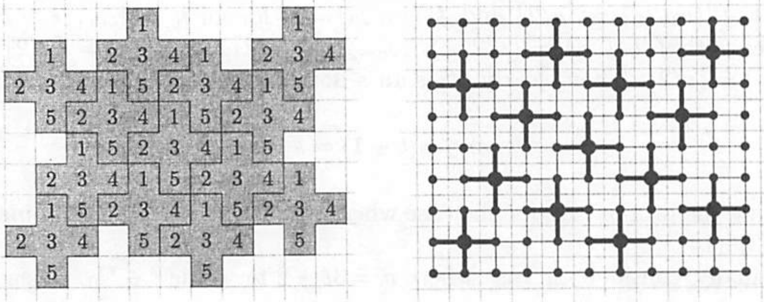
It seems to be a very hard problem to find a simple, closed formula for $\gamma\langle r, s\rangle$ for the grid graph $C_m \times C_n$ (in fact, such a formula is not even known for $\gamma\langle 1, 1\rangle$). We therefore merely establish an upper bound on $\gamma\langle r, s\rangle$ in this case. However, we then go on to show that this upper bound is good (and, in fact, often exact).

Proposition 6 (Grid graphs on a torus)

$$\gamma\langle r, s, C_m \times C_n\rangle \leq \left\lceil \frac{mns}{5} \right\rceil + g(m,n,s) + g(n,m,s) + h(m,n,s), \tag{17}$$

where

$$g(x,y,s) = \begin{cases} 0 & \text{if } s \equiv 0 \pmod 5 \text{ or } y \equiv 0 \pmod 5 \\ \left\lceil \frac{2(x-1)}{5} \right\rceil & \text{if } ys \equiv 3 \pmod 5 \\ \left\lceil \frac{x-1}{5} \right\rceil & \text{otherwise} \end{cases}$$



(a) The tiling pattern (b) Corresponding crosses

Figure 3.2: Upper bound construction on $\gamma\langle r, s, C_m \times C_n\rangle$.

and

$$h(m,n,s) = \begin{cases} \left\lceil \frac{2(n-1)}{5} \right\rceil - \left\lceil \frac{n-1}{5} \right\rceil & \text{if } m,n,s \equiv 2,3 \pmod 5 \\ & \text{or } s \equiv 2 \pmod 5 \;\&\; m \equiv n \equiv 1 \pmod 5 \\ & \text{or } s \equiv 3 \pmod 5 \;\&\; m \equiv n \equiv 4 \pmod 5 \\ 0 & \text{otherwise} \end{cases}$$

for all $r, s, m, n \in \mathbb{N}$ *satisfying* $4 \leq n \leq m$ *and* $s \leq 5r$.

Proof: We use the infinite tiling pattern (of crosses) in Figure 3.2(a) to assign values to an r-function $f^* : \mathcal{V}(P_\infty \times P_\infty) \mapsto \mathbb{N}_0$. The corresponding pattern of crosses is super-imposed on a portion of the graph in Figure 3.2(b) — the centre of each (bold faced) cross is shown slightly larger. Our approach is to ensure that f^* is an s-dominating r-function of the infinite grid graph and to use it to form an s-dominating r-function f of a finite grid graph (on the torus) by selecting an $m \times n$ subpattern of the infinite tiling pattern in Figure 3.2 and by wrapping the ends horizontally and vertically to form a copy of $C_m \times C_n$ with vertex set $\mathcal{V}$. We do this by letting $f(v) = f^*(v)$ for any vertex $v \in \mathcal{V}$ that is not adjacent to a wrapping seam and by applying correction terms to the function value at any vertex adjacent to a wrapping seam (the two functions $g(m,n,s)$ and $g(n,m,s)$ in (17)). Another correction term (the function $h(m,n,s)$ in (17)) arises because the best row patterns for the horizontal seam is not necessarily the best pattern for the vertical seam.

Notice that if values of f^* are chosen arbitrarily such that

$$\left.\begin{array}{c}\text{for any two vertices } u, v \text{ corresponding to the same} \\ \text{position within two crosses, } f^*(u) = f^*(v)\end{array}\right\} \quad (18)$$

and such that

$$\left.\begin{array}{c}\text{the values of } f^* \text{ add up to } s \text{ when evaluated over} \\ \text{all five vertices comprising any single cross,}\end{array}\right\} \quad (19)$$

then $\sum_{u \in N[v]} f^*(u) = s$ for every vertex v and hence f^* is an s-dominating r-function of the infinite grid graph if $f^*(v) \leq r$ for all vertices v. Furthermore, the tiling pattern consists of five row patterns $\boldsymbol{p}^{(0)}, \ldots, \boldsymbol{p}^{(4)}$ which may be obtained from one another by means of horizontal shifts. More specifically, if $p_i^{(j)}$ denotes the i-th entry of $\boldsymbol{p}^{(j)}$, then $p_i^{(j)} = p_{i-2j}^{(0)}$ for all $j = 1, 2, 3, 4$ and $i \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers. We shall assign either the value $\lceil s/5 \rceil$ or the value $\lfloor s/5 \rfloor$ to $f(v)$ for any $v \in \mathcal{V}$, as described below.

The sequence formed by the values of f^* along any row or column of the grid graph is periodic, with period 5. Therefore, if $n \equiv 0 \pmod 5$, then no correction term is required for the value of $f(v)$ at any vertex v adjacent to the wrapping seam of length m. A similar observation holds for the wrapping seem of length n if $m \equiv 0 \pmod 5$. Also, if $s \equiv 0 \pmod 5$, then we may choose $f(v) = s/5$ for all vertices $v \in \mathcal{V}$, in which case $g(m, n, s) = g(n, m, s) = h(m, n, s) = 0$ and $\gamma(r, s) = |f| = mns/5$.

We consider the other values of $s \pmod 5$ separately, first focussing on wrapping in one direction so as to form a single seam. The second wrapping seam is considered in more detail at the end of the proof.

The case $s \equiv 1 \pmod 5$. Let the row patterns for f^* be

$$\begin{aligned} p_i^{(0)} &= \begin{cases} \lceil s/5 \rceil & \text{if } i \equiv 0 \pmod 5 \\ \lfloor s/5 \rfloor & \text{otherwise,} \end{cases} \\ p_i^{(j)} &= p_{i-2j}^{(0)}, \quad j = 1, 2, 3, 4. \end{aligned}$$

Number the rows $0, \ldots, n-1$ with the horizontal wrapping seam between rows 0 and $n-1$. Similarly number the columns $0, \ldots, m-1$ with the vertical wrapping seam between columns 0 and $m-1$. Let row j have pattern $\boldsymbol{p}^{(j \pmod 5)}$ for all $j = 1, \ldots, n-1$. Let each value in row 0 be the maximum of the corresponding values in pattern $\boldsymbol{p}^{(0)}$ and pattern $\boldsymbol{p}^{(n \pmod 5)}$, essentially causing an overlap of the periodic tiling pattern in Figure 3.2 in the n-direction by one row. This ensures that $\sum_{u \in N[v]} f(u) \geq s$ for every vertex v in row $n-1$. However, a similar inequality does not necessarily hold for row 0. If not, then we replace the value of f for every vertex in row $n-1$ by the maximum of the corresponding values of f

(a)

row $n-1$	0	0	1	0	0	0	0	1	0	0	0	0	...
row 0	1	0	0	0	1	1	0	0	0	1	1	0	...
row 1	0	0	1	0	0	0	0	1	0	0	0	0	...
row 2	0	0	0	0	1	0	0	0	0	1	0	0	...
row 3	0	1	0	0	0	0	1	0	0	0	0	1	...
row 4	0	0	0	1	0	0	0	0	1	0	0	0	...

(b)

row $n-1$	0	0	0	1	1	0	0	0	1	1	0	0	...
row 0	1	1	0	0	0	1	1	0	0	0	1	1	...
row 1	0	0	1	0	0	0	0	1	0	0	0	0	...
row 2	0	0	0	0	1	0	0	0	0	1	0	0	...
row 3	0	1	0	0	0	0	1	0	0	0	0	1	...
row 4	0	0	0	1	0	0	0	0	1	0	0	0	...

Table 3.1: The procedure of overlapping described in the case $s \equiv 1 \pmod 5$ when (a) $n \equiv 2 \pmod 5$ or (b) $n \equiv 3 \pmod 5$.

in pattern $p^{(n-1 \,(\mathrm{mod}\, 5))}$ and pattern $p^{(4)}$, essentially extending the overlap by one more row. The procedure is illustrated in Table 3.1 for the cases (a) $n \equiv 2 \pmod 5$ and (b) $n \equiv 3 \pmod 5$. The table entries represent the values of $f(v) - \lfloor s/5 \rfloor$. Values of f due to the overlapping of the tiling pattern during the wrapping process are indicated in bold face. In (a) row 0 is replaced by max{pattern $p^{(0)}$, pattern $p^{(2)}$}. In (b) row 0 is replaced by max{pattern $p^{(0)}$, pattern $p^{(3)}$} and row $n-1$ is replaced by max{pattern $p^{(n-1 \,(\mathrm{mod}\, 5))}$, pattern $p^{(4)}$}. For the case $n \equiv 2 \pmod 5$ an overlap of just one row is sufficient, but for the case $n \equiv 3 \pmod 5$ a double overlap is required. It is easy to verify that for $n \equiv 1, 2, 4 \pmod 5$ one overlap is sufficient, in which case the correction term is approximately $n/5$. However, for $n \equiv 3 \pmod 5$ the correction term is approximately $2n/5$ (along the concerned seam). Since we ensure that $f(v^*) = \lceil s/5 \rceil$ for the first vertex v^* in row 0, the correction term will be at most $\lceil (n-1)/5 \rceil$ or $\lceil 2(n-1)/5 \rceil$ for the two cases respectively.

The case $s \equiv 4 \pmod 5$. This case is similar to the case $s \equiv 1 \pmod 5$, except that

$$p_i^{(0)} = \begin{cases} \lfloor s/5 \rfloor & \text{if } i \equiv 4 \pmod 5 \\ \lceil s/5 \rceil & \text{otherwise.} \end{cases}$$

Here the the correction term is at most $\lceil (n-1)/5 \rceil$ for the cases $n \equiv 1, 3, 4 \pmod 5$, and at most $\lceil 2(n-1)/5 \rceil$ for the case $n \equiv 2 \pmod 5$.

The cases $s \equiv 2, 3 \pmod 5$. These two cases differ from the previous two in the sense that there are two possible tiling patterns, and the vertical and horizontal (cyclic) patterns are necessarily different. For example, if the horizontal row pattern in the equivalent of Table 3.1 is $0,0,0,1,1,0,0,0,1,1,\ldots$ then the vertical pattern will be $0,0,1,0,1,0,0,1,0,1,\ldots$ (and *vice versa*). We may therefore choose the pattern adjacent to the longer seam that results in the smallest correction term, but that will

(a) $s \equiv 2 \pmod 5$

$n \pmod 5 \rightarrow$	1	2	3	4
$1,0,1,0,0,\ldots$	$\lceil\frac{m-1}{5}\rceil$	$\lceil\frac{2m-2}{5}\rceil$	$\lceil\frac{2m-2}{5}\rceil$	$\lceil\frac{2m-2}{5}\rceil$
$1,1,0,0,0,\ldots$	$\lceil\frac{2m-2}{5}\rceil$	$\lceil\frac{m-1}{5}\rceil$	$\lceil\frac{m-1}{5}\rceil$	$\lceil\frac{2m-2}{5}\rceil$

(b) $s \equiv 3 \pmod 5$

$n \pmod 5 \rightarrow$	1	2	3	4
$1,1,1,0,0,\ldots$	$\lceil\frac{2m-2}{5}\rceil$	$\lceil\frac{m-1}{5}\rceil$	$\lceil\frac{m-1}{5}\rceil$	$\lceil\frac{2m-2}{5}\rceil$
$1,1,0,1,0,\ldots$	$\lceil\frac{2m-2}{5}\rceil$	$\lceil\frac{2m-2}{5}\rceil$	$\lceil\frac{2m-2}{5}\rceil$	$\lceil\frac{m-1}{5}\rceil$

Table 3.2: Upper bounds on the correction term for each type of row pattern.

fix the pattern along the shorter seam, which may be undesirable. The function h in (17) compensates for such undesirability in cases where the correction term for the shorter seam is forced to be $\lceil 2(n-1)/5\rceil$ instead a sufficient increase of $\lceil (n-1)/5\rceil$. Upper bounds on the correction term as a result of overlapping during the wrapping process are shown in Table 3.2 for each type of row pattern. Notice that it is always sufficient to have a correction term of at most $\lceil (m-1)/5\rceil$ for both seams, except when $m \equiv 4 \pmod 5$ and $s \equiv 2 \pmod 5$, or when $m \equiv 1 \pmod 5$ and $s \equiv 3 \pmod 5$. Thus, in all cases when the upper bound on the correction terms for both seams is $\lceil 2(m-1)/5\rceil$, it holds that $ms \equiv 3 \pmod 5$. The function h in (17) may be determined from Table 3.2. ■

We demonstrate the result of Proposition 6 by means of a numerical example.

Example 1 $\gamma\langle r, 8, C_9 \times C_4\rangle \leq \lceil 9 \times 4 \times 8/5\rceil + 2 + 1 + 1 = 62$ *as shown in Table 3.3(a). Similarly,* $\gamma\langle r, 1, C_9 \times C_4\rangle \leq \lceil 9 \times 4 \times 1/5\rceil + 2 + 1 + 0 = 11$ *as shown in Table 3.3(b), which compares favourably with the exact value* $\gamma\langle 1, 1, C_9 \times C_4\rangle = 9$ *established by Klavžar and Seifter [4, Theorem 2.5].* □

In fact, in the classical domination setting (where $r = s = 1$) our construction in Proposition 6 gives the correct result, namely $\gamma\langle 1, 1, C_m \times C_4\rangle = m$ as determined by Klavžar and Seifter [4, Theorem 2.5] for the case $m \equiv 0 \pmod 5$, and overestimates this value by $1, 2, 3, 2$ for the cases $m \equiv 1, 2, 3, 4 \pmod 5$ respectively. This overestimation of course becomes negligible as m increases.

Furthermore, Proposition 6 gives the exact value of $\gamma\langle 1, 1, C_m \times C_5\rangle$ for the cases $m \equiv 0, 1, 2, 4 \pmod 5$ and matches the upper bound on $\gamma\langle 1, 1, C_m \times C_5\rangle$ established by Klavžar and Seifter [4, Theorem 2.6] for the case $m \equiv 3 \pmod 5$.

(a) $s = 8$

column	1	2	3	4	5	6	7	8	9
row 0	2	2	1	2	2	2	2	1	2
row 1	2	1	2	2	1	2	1	2	2
row 2	2	1	2	1	2	2	1	2	1
row 3	2	2	2	1	2	1	2	2	1

(b) $s = 1$

column	1	2	3	4	5	6	7	8	9
row 0	1	0	0	1	0	1	0	0	1
row 1	0	0	1	0	0	0	0	1	0
row 2	0	0	0	0	1	0	0	0	0
row 3	0	1	0	0	0	0	1	0	0

Table 3.3: Values of an s-dominating r-function f for the graph $C_9 \times C_4$.

4 Further work

Further, related work may include determining the value of $\gamma\langle r, s\rangle$ for other graph classes, such as complete multipartite graphs, circulant graphs, and other Cartesian graph products, such as $P_m \times P_n$, $P_m \times C_n$, $K_m \times K_n$, $K_m \times P_n$ and $K_m \times C_n$. The upper domination parameter $\Gamma\langle r, s\rangle$ (the largest weight of a *minimal* s-dominating r-function) is also of interest. Establishing general lower and upper bounds on $\Gamma\langle r, s\rangle$, as well as determining the value of $\Gamma\langle r, s\rangle$ for special graph classes, such as those mentioned above and considered in this paper, may also be a worth-while endeavour.

Acknowledgements

The authors are indebted to Dr Werner Gründlingh, who produced the graphics in this paper, and to the anonymous referees who made valuable suggestions with respect to improvement of the paper. Work towards this paper was funded by Research Sub-committee B of the University of Stellenbosch in the form of a post-doctoral fellowship for the first author, as well as by the South African National Research Foundation in the form of a research grant (GUN 2072999) for the second author.

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